Reconstructed b_3 Hyperdissipative Dynamics for Generalized Affine One-Phase Navier Stokes equations on $\mathbb{T}^3$

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Abstract

This manuscript develops and systematically audits a finite-N mechanism for terminal amplification of the reconstructed scalar product $b_3 = u_x u_y$ for generalized three-dimensional incompressible Navier–Stokes equations on $\mathbb{T}^3$ with fractional dissipation $1 \leq \alpha < 5/4$. The approach combines an exact scalar product equation, affine one-phase reduction of the fractional Laplacian, Mellin analysis, Lambert-W regularization, three-dimensional velocity reconstruction, incompressibility and pressure compatibility, moving coterminal geometry, and finite-dimensional spectral certification. A central analytic step begins with the explicit regular profile at the classical endpoint $\alpha = 1$ and continues the profile equation to a nonempty interval $1 < \alpha < 1 + \delta$, providing the profile structure needed to reach and control the terminal similarity layer. Within that layer, persistence yields the characteristic scaling

$$b_{3,N}(s) \asymp s^{-(2\alpha-1)};$$

combined with the characteristic relation

$$\dot{s} = 1 - b_{3,N},$$

this produces finite-time blowup of the finite-N scalar $b_{3,N}$, with

$$b_{3,N}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}},$$

under the manuscript's stated certification and persistence hypotheses.

1 Introduction

The three-dimensional incompressible Navier–Stokes equations occupy a central position in mathematical fluid mechanics because they combine a comparatively simple physical formulation with an exceptionally difficult nonlinear regularity problem. The interaction of transport, pressure, incompressibility, and viscous dissipation permits substantial transfer of energy and structure across spatial scales, while at the same time imposing strong geometric

constraints on any admissible velocity field. The principal difficulty is therefore not merely to identify a quantity that can become large, but to determine whether such growth can occur while all parts of the fluid system remain mutually compatible.

The present manuscript develops and audits a particular mechanism for finite-time amplification built around a scalar product of two velocity components, a one-phase similarity reduction, a Lambert–W regularization, and a reconstruction of the underlying three-dimensional velocity field. The analysis is carried out primarily on the periodic domain $\mathbb{T}^3$, with generalized dissipation

$$(-\Delta)^{\alpha}, \qquad 1 \le \alpha < \frac{5}{4},$$

and with special attention to a small interval immediately above the classical Navier–Stokes exponent $\alpha = 1$.

The guiding quantity throughout the construction is

$$b_3 = u_x u_y.$$

This product is not introduced as an unrelated scalar model. It is obtained from the velocity equations themselves by combining the first two momentum equations. Its evolution consequently retains contributions from transport, pressure, forcing, dissipation, and the nonlinear interaction between the velocity components. This makes b_3 useful as a diagnostic variable: growth of b_3 records genuine growth in the reconstructed velocity components, but such growth can be relevant to the Navier–Stokes problem only if the accompanying velocity, pressure, incompressibility, and dissipative structures can also be reconstructed consistently.

This distinction between a scalar amplification mechanism and a complete fluid solution is fundamental to the organization of the paper. Much of the manuscript is devoted not to producing additional formal singular formulas, but to determining which parts of the proposed mechanism survive increasingly strict compatibility tests. Earlier reductions are repeatedly revisited when continuity, pressure, fractional dissipation, reconstruction geometry, operator domains, transverse forcing, or finite-N effects introduce additional constraints. The resulting document is therefore both a construction and a systematic audit of that construction.

The classical endpoint as an analytic anchor

A major organizing principle is to begin at the classical dissipation exponent

$$\alpha = 1,$$

where the profile problem becomes substantially more explicit. At this endpoint the relevant Mellin-conjugated fractional operator reduces to an ordinary differential operator, an explicit profile is available, and the linearized equation can be studied directly.

The manuscript uses this solvable endpoint as an anchor rather than as an isolated special case. The objective is to continue the profile into a nonempty interval

$$1 < \alpha < 1 + \delta < \frac{5}{4}.$$

Doing this rigorously requires considerably more than formal dependence on α. The operators for different values of the exponent must act on compatible spaces, their domains must be compared, the linearized kernel must be understood, and the inverse problem must remain stable as the exponent varies.

This leads naturally to the Mellin representation used throughout the first major part of the paper. Logarithmic variables convert the profile problem into a form in which endpoint behavior, translation invariance, and fractional differentiation can be examined simultaneously. The exact linearization at $\alpha = 1$ is analyzed, its translation mode is identified, and a second homogeneous solution is constructed. Weighted spaces are then chosen to distinguish admissible from inadmissible endpoint behavior.

A significant feature of this analysis is the correction of an apparent Fredholm obstruction that arises in an intermediate formulation. After the weighted dual spaces and endpoint behavior are treated correctly, the relevant admissible kernel is the translation mode. A phase condition removes this degeneracy. The continuation problem can therefore be formulated as a phase-fixed inverse problem rather than one requiring an artificial additional amplitude parameter.

The manuscript subsequently establishes the symbol estimates and common-domain control needed to compare the fractional operators for α near one. The purpose of these sections is to place the continuation argument on a fixed analytic foundation before any terminal blow-up mechanism is invoked.

One-phase geometry and the terminal profile

The next structural ingredient is a strict one-phase architecture. The relevant fields are first considered as functions of a single affine phase. For such functions, the three-dimensional fractional Laplacian reduces exactly to a one-dimensional fractional operator along the Fourier line generated by the phase direction. This is not a heuristic dimensional reduction: on the strict one-phase subspace it follows from the Fourier support of the function.

A distinguished phase of the form

$$s = t - z - \theta_{*,N}$$

plays a particularly important role. Along the corresponding reduced characteristics its evolution is tied directly to the scalar product variable. This relationship provides the kinematic bridge between a singularity expressed in the phase variable and finite physical time.

Similarity analysis then identifies the natural terminal power law. For dissipation exponent α, the scalar profile has the characteristic strength

$$b(s) \sim s^{-(2\alpha-1)}.$$

The exponent is determined by balancing transport and fractional dissipation in the reduced equation. The manuscript does not rely solely on this formal balance. It derives the associated Mellin profile equation, studies its endpoint equilibria, continues the profile away from the explicit $\alpha = 1$ anchor, and develops a persistence framework for the resulting terminal regime.

Once the phase profile is combined with the characteristic relation, the terminal phase distance is converted into physical time. This produces the finite-time scaling

$$b(t) \asymp (T_* - t)^{-\frac{2\alpha-1}{2\alpha}},$$

provided that the actual reconstructed trajectory enters and remains in the terminal profile regime. Establishing precisely that entry and persistence is one of the main tasks of the later portions of the manuscript.

Lambert–W regularization and finite-N dynamics

The terminal construction is regularized at finite N through a Lambert–W cascade. The purpose of this cascade is not merely to supply an explicit special-function representation. It provides a controlled finite approximation to the branch geometry associated with the terminal regime.

The Lambert branch point introduces a natural small distance variable. Successive renormalized iterations describe how this distance changes through the cascade and make it possible to distinguish several limiting procedures that would otherwise be easily confused. In particular, fixed cascade depth, increasing depth, large N, and approach to the Lambert branch point do not represent interchangeable limits.

This distinction is important for the interpretation of the velocity field. A quantity that becomes small at a fixed depth need not behave in the same way when the depth is allowed to increase with N. The manuscript therefore treats the finite-N problem first and keeps the eventual large-N question logically separate.

The finite-N viewpoint has another advantage: all quantities remain regular before the terminal time, allowing the reconstruction, continuity equation, characteristic flow, spectral problem, and numerical certification to be formulated as finite mathematical problems. The terminal mechanism is then approached through controlled finite objects rather than by assuming a singular limiting field from the outset.

Reconstruction of the three-dimensional velocity

A scalar profile for b_3 does not determine a Navier–Stokes velocity field. The manuscript therefore reconstructs the individual components u_x, u_y, u_z subject to the product relation

$$u_x u_y = b_3$$

and the incompressibility condition.

This reconstruction introduces an important second layer of analysis. A choice that reproduces the desired scalar product may nevertheless create unacceptable spatial derivatives, violate periodicity, fail to satisfy continuity, or generate incompatible pressure terms. Consequently the reconstruction is treated as a mathematical problem in its own right rather than as a purely algebraic afterthought.

Geometric relations are used to distribute the scalar amplification between the individual velocity components. Near distinguished spatial sets these relations lead to square-root-type reconstruction laws. Such laws can produce a velocity field with finite local kinetic energy

even when certain spatial derivatives become much larger. This difference between L^2 control of the velocity and H^1-type control of its derivatives becomes especially important in the later Leray audit.

Continuity, pressure, and the danger of scalar closure

The incompressibility equation is one of the principal constraints on the reconstruction. It determines part of the correction required to convert the scalar ansatz into a three-dimensional velocity field. Importantly, the manuscript does not ultimately treat this correction as an arbitrary algebraic term. Its linearization leads to a first-order transport problem, whose inverse is governed by the characteristic flow of the associated transport field.

This observation corrects a stronger isotropic inverse estimate that would not follow merely from a zero-mean condition. The appropriate estimate is instead constructed in flow-box coordinates. The resulting characteristic description provides explicit finite-N control of the continuity response and its derivatives.

Pressure introduces a related compatibility problem. A scalar product equation can be satisfied without the three reconstructed momentum equations possessing one common periodic pressure. Pressure compatibility must therefore be checked rather than inferred from the scalar dynamics. Several later audits in the manuscript are devoted precisely to separating identities that follow from the product equation from those that require the full velocity–pressure system.

This distinction is one of the central methodological lessons of the paper: scalar closure is necessary for the proposed mechanism, but it is not equivalent to closure of the three-dimensional Navier–Stokes equations.

From terminal asymptotics to actual finite-N entry

A terminal profile is useful only if the evolution generated by the prescribed finite-N data actually reaches it. The manuscript therefore develops a first-crossing and bridge analysis connecting the preterminal reconstructed flow to the asymptotic terminal regime.

The scalar part of the normalized dynamics possesses a restoring structure on a bounded corridor around the desired profile. Scalar control alone, however, is insufficient because perturbations can also occur in shape and transverse directions. The corresponding differentiated equations lead to a spectral stability problem.

A major portion of the later manuscript is devoted to identifying the correct state space for this problem. Several initially plausible formulations are replaced after direct substitution into the raw reconstruction reveals additional continuity or transverse terms. The final architecture treats the continuity correction as a slaved transport response and separates the principal fractional damping from a finite lower-order perturbation budget.

The purpose of this bridge is to prove that the actual finite-N reconstructed trajectory enters the neighborhood in which the terminal bootstrap applies. Once entry occurs, the invariant-region and persistence estimates propagate the terminal power law to the finite terminal time.

Why the prescribed forcing changes the bridge

The forcing used in the complete reconstruction contains genuine transverse dependence. This has an important consequence: even though the strict one-phase manifold is analytically convenient and is exactly preserved by the reduced fractional operator, it need not be invariant under the full forced dynamics.

The manuscript therefore does not ultimately require the forced solution to converge to a purely one-phase state. Instead, the persistent transverse forcing is incorporated into a forced principal bridge. The one-phase profile supplies the longitudinal terminal mechanism, while the transverse response generated by the forcing is included in the principal comparison field rather than being incorrectly classified as a decaying error.

The remainder is then measured relative to this forced bridge. This reformulation removes a potential circularity: one no longer assumes decay toward a manifold that the prescribed forcing continuously drives the solution away from.

Spectral decomposition and finite certification

The final stability problem naturally separates into compact and high frequencies.

At high frequency, the leading fractional operator has a coercive symbol of order 2α. The lower-order reconstruction and continuity terms grow more slowly in frequency. This permits the high-frequency tail to be controlled analytically.

The bounded-frequency sector is genuinely finite dimensional. After Fourier truncation, the Hermitian part of the linearized operator is represented by a finite matrix. Its spectral gap can therefore be enclosed by outward-rounded interval arithmetic. The manuscript combines this finite enclosure with the analytic high-frequency estimate to obtain a rigorous stability test.

This is an important conceptual reduction. The decisive stability condition is no longer an informal numerical observation about a plotted spectrum or a single approximate eigenvalue. It becomes a finite collection of inequalities whose inputs, truncation levels, coefficient bounds, and rounding errors can in principle be recorded and reproduced.

The same philosophy is applied to the continuity inverse, the forced bridge, the coefficient-decay estimates, and the first-crossing conditions. The result is a finite-N certification program in which all quantities needed for entry into the terminal regime are associated with one consistent parameter record.

The proof architecture

The manuscript can therefore be understood as a sequence of logically distinct layers.

The first layer derives the scalar product equation directly from the fluid system and identifies the one-phase terminal scaling.

The second layer constructs the profile at the classical endpoint and continues it to a small interval of fractional exponents through Mellin analysis, phase fixing, weighted spaces, and perturbation of the fractional symbol.

The third layer introduces the finite-N Lambert regularization and reconstructs the individual velocity components.

The fourth layer audits incompressibility, geometric compatibility, coterminal behavior, derivative control, and pressure.

The fifth layer connects the prescribed finite-N dynamics to the terminal profile through a bridge and first-crossing argument.

The sixth layer incorporates the persistent transverse forcing into the principal comparison field.

The final layer reduces the remaining finite-N stability question to a reproducible certification problem consisting of a finite compact-frequency matrix calculation, an analytic high-frequency tail estimate, characteristic continuity bounds, and coefficient-decay inequalities.

In compressed form, the intended finite-N chain is

$$\boxed{\text{reconstructed finite-}N\text{ data} \longrightarrow \text{forced principal bridge} \longrightarrow \text{first crossing} \longrightarrow \text{terminal persistence} \longrightarrow \text{finite-time scalar amplification.}}$$

Each arrow represents a separate mathematical obligation. A substantial purpose of the manuscript is to prevent one of these implications from being silently substituted for another.

What the finite-N conclusion means

At fixed finite N, the terminal mechanism becomes conditional on a specific and reproducible set of inequalities. Once the reconstruction is nondegenerate, the continuity inverse is controlled, the forced bridge exists, the compact and high-frequency stability margins are positive, and the required coefficient differences decay, the first-crossing argument places the actual scalar trajectory inside the terminal regime. The persistence analysis then gives the terminal power law and converts it into finite physical time.

This conclusion is stronger than a purely formal singular ansatz because it specifies how the prescribed finite-N dynamics are to be connected to the terminal profile and identifies the finite quantities that must be certified.

It is nevertheless essential to distinguish this statement from a theorem asserting finite-time singularity of a limiting three-dimensional Navier–Stokes solution.

A fixed-N divergence of b_3 does not automatically survive a coupled limit in N, nor does the scalar product equation alone guarantee that all three momentum equations possess a common admissible pressure. A full Navier–Stokes conclusion additionally requires uniform control of the reconstructed velocity in fixed spatial function spaces, global incompressibility, pressure compatibility, sufficient compactness for the nonlinear and dissipative terms, and a limiting procedure that preserves the estimates on which the finite-N construction depends.

The manuscript therefore maintains a deliberate separation between three levels of conclusion:

1. the analytic terminal profile and its finite-time scalar amplification mechanism;
2. the finite-N reconstructed dynamics and the finite certification needed to connect prescribed data to that terminal regime;
3. the substantially stronger problem of passing to a genuine limiting three-dimensional Navier–Stokes solution.

This separation is not merely a matter of presentation. It is necessary for identifying exactly which statements have been established analytically, which depend on finite interval certification, and which require additional global velocity–pressure and compactness arguments.

Role of the repeated audits

The length and iterative structure of the manuscript reflect this distinction. Several parts revisit conclusions obtained earlier in the development. These revisions are mathematically significant. They expose places where a formal one-phase argument, a naive Fredholm count, an isotropic continuity inverse, an assumed transverse decay, or an incomplete spectral reduction would have been stronger than the displayed equations justified.

Rather than concealing these corrections, the manuscript records them and uses them to narrow the remaining proof obligations. The continuation problem is reformulated on a common operator domain; the translation degeneracy is treated by phase fixing; the continuity correction is inverted along its actual characteristics; the transverse forcing is incorporated into the principal bridge; and the stability problem is divided into a finite compact-frequency certificate and an analytic high-frequency tail.

The repeated proof ledgers near the ends of the major audits should therefore be read as part of the logical structure of the paper. Their purpose is to distinguish established identities, conditional implications, finite computational obligations, and unresolved global consequences.

Scope and perspective

The central contribution of the manuscript is thus not a single explicit Lambert–W formula or a single similarity exponent. It is the assembly of several mechanisms that must operate simultaneously: an exact scalar product identity inherited from the velocity equations; a fractional one-phase profile continued from the classical endpoint; a finite-N regularization of the terminal geometry; a reconstruction compatible with continuity and periodicity; a moving coterminal geometry capable of avoiding the local derivative obstruction of the nonmoving construction; a characteristic treatment of the continuity response; a forced bridge that respects the actual transverse forcing; and a spectral certification procedure that separates finite frequencies from the coercive fractional tail.

The analysis also clarifies where the principal difficulties lie. Producing a formal terminal power law is comparatively straightforward. The harder questions concern entry into that regime from prescribed data, preservation of the required geometry under the complete dynamics, control of transverse and reconstruction errors, and compatibility of the scalar mechanism with the full three-dimensional velocity–pressure system.

Accordingly, the manuscript should be read as a layered finite-N construction with an explicit certification program, together with a continuing audit of the additional conditions required for a full Navier–Stokes interpretation. Its strongest conclusions are those for which the analytic and finite certification hypotheses are stated explicitly. Where a limiting three-dimensional conclusion requires additional uniform estimates or compatibility arguments,

those requirements are kept separate rather than absorbed into the scalar blow-up mechanism.

The resulting framework provides a precise route by which the proposed terminal dynamics can be tested. If the required finite-N certificates are verified for a consistent parameter set, the bridge and first-crossing arguments connect the reconstructed dynamics to the terminal scalar regime. If a certificate fails, the failure identifies a concrete obstruction rather than an ambiguous breakdown of the ansatz. Any subsequent passage from this finite-N mechanism to a singular solution of the complete three-dimensional Navier–Stokes equations must then address the remaining uniform reconstruction, pressure, compactness, and limiting questions directly.

In this sense, the manuscript has two closely related purposes. It develops a candidate mechanism for finite-time scalar amplification, and at the same time it establishes a framework for determining rigorously how much of that mechanism survives reconstruction into the full fluid equations. The interaction of these two objectives—construction and verification—is the organizing principle of the work that follows.

Separation Between Profile Continuation and Terminal Blowup

An important structural distinction should be emphasized before entering the technical development of the paper. Two analytically related, but logically different, mechanisms occur in the construction. The first is the continuation of a regular one-dimensional profile with respect to the dissipation exponent α. The second is the terminal amplification and finite-time blowup mechanism for the reconstructed product variable

$$b_3 = u_x u_y.$$

These two mechanisms should not be identified with one another. In particular, the regularity of the profile used as the anchor for continuation does not imply regularity of b_3 at the terminal point.

At the ordinary-viscosity value

$$\alpha = 1,$$

the profile problem admits the explicit family

$$F_1(s) = \left(\frac{3\nu}{2(s + s_0)} \right)^{1/2}.$$

For the continuation argument developed in this work, the parameter is chosen so that

$$s_0 > 0.$$

Consequently,

$$F_1(0) = \left(\frac{3\nu}{2s_0} \right)^{1/2} < \infty.$$

This choice is intentional. The role of $s_0 > 0$ is to provide a regular base profile at $\alpha = 1$ from which the profile equation may be continued to nearby values of the dissipation exponent. Thus the relevant continuation problem has the schematic form

$$(1, F_1) \longmapsto (\alpha, F_\alpha), \qquad 1 < \alpha < 1 + \delta,$$

for sufficiently small $\delta > 0$, within the weighted functional framework developed below. The continuation argument is therefore a statement about existence, regularity, and stability of the profile family F_α as the operator $(-\Delta)^\alpha$ is varied away from the ordinary Laplacian.

The terminal behavior of b_3, however, is organized differently. In the terminal scalar analysis the singular power is separated explicitly from a normalized amplitude. The relevant form is

$$b_{3,N}(s) = B_{*,N} s^{-(2\alpha-1)} F_N(s),$$

where the terminal amplitude is normalized so that

$$F_N(s) \longrightarrow 1 \qquad \text{as } s \downarrow 0$$

within the terminal regime established later in the paper. Thus the normalized factor F_N may remain bounded and regular while the physical product $b_{3,N}$ diverges. The singularity is carried by the explicit phase factor

$$s^{-(2\alpha-1)},$$

rather than by requiring the normalized amplitude itself to become singular.

This separation is especially transparent at $\alpha = 1$. In that case,

$$2\alpha - 1 = 1,$$

and the terminal law takes the form

$$b_{3,N}(s) = B_{*,N} s^{-1}\big(1 + o(1)\big).$$

Hence, provided $B_{*,N} > 0$ and the reconstructed trajectory has entered the terminal regime in which the asymptotic analysis applies,

$$b_{3,N}(s) \longrightarrow +\infty \qquad \text{as } s \downarrow 0.$$

There is therefore no contradiction between using a regular profile with $s_0 > 0$ in the continuation theory and obtaining singular terminal behavior for b_3. They serve different purposes in the proof.

The distinction can be summarized schematically as

$$\boxed{\begin{array}{c} s_0 > 0 \\ \Downarrow \\ \text{regular base profile } F_1 \\ \Downarrow \\ \text{continuation } F_1 \longrightarrow F_\alpha \\ \textit{separate terminal reconstruction} \\ \Downarrow \\ b_{3,N}(s) = B_{*,N} s^{-(2\alpha-1)} F_N(s) \\ \Downarrow \\ F_N(s) \to 1 \quad \text{while} \quad s^{-(2\alpha-1)} \to \infty \\ \Downarrow \\ b_{3,N}(s) \to \infty. \end{array}}$$

The phase variable s is then related to physical time through the characteristic dynamics. In the terminal regime considered in this work,

$$\frac{ds}{dt} = 1 - b_{3,N}.$$

If

$$b_{3,N}(s) = B_{*,N} s^{-(2\alpha-1)}\big(1 + o(1)\big),$$

then $b_{3,N} \gg 1$ as $s \downarrow 0$, and therefore

$$\frac{ds}{dt} = -B_{*,N} s^{-(2\alpha-1)}\big(1 + o(1)\big).$$

Equivalently,

$$dt = -\frac{1}{B_{*,N}} s^{2\alpha-1}\big(1 + o(1)\big)\, ds.$$

Integration from $s(t)$ to the terminal value $s = 0$ gives

$$T_{*,N} - t = \frac{s(t)^{2\alpha}}{2\alpha B_{*,N}}\big(1 + o(1)\big).$$

Since $\alpha > 0$, the right-hand side is finite for every positive $s(t)$. Thus the phase endpoint $s = 0$ is reached in finite physical time within the terminal characteristic regime.

Solving for $s(t)$ and substituting back into the terminal law gives

$$s(t) = [2\alpha B_{*,N}(T_{*,N} - t)]^{1/(2\alpha)}\,\big(1 + o(1)\big),$$

and hence

$$b_{3,N}(t) = B_{*,N}^{1/(2\alpha)}(2\alpha)^{-(2\alpha-1)/(2\alpha)}(T_{*,N} - t)^{-(2\alpha-1)/(2\alpha)}\big(1 + o(1)\big).$$

At the ordinary-viscosity endpoint $\alpha = 1$, this becomes

$$\boxed{b_{3,N}(t) \sim \left(\frac{B_{*,N}}{2}\right)^{1/2} (T_{*,N} - t)^{-1/2}.}$$

Accordingly, the terminal scalar mechanism at $\alpha = 1$ is a finite-time divergence of $b_{3,N} = u_x u_y$, even though the profile used to anchor the continuation in α is regular.

The same distinction is essential when α is moved slightly above one. The purpose of the continuation theory is not to force the continued profile F_α itself to become singular. Rather, it provides control of the profile-dependent structure entering the nearby-α problem while the terminal scalar analysis determines whether the explicitly extracted singular factor

$$s^{-(2\alpha-1)}$$

persists in the reconstructed dynamics. Thus the continuation and terminal arguments are complementary:

regular continuation in α + terminal singular reconstruction $\Longrightarrow$ candidate finite-time b_3 amplification mechanism.

There is also a notational point that is important throughout the paper. Profile functions and normalized terminal amplitudes may both be denoted by symbols of the form F in different stages of the analysis. Their roles must nevertheless remain conceptually distinct. The continuation profile F_α belongs to the fractional profile equation and is used to control variation with respect to α. The normalized terminal factor F_N appearing in

$$b_{3,N} = B_{*,N} s^{-(2\alpha-1)} F_N$$

belongs to the terminal reconstruction and is normalized relative to the explicit singular power. Any identification between these objects must be made through the transformations established in the corresponding sections; it is not assumed merely from the similarity of notation.

Finally, the finite-time divergence of the product variable b_3 should be distinguished from the final three-dimensional Navier–Stokes conclusion. The scalar terminal law establishes the behavior of the reconstructed product variable along the terminal characteristic once the hypotheses of the local persistence regime have been verified. Completion of the full argument still requires the reconstruction of the velocity components, exact incompressibility, compatibility with a common pressure, control of the moving coterminal and matching regions, the required Leray estimates, and the finite-N entry and limiting arguments developed in the subsequent parts of the paper. Maintaining this separation allows the continuation theorem, the scalar blowup mechanism, and the full velocity-pressure reconstruction to be assessed independently and then assembled in their proper logical order.

1.1 Background and motivation

The three-dimensional incompressible Navier–Stokes equations are among the central nonlinear partial differential equations of mathematical fluid mechanics. In their classical viscous form they describe the evolution of an incompressible velocity field

$$u = u(x,t) = (u_x, u_y, u_z)$$

and a scalar pressure field $P = P(x,t)$, subject to the divergence-free constraint

$$\nabla \cdot u = 0.$$

On a periodic three-dimensional domain, which throughout this manuscript is taken principally to be the three-torus

$$\mathbb{T}^3,$$

the equations may be written in the generalized dissipative form

$$\partial_t u + (u \cdot \nabla)u + \nabla P + \nu(-\Delta)^\alpha u = f, \qquad \nabla \cdot u = 0,$$

where

$$\nu > 0$$

is the viscosity parameter, f is an external forcing, and

$$(-\Delta)^\alpha$$

denotes the fractional power of the periodic Laplacian.

The classical Navier–Stokes equations correspond to

$$\alpha = 1.$$

A central theme of the present work is the behavior of the system for dissipation exponents lying just above this classical endpoint. Rather than restricting the analysis from the beginning to one isolated exponent, the manuscript develops a small-supercritical continuation framework of the form

$$\boxed{1 < \alpha < 1 + \delta < \frac{5}{4},}$$

where the size of the interval is to be determined analytically from continuation and stability estimates near the exactly accessible endpoint $\alpha = 1$.

The purpose of this viewpoint is not merely to perturb a known calculation at one value of α. The objective is to identify which parts of the construction are genuinely stable under variation of the fractional dissipation exponent and which parts require new estimates. This distinction becomes especially important in a long proof architecture in which exact algebraic identities, asymptotic profile calculations, finite-N regularizations, geometric reconstruction, spectral estimates, and computer-assisted finite-dimensional certification all interact.

The manuscript therefore has two complementary aims. The first is analytic: to derive and study a scalar quantity extracted from the velocity equations whose dynamics can exhibit a terminal power-law amplification. The second is structural: to audit, one by one, the additional compatibility conditions required before such scalar behavior can be interpreted as part of a genuine solution of the complete reconstructed system.

This second aim explains the unusually detailed character of the work. Many intermediate constructions that initially appear plausible are subjected to additional continuity, pressure, fractional-operator, geometric, fixed-space, spectral, and characteristic checks. When an earlier reduction is found to require an additional hypothesis, the later parts of the manuscript replace it by a more precise statement rather than silently retaining the stronger conclusion. The resulting document is consequently both a construction and an extended mathematical audit of that construction.

1.2 The scalar product variable $b_3 = u_x u_y$

The principal scalar quantity considered throughout the paper is

$$\boxed{b = b_3 := u_x u_y.}$$

This variable is not introduced as an independent scalar model. It is obtained directly from the velocity components of the underlying three-dimensional system.

Multiplying the x-momentum equation by u_y, multiplying the y-momentum equation by u_x, and adding gives the exact identity

$$\boxed{b_t + u \cdot \nabla b + u_y P_x + u_x P_y + \nu \left[u_y (-\Delta)^\alpha u_x + u_x (-\Delta)^\alpha u_y \right] = u_y f_x + u_x f_y.}$$

Thus the evolution of b_3 retains information about transport, pressure, forcing, dissipation, and the nonlinear coupling of the first two velocity components.

For a fractional Laplacian, the dissipative product cannot in general be replaced by an ordinary Leibniz rule. The appropriate object is the exact fractional product commutator

$$\mathcal{R}_\alpha(f,g) := (-\Delta)^\alpha(fg) - f(-\Delta)^\alpha g - g(-\Delta)^\alpha f.$$

Consequently,

$$u_y(-\Delta)^\alpha u_x + u_x(-\Delta)^\alpha u_y = (-\Delta)^\alpha b - \mathcal{R}_\alpha(u_x, u_y),$$

and the exact scalar equation becomes

$$\boxed{b_t + u\cdot\nabla b + u_y P_x + u_x P_y + \nu(-\Delta)^\alpha b = u_y f_x + u_x f_y + \nu\mathcal{R}_\alpha(u_x,u_y).}$$

This identity is one of the fundamental starting points of the manuscript. It also explains why a scalar blow-up calculation alone is insufficient. Any proposed terminal behavior for b_3 must ultimately remain compatible with the reconstruction of u_x, u_y, u_z, incompressibility, pressure, the fractional commutator, and the complete momentum equations.

1.3 Why the manuscript begins near $\alpha = 1$

A major analytic advantage occurs at

$$\alpha = 1.$$

At this endpoint the fractional profile operator reduces to an ordinary second-order differential operator, and an explicit profile is available. The manuscript uses this solvable endpoint as an anchor for continuation in α.

The guiding principle is therefore

$$\boxed{\text{explicit anchor at } \alpha = 1 \quad \Longrightarrow \quad \text{continuation to } 1 < \alpha < 1+\delta.}$$

This requires considerably more than formal continuity in the exponent. The nonlinear profile equation must be placed on function spaces having a common operator domain as α varies. The linearization at the anchor must be understood, its translation mode must be removed by a phase condition, and the resulting phase-fixed operator must admit an inverse with estimates that persist under small changes of α.

A substantial early portion of the manuscript is devoted to precisely this problem. In particular, the analysis passes through the Mellin representation of the fractional operator, weighted spaces at the two profile endpoints, the exact kernel of the anchor linearization, a second homogeneous solution, Green-function representations, Fredholm index calculations, phase fixing, and perturbation estimates for the Mellin symbol.

An important outcome of this audit is that the admissible kernel at the anchor is the translation mode. The formal adjoint expression that appears in an intermediate Fredholm calculation does not generate an additional cokernel in the weighted dual space used in the final formulation. Thus the continuation problem is phase-fixed by removing translation rather than by introducing an artificial amplitude parameter to compensate for a nonexistent obstruction.

1.4 Strict one-phase reduction

A second central ingredient is the use of an affine one-phase variable

$$s_N = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N}.$$

For a strict one-phase function

$$F = F(s_N),$$

the Fourier support lies on the one-dimensional lattice generated by

$$(C_1, C_2, C_3).$$

If

$$\kappa^2 = C_1^2 + C_2^2 + C_3^2,$$

then the three-dimensional fractional Laplacian reduces exactly to

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha F(s_N) = \kappa^{2\alpha}(-\partial_{ss})^\alpha F(s_N).}$$

The corresponding fractional product commutator reduces in the same way:

$$\boxed{\mathcal{R}_\alpha^{3D}(F(s_N), G(s_N)) = \kappa^{2\alpha}\mathcal{R}_\alpha^{1D}(F, G)(s_N).}$$

This is an exact Fourier-line identity rather than an approximation of the three-dimensional symbol.

Particular attention is given to the distinguished phase

$$\boxed{s_N = t - z - \theta_{*,N}.}$$

Along a characteristic satisfying

$$\dot{z} = b_N, \qquad \dot{t} = 1,$$

one obtains the exact kinematic relation

$$\boxed{\dot{s}_N = 1 - b_N.}$$

This simple identity becomes decisive when a terminal power law in the phase variable is converted into a finite physical time.

1.5 Similarity scaling and the terminal exponent

The similarity exponent depends on the fractional dissipation exponent. The manuscript derives

$$\boxed{r_\alpha = 2\alpha - 1.}$$

Accordingly, the candidate terminal behavior has the form

$$\boxed{b_{N,\alpha}(s) \asymp s^{-(2\alpha-1)}.}$$

This exponent is not kept fixed at the value inherited from a single specialized choice of α. One of the central revisions carried out throughout the manuscript is to preserve the full dependence

$$2\alpha - 1$$

over the small-supercritical interval.

If two-sided bounds

$$c_- s^{-(2\alpha-1)} \le b_{N,\alpha}(s) \le c_+ s^{-(2\alpha-1)}$$

hold in a terminal region, then the characteristic equation

$$\dot{s} = 1 - b_{N,\alpha}$$

converts the phase singularity into finite physical time.

Indeed, for sufficiently small s, one has $b_{N,\alpha} \ge 2$, and therefore

$$\frac{1}{2} b_{N,\alpha} \le -\dot{s} = b_{N,\alpha} - 1 \le b_{N,\alpha}.$$

Consequently,

$$\frac{1}{c_+} s^{2\alpha-1} \le \frac{dt}{-ds} \le \frac{2}{c_-} s^{2\alpha-1}.$$

Integration gives

$$\boxed{T_{*,N} - t \asymp s^{2\alpha},}$$

and hence

$$\boxed{b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}}.}$$

Thus the finite-time conversion itself is elementary once the two-sided terminal bounds have been rigorously established. Much of the later manuscript is concerned precisely with the much harder problem of proving that the complete reconstructed finite-N trajectory actually enters and remains in the region in which these bounds are valid.

1.6 The Mellin profile equation

The terminal profile analysis is naturally expressed through Mellin variables. For a half-line power s_+^λ, the fractional operator takes the form

$$(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda-2\alpha},$$

with Mellin coefficient

$$\boxed{K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi} \sin\big(\pi(\alpha-\lambda)\big).}$$

Writing

$$F(s) = s^{-a} G(\log s), \qquad a = \alpha - \frac{1}{2},$$

conjugates the fractional operator to a translation-invariant Mellin operator:

$$(-\partial_{ss})^{\alpha} F = s^{-a-2\alpha} \mathcal{M}_a G.$$

The resulting autonomous profile equation studied in the manuscript is of the form

$$\boxed{\nu \mathcal{M}_a G = G^2 G' - c G^3,}$$

with an appropriate positive coefficient c, together with heteroclinic endpoint conditions

$$G(-\infty) = 0, \qquad G(+\infty) = H_*.$$

At $\alpha = 1$, an explicit anchor profile is available:

$$F_1(s) = \sqrt{\frac{3\nu}{2(s + s_0)}}.$$

After logarithmic normalization this leads to the convenient anchor

$$Z_1(x) = (1 + e^{-x})^{-1/2},$$

up to the normalization adopted in the corresponding section.

We then derive the exact Mellin symbol, study its dependence on α, prove weighted estimates, identify the translation kernel, construct the second homogeneous solution, and obtain a phase-fixed inverse. These calculations form the analytic foundation of the local profile continuation theorem near $\alpha = 1$.

1.7 Lambert W, finite-N regularization, and the cascade

A distinctive feature of the construction is the use of the principal real branch of the Lambert W-function,

$$W_0.$$

The finite-N regularization and the associated iterated Lambert dynamics are used to control coefficients entering the reconstructed scalar equation.

A recurring renormalized variable is governed by the map

$$\boxed{\Delta_{N,k+1} = 1 + W_0\big(-e^{-1-\Delta_{N,k}}\big)\,.}$$

Near the Lambert branch point the map has a square-root structure, while away from the branch core it approaches a positive attracting fixed point. The manuscript develops this dynamics in considerable detail because derivatives of the iterates enter the transport and coefficient equations.

The logarithmic Euler sensitivity

$$S_{N,k} = E \log \Delta_{N,k}, \qquad E = s\partial_s,$$

satisfies an exact multiplicative recurrence

$$S_{N,k+1} = A(\Delta_{N,k})S_{N,k},$$

with

$$0 < A(\Delta) < \frac{1}{2}$$

on the relevant real interval. This gives the contraction estimate

$$|S_{N,k}| \le 2^{-k}|S_{N,0}|,$$

together with corresponding higher Euler-derivative bounds.

These identities are subsequently used to control the finite-N coefficient hierarchy, distinguish fixed-depth and increasing-depth regimes, analyze the branch core, and formulate the finite-N bridge estimates needed for entry into the terminal similarity region.

1.8 Reconstruction of the velocity field

The scalar b_3 cannot be studied in isolation if the final goal is a statement about a reconstructed fluid system. A substantial part of the paper therefore concerns reconstruction.

Several intermediate reconstruction architectures are examined and audited. The later raw two-correction architecture is written in the form

$$\boxed{u_y = A = Q + S - D, \qquad u_z = C = Q + S + D, \qquad u_x = \frac{b}{A}.}$$

Writing

$$V = Q + S,$$

one has

$$A = V - D, \qquad C = V + D.$$

The algebraic correction D is related to the prescribed Lambert coefficient through the reconstruction relation, while S is constrained by incompressibility. The exact continuity equation becomes

$$\boxed{\partial_x\left(\frac{b}{A}\right) + A_y + C_z = 0,}$$

or, in the raw variables,

$$S_y + S_z + D_z - D_y = (C_2 + C_3)Q' - \partial_x\left(\frac{b}{A}\right).$$

This equation plays a major role in the later parts of the manuscript. Because it is a first-order transport-type equation, its inversion requires a normalization and a characteristic analysis. The paper therefore studies the corresponding continuity-response operator rather than assuming that the correction S can be chosen arbitrarily.

1.9 Geometric reconstruction and why it is audited separately

Another major thread concerns the geometric coefficient

$$\chi(x) = \sin(x - x_*),$$

and the intrinsic nonnegative variable

$$\boxed{\sigma = \chi^2.}$$

Near a zero of χ, the relation

$$\chi = -\sqrt{\sigma}$$

on a chosen branch introduces fractional geometric powers into the reconstruction.

The paper derives the corresponding characteristic equation for σ, studies the induced fractional operator in this nonlinear coordinate, and compares it with the flat Mellin operator in a terminal chart.

This analysis is important for two reasons. First, a nonlinear geometric coordinate cannot simply be treated as an affine phase variable when applying the fractional Laplacian. Second, a reconstruction that is regular along a single characteristic need not be regular as a function on the entire three-dimensional torus.

The later fixed-space audits therefore examine local kinetic energy and Sobolev regularity near the zeros of the geometric coefficient. They also study whether concentration in a shrinking transverse region can compensate for the growth of the reconstructed velocity components. These checks separate a scalar characteristic singularity from a genuine fixed-space velocity singularity and are essential to the final interpretation of the construction.

1.10 From formal terminal asymptotics to finite-N entry

A recurring logical distinction in the manuscript is the difference between

existence of a terminal profile

and

proof that the prescribed finite-N trajectory reaches that profile.

The profile continuation theorem addresses the first question. The finite-N bridge, first-crossing, and conormal propagation arguments address the second.

The normalized logarithmic amplitude is compared with the terminal profile, and an entry mismatch is measured through finitely many Euler derivatives. Once the solution lies inside the terminal conormal tube, the manuscript derives a propagation mechanism that keeps it there and yields the terminal power law.

The difficult point is entry. Smoothness of the finite-N solution on a preterminal interval does not imply closeness to the terminal profile. Accordingly, the paper develops an explicit first-crossing mechanism based on a restoring scalar field, a finite shape jet, a spectral corridor, and decaying residual estimates.

This distinction prevents the terminal bootstrap from being used circularly to prove the assumptions required for entering the terminal bootstrap.

1.11 The first-crossing mechanism

If Y_N denotes the normalized logarithmic amplitude, the scalar principal field has the restoring form

$$\mathscr{P}_N(Y) = a_N(e^{-Y} - 1), \qquad a_N = 2\alpha - 1 - 2^{-k_N} > 0.$$

On a bounded bridge corridor this yields a differential inequality of the form

$$\partial_\tau |Y_N| \le -\lambda_N^{\mathrm{br}} |Y_N| + \rho_N(\tau),$$

and hence

$$|Y_N(\tau)| \le e^{-\lambda_N^{\mathrm{br}}(\tau-\tau_0)} |Y_N(\tau_0)| + \int_{\tau_0}^{\tau} e^{-\lambda_N^{\mathrm{br}}(\tau-r)} \rho_N(r)\, dr.$$

If the complete residual decays sufficiently rapidly, this produces a finite logarithmic first-crossing time into the certified spectral corridor.

The manuscript emphasizes that amplitude crossing alone is insufficient. The finite Euler-derivative shape jet must also enter its contraction region. This leads naturally to the spectral analysis of the complete linearized shape operator.

1.12 Spectral analysis and the raw continuity response

The spectral part of the paper undergoes several successive refinements. Rather than mixing reconstruction formulas from different stages of the development, the later sections rederive the bridge operator directly for the raw two-correction architecture.

The exact viscous product block is

$$\boxed{\mathfrak{V}_{N,\alpha} = A_N L_\alpha \left(\frac{b_N}{A_N} \right) + \frac{b_N}{A_N} L_\alpha A_N,}$$

where

$$L_\alpha = (-\Delta)^\alpha.$$

Its normalized linearization has the universal principal symbol

$$\boxed{|\xi|^{2\alpha}.}$$

This supplies the favorable high-frequency ellipticity.

The remaining terms arise from continuity response, algebraic reconstruction, transport, forcing, and finite coefficient variation. These are treated as finite lower-order corrections rather than being silently absorbed into the principal fractional term.

Linearization of the raw continuity equation produces a first-order operator

$$\mathcal{B}_N \Sigma = -\partial_x(p_N \Sigma) + \partial_y(q_N \Sigma) + \partial_z(r_N \Sigma).$$

The associated transport direction is

$$v_N = (-p_N, q_N, r_N).$$

The manuscript computes this direction explicitly and finds, in particular, the useful identity

$$q_N + r_N = 2.$$

For the transverse coordinate

$$\omega = y + z,$$

the characteristic speed is therefore constant:

$$\frac{d\omega}{d\ell} = 2.$$

On a 2π transverse slab this gives the exact flow-box flight time

$$\boxed{L_N^{\mathrm{fl}} = \pi.}$$

This calculation replaces an overly strong isotropic inverse assumption by a characteristic inversion adapted to the actual transport geometry.

1.13 The role of the forcing and the forced transverse bridge

The forcing used in the reconstruction contains a genuinely three-dimensional component of the form

$$G_\perp(x, y, z) = \sin^2 x \, \sin^2 y \, \sin^2 z.$$

Consequently, even if one begins with a strictly one-phase scalar datum, the complete forced evolution need not remain in the one-phase class.

This observation leads to another important refinement. The strict one-phase profile remains useful for understanding the principal terminal mechanism and the exact fractional reduction, but the actual pre-entry bridge must account for the persistent transverse forcing.

The later manuscript therefore constructs a forced transverse principal bridge and ultimately a spectrally complete nonlinear forced bridge, denoted schematically by

$$b_N^{\mathrm{F},*}.$$

The actual finite-N solution is then compared with this forced bridge, rather than with an invariant one-phase manifold that the prescribed forcing does not preserve.

This modification absorbs the persistent transverse response into the principal object and leaves only genuinely decaying coefficient and reconstruction defects to be estimated.

1.14 Compact frequencies, high frequencies, and the certified gap

The stability problem is divided into a compact-frequency block and an analytic high-frequency tail.

For a Fourier cutoff K, the compact block is represented by a finite Hermitian matrix

$$\mathbf{H}_{N,K,j} = -\frac{1}{2} P_K \left(\mathbf{L}_{N,\mathrm{prin},j} + \mathbf{L}_{N,\mathrm{prin},j}^* \right) P_K.$$

A rigorous lower bound for its smallest eigenvalue is obtained by enclosing the matrix entries and the eigenvalue error with directed interval arithmetic.

For the high-frequency tail, the leading fractional dissipation

$$\nu|k|^{2\alpha}$$

dominates lower-order symbols whose powers are strictly below 2α. This gives an analytic lower bound of the form

$$\lambda_{N,K}^{\mathrm{tail},-} = \nu(K+1)^{2\alpha} - \sum_j R_{N,j}(K+1)^{\rho_j}, \qquad \rho_j < 2\alpha.$$

The certified principal spectral gap is therefore

$$\boxed{\underline{\lambda}_N^{0,-}(K) = \min\left\{\lambda_{N,K}^{\mathrm{comp},-}, \lambda_{N,K}^{\mathrm{tail},-}\right\}.}$$

1.15 The final interval-arithmetic certificate

One of the principal achievements of the final portion of the manuscript is to replace an unspecified phrase such as "the lower-order terms are small" by a single reproducible finite-N certificate.

The complete lower-order corrections are enclosed separately on the compact frequency block and on the high-frequency tail:

$$\mathfrak{C}_{N,M,K}^{\mathrm{comp},+}, \qquad \mathfrak{C}_{N,M,K}^{\mathrm{tail},+}.$$

The final signed stability margin is

$$\boxed{\mathfrak{m}_{N,M,K}^{\mathrm{cert}} = \underline{\lambda}_N^{0,-}(K) - \max\left\{\mathfrak{C}_{N,M,K}^{\mathrm{comp},+}, \mathfrak{C}_{N,M,K}^{\mathrm{tail},+}\right\}.}$$

Thus the formerly unresolved strict stability inequality is certified precisely when

$$\boxed{\mathfrak{m}_{N,M,K}^{\mathrm{cert}} > 0.}$$

This condition is intentionally formulated for interval arithmetic. A floating-point calculation whose uncertainty crosses zero is not treated as a negative result; it is treated as inconclusive and must be refined by increasing precision, subdividing the bridge interval, improving coefficient enclosures, or changing the spectral cutoff.

The complete finite-N certificate also contains the raw nondegeneracy conditions, the continuity-response bounds, the forced-bridge contraction criterion, and the raw/Lambert coefficient-decay estimate. Once these are all verified for one internally consistent parameter record, the manuscript obtains

$$\boxed{\|b_N(\tau) - b_N^{\mathrm{F},*}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M} e^{-\eta_N \tau}.}$$

This supplies the pre-entry jet hypothesis required by the first-crossing argument.

1.16 Logical chain of the finite-N argument

The final finite-N proof architecture can therefore be summarized by the chain

$$
\boxed{
\begin{array}{c}
\text{complete reconstructed finite-}N\text{ data} \\
\Downarrow \\
\text{raw continuity and reconstruction certification} \\
\Downarrow \\
\text{existence of the complete forced bridge} \\
\Downarrow \\
\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0 \quad \text{and coefficient-decay certification} \\
\Downarrow \\
\text{pre-entry raw-jet convergence} \\
\Downarrow \\
\text{first crossing into the terminal spectral corridor} \\
\Downarrow \\
\text{terminal conormal persistence} \\
\Downarrow \\
b_{N,\alpha}(s) \asymp s^{-(2\alpha-1)} \\
\Downarrow \\
T_{*,N} < \infty, \qquad b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}}.
\end{array}
}
$$

The positivity of

$$\mathfrak{m}^{\mathrm{cert}}_{N,M,K}$$

is therefore an important component of the argument, but it is not by itself the complete finite-time conclusion. It certifies the required spectral stability margin. The remaining entries of the complete finite-N certificate must also be verified before the first-crossing and persistence theorems can be invoked.

1.17 What the manuscript proves conditionally at fixed finite N

The appropriate finite-N conclusion developed by the manuscript may be stated schematically as follows.

Theorem 1 (Finite-N terminal blow-up implication)**.** *Fix $N < \infty$ and a complete finite-N parameter record for which the raw reconstruction and continuity bounds, the complete*

forced-bridge contraction condition, the strict interval stability certificate

$$\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0,$$

and the required raw/Lambert coefficient-decay estimates are all rigorously verified.

Assume that these certified estimates supply the hypotheses of the finite-N first-crossing theorem and of the terminal persistence theorem. Then the corresponding reconstructed scalar quantity $b_{N,\alpha}$ enters the terminal similarity regime and satisfies

$$c_- s^{-(2\alpha-1)} \le b_{N,\alpha}(s) \le c_+ s^{-(2\alpha-1)}$$

for sufficiently small $s > 0$.

Consequently there exists a finite time

$$T_{*,N} < \infty$$

such that

$$\boxed{b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}}}$$

as

$$t \uparrow T_{*,N}.$$

In particular,

$$\boxed{b_{N,\alpha}(t) \longrightarrow +\infty \qquad \text{as } t \uparrow T_{*,N}.}$$

The theorem is deliberately formulated at fixed finite N. The finite-N regularization is essential to many of the algebraic, characteristic, and interval-certification steps.

1.18 Finite-N scalar blow-up versus a full Navier–Stokes singularity theorem

A central point of interpretation must be stated explicitly.

The implication

$$\boxed{\text{complete positive finite-}N\text{ certificate} \Longrightarrow \text{first crossing} \Longrightarrow \text{terminal persistence} \Longrightarrow T_{*,N} < \infty, \quad b_{N,\alpha}(t) \to \infty}$$

is a statement about the certified finite-N reconstructed scalar dynamics.

It is not, by itself, a theorem asserting finite-time singularity of a limiting three-dimensional Navier–Stokes solution.

To obtain such a stronger conclusion one would additionally have to justify the limiting passage and the complete velocity-pressure reconstruction in a topology strong enough to preserve the required nonlinear and fractional relations. Among the issues examined throughout the manuscript are:

1. exact incompressibility of the reconstructed velocity;
2. solvability and normalization of the continuity correction;
3. compatibility of the reconstructed velocity with the momentum equations;

4. existence and compatibility of the pressure;

5. control of the fractional commutator and of nonlocal tails;

6. fixed-space regularity near zeros of the geometric coefficients;

7. finite kinetic energy and the required Sobolev bounds;

8. compactness as the regularization parameters vary;

9. preservation of the nonlinear product relation

$$b_3 = u_x u_y$$

   under the limiting procedure;

10. and sufficient convergence of the fractional dissipative terms to pass to a limiting equation.

Several of these questions are addressed or reduced to explicit finite conditions in later parts of the paper.

1.19 Why the manuscript contains repeated audits and revisions

The length of the manuscript reflects the fact that the construction is not developed as one uninterrupted formal asymptotic calculation. Instead, successive sections revisit earlier assumptions whenever a new compatibility condition becomes visible.

Examples include the distinction between literal Lambert iteration and the renormalized Lambert cascade; the replacement of an isolated $\alpha = 9/8$ calculation by an interval continuation from $\alpha = 1$; the correction of the Fredholm and adjoint-cokernel analysis; the distinction between an affine phase and the nonlinear intrinsic geometric variable; the fixed-space audit of square-root reconstruction; the distinction between profile existence and actual trajectory entry; the direct rederivation of the spectral operator for the raw two-correction architecture; the characteristic inversion of the continuity response; and the replacement of a one-phase pre-entry bridge by a forced transverse bridge when the actual three-dimensional forcing is taken into account.

These are not merely changes of notation. They represent logically distinct questions:

Does the formal profile exist?
Does the finite-N trajectory approach it?
Does the reconstructed velocity remain divergence free?
Does the fractional operator have the claimed leading behavior?
Are the lower-order terms genuinely dominated?
Does the characteristic reach the terminal point in finite time?
Can the result be passed to a limiting three-dimensional field?

The purpose of the audit structure is to prevent an affirmative answer to one of these questions from being used automatically as an affirmative answer to the others.

1.20 Organization of the manuscript

The more than four hundred pages that follow may be viewed as a sequence of interlocking stages.

The opening parts derive the exact b_3 product equation, introduce the fractional commutator, formulate the strict one-phase reduction, identify the similarity exponent

$$2\alpha - 1,$$

and set up continuation from the explicit $\alpha = 1$ profile.

The next major group of sections develops the Mellin formulation of the fractional profile problem. It computes the Mellin symbol, establishes its dependence on α, analyzes the explicit anchor, determines the translation kernel and second homogeneous solution, constructs Green representations, fixes the phase, and establishes the functional-analytic framework for continuation on a nonempty interval above $\alpha = 1$.

The manuscript then turns from the abstract profile to the finite-N dynamics. It develops entry estimates, terminal bootstrap inequalities, coefficient gaps, invariant regions, compactness requirements, and the exact Lambert coefficient calculus needed to control the regularized coefficients.

A further sequence of parts audits the increasing-depth Lambert cascade, coefficient convergence, cutoff interpretations, periodic geometry, the intrinsic variable

$$\sigma = \chi^2,$$

the induced fractional operator, weighted geometric remainders, upper invariant-boundary signs, and the complete finite-amplitude bracket.

The next stage examines the reconstruction in fixed physical space. It derives the exact signed transverse-derivative terms, characteristic maps, coefficient hierarchy, Lambert-product estimates, fixed-space velocity behavior, transverse concentration widths, scalar fractional dissipation, and critical coupled N-terminal paths.

The later parts revisit the construction using the raw two-correction variables

$$A = Q + S - D, \qquad C = Q + S + D, \qquad u_x = \frac{b}{A}.$$

This leads to the direct raw spectral rederivation, the continuity-response operator, its normalization and characteristic inverse, pressure compatibility, the outer-bridge linearization, and the scalar slaved-continuity stability problem.

The final stages incorporate the genuinely three-dimensional forcing into a forced transverse principal bridge. They then construct a spectrally complete forced bridge, split the stability problem into compact and high-frequency sectors, and formulate a reproducible interval-arithmetic certificate.

The culminating finite quantity is

$$\mathfrak{m}^{\mathrm{cert}}_{N,M,K}.$$

Its positivity, together with the other finite certificate conditions, provides the pre-entry stability information required by the first-crossing and terminal-persistence machinery.

1.21 A guide to the main mathematical themes

For readers approaching the manuscript for the first time, the analysis can be organized around the following interacting themes:

1. **Exact scalar reduction.** The quantity

$$b_3 = u_x u_y$$

   is derived from the momentum equations rather than postulated as an independent model.

2. **Fractional product structure.** The nonlocal dissipative term is handled through the exact fractional commutator rather than an ordinary Leibniz rule.

3. **One-phase Fourier reduction.** Strict affine one-phase fields reduce the periodic fractional Laplacian exactly to a one-dimensional fractional operator.

4. **Mellin analysis.** Power-law terminal behavior becomes translation invariant after logarithmic conjugation, making the Mellin symbol the natural object for continuation.

5. **Continuation from $\alpha = 1$.** The explicit classical anchor is used to construct profiles for a nonempty interval of exponents above one.

6. **Lambert W regularization.** The finite-N coefficient hierarchy is generated and controlled through a renormalized Lambert cascade and its Euler sensitivities.

7. **Reconstruction and incompressibility.** The scalar variable is embedded into velocity components whose continuity correction must satisfy a genuine transport equation.

8. **Intrinsic geometry.** Zeros of the periodic geometric coefficient require a separate nonlinear coordinate analysis and a fixed-space regularity audit.

9. **Entry versus persistence.** Existence of a terminal profile is separated from the proof that the prescribed finite-N trajectory actually reaches it.

10. **Spectral coercivity.** The complete linearized operator is decomposed into a favorable fractional principal part and finite lower-order reconstruction terms.

11. **Forced transverse dynamics.** The actual three-dimensional forcing is incorporated into the principal bridge rather than being incorrectly treated as a decaying one-phase error.

12. **Interval certification.** The final strict inequalities are formulated so that they can be verified by directed interval arithmetic for a fully specified finite-N parameter record.

13. **Finite-time conversion.** Once terminal persistence gives

$$b_{N,\alpha}(s) \asymp s^{-(2\alpha-1)},$$

the characteristic law

$$\dot{s} = 1 - b_{N,\alpha}$$

implies

$$T_{*,N} < \infty$$

and the temporal rate

$$b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}}.$$

14. **Separation of finite-N and limiting conclusions.** The paper distinguishes carefully between a certified finite-N scalar terminal law and the additional work required for a full limiting three-dimensional Navier–Stokes singularity statement.

1.22 Perspective

The central methodological principle of this manuscript is that a candidate singular mechanism should survive a sequence of increasingly stringent tests.

A formal power law is not enough. It must solve the appropriate profile equation.

A profile solution is not enough. Its linearization must have the correct Fredholm structure and the profile must persist as α varies.

Continuation of the profile is not enough. The actual finite-N reconstructed trajectory must enter its neighborhood.

Entry of a scalar variable is not enough. The continuity correction, fractional commutator, forcing, pressure, and reconstructed velocity must remain compatible.

A favorable principal symbol is not enough. Compact frequencies and lower-order terms must satisfy a strict quantitative gap.

A numerical indication of such a gap is not enough. The final sign must be certified with controlled numerical error.

And finally, a finite-N scalar terminal law is not automatically a singularity theorem for a limiting three-dimensional Navier–Stokes solution. The limiting reconstruction and the required compactness must be proved separately.

For this reason the manuscript culminates not in an unspecified assertion that all error terms are sufficiently small, but in a finite reproducible certification problem. For a complete parameter record, the principal stability margin is encoded by

$$\boxed{\mathfrak{m}^{\mathrm{cert}}_{N,M,K} = \underline{\lambda}^{0,-}_{N}(K) - \max\left\{\mathfrak{C}^{\mathrm{comp},+}_{N,M,K}, \mathfrak{C}^{\mathrm{tail},+}_{N,M,K}\right\}.}$$

When the complete collection of finite-N certificate conditions is verified, including

$$\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0,$$

the first-crossing and terminal-persistence framework yields the finite-N scalar terminal law

$$\boxed{b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}}, \qquad T_{*,N} < \infty.}$$

Accordingly, the more than four hundred pages that follow should be read as a progressive verification of the implication

$$\boxed{\begin{array}{c}\text{exact reconstructed dynamics} \Longrightarrow \text{continued terminal profile} \Longrightarrow \text{finite-}N\text{ coefficient control} \\ \Longrightarrow \text{certified pre-entry bridge} \Longrightarrow \text{first crossing} \Longrightarrow \text{terminal persistence} \Longrightarrow \text{finite-time scalar amplification},\end{array}}$$

together with a detailed accounting of the additional reconstruction and limiting questions that separate this finite-N conclusion from a full three-dimensional Navier–Stokes singularity theorem.

This organization is intended to make the logical dependencies of the construction explicit. Each later certification should be read not as an independent auxiliary calculation, but as an attempt to close one of the specific implications required by this chain.

2 Part A

We start with a **proof program for an interval** first, and only state unconditional finite-time blow-up after the continuation, entry/bootstrap, coefficient-gap, and limit estimates have actually been established uniformly for $1 < \alpha < 1 + \delta$.

Part I. The b_3 equation uniformly for $1 < \alpha < 1 + \delta$

Fix an as-yet-undetermined

$$\boxed{\delta > 0}$$

and consider

$$\boxed{1 < \alpha < 1 + \delta < \frac{5}{4}.}$$

The point is that δ will ultimately be produced by a continuation argument from the exactly solvable endpoint $\alpha = 1$, rather than chosen numerically in advance.

Start with the hyperdissipative velocity equations on $\mathbb{T}^3$,

$$\partial_t u + (u \cdot \nabla)u + \nabla P + \nu(-\Delta)^\alpha u = f, \qquad \nabla \cdot u = 0,$$

where

$$1 < \alpha < 1 + \delta.$$

Define

$$\boxed{b = b_3 := u_x u_y.}$$

This is exactly the physical product used by the manuscript.

Taking a time derivative gives

$$b_t = u_y u_{x,t} + u_x u_{y,t},$$

while

$$u \cdot \nabla b = u_y(u \cdot \nabla u_x) + u_x(u \cdot \nabla u_y).$$

Multiply the x-momentum equation by u_y, the y-momentum equation by u_x, and add. This gives the **exact identity**

$$\boxed{b_t + u \cdot \nabla b + u_y P_x + u_x P_y + \nu\left[u_y(-\Delta)^\alpha u_x + u_x(-\Delta)^\alpha u_y\right] = u_y f_x + u_x f_y.} \tag{1}$$

This equation is already derived exactly in the manuscript.

The important change is that we now regard (1) as holding **uniformly over an interval of exponents**

$$\boxed{\alpha \in (1, 1+\delta)}$$

rather than immediately specializing to

$$\alpha = \frac{9}{8}.$$

Fractional product identity

For $\alpha > 1$, one cannot use the ordinary Leibniz rule. Define the exact commutator

$$\boxed{\mathcal{R}_\alpha(f,g) := (-\Delta)^\alpha(fg) - f(-\Delta)^\alpha g - g(-\Delta)^\alpha f.}$$

Thus

$$(-\Delta)^\alpha(u_x u_y) = u_y(-\Delta)^\alpha u_x + u_x(-\Delta)^\alpha u_y + \mathcal{R}_\alpha(u_x, u_y).$$

Since $b = u_x u_y$,

$$u_y(-\Delta)^\alpha u_x + u_x(-\Delta)^\alpha u_y = (-\Delta)^\alpha b - \mathcal{R}_\alpha(u_x, u_y).$$

Substitution into (1) therefore gives

$$\boxed{b_t + u \cdot \nabla b + u_y P_x + u_x P_y + \nu(-\Delta)^\alpha b = u_y f_x + u_x f_y + \nu\mathcal{R}_\alpha(u_x, u_y).} \tag{2}$$

This is the correct no-double-counting form. The manuscript derives precisely this structural separation: one principal scalar fractional term and a separate product-commutator contribution.

In Fourier variables,

$$\boxed{\widehat{\mathcal{R}_\alpha(f,g)}(k) = \sum_{\ell \in \mathbb{Z}^3} \left(|k|^{2\alpha} - |\ell|^{2\alpha} - |k-\ell|^{2\alpha}\right) \widehat{f}(\ell)\widehat{g}(k-\ell).} \tag{3}$$

The manuscript already gives this exact formula.

Nothing in (1)–(3) depends on choosing $\alpha = 9/8$.

Reconstruction variables

Retain the manuscript's variables

$$\boxed{A := u_y, \qquad C := u_z, \qquad u_x = \frac{b}{A}.}$$

Then

$$u = \left(\frac{b}{A}, A, C\right).$$

Consequently,

$$u \cdot \nabla b = \frac{b}{A} b_x + A b_y + C b_z.$$

After the completed reconstruction, pressure rearrangement, Lambert coefficient, forcing and fractional decomposition are inserted, its corrected scalar equation is

$$\boxed{\begin{aligned} &b_t + b b_z + A C^2 b_x + \frac{b C^2}{A} b_y + b \frac{C_t + C_z}{C} + 2AbC_x + 2\frac{b^2}{A} C_y \\ &\quad = L_N b^2 + 2CG_{\text{forc}} - \nu(-\Delta)^\alpha b + \mathcal{F}^{\text{rem}}_{\text{frac}}. \end{aligned}} \tag{4}$$

This is this paper's b_3-dynamics equation.

Equation (4), rather than a $\alpha = 9/8$ specialization,(attempted it separately) should become the starting scalar PDE to work with.

Now impose the strict one-phase architecture

Take

$$s_N = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N}$$

and

$$u_N(x, y, z, t) = U_N(s_N).$$

Hence

$$b_N = B_N(s_N), \qquad A_N = A_N(s_N), \qquad C_N = C_N(s_N).$$

This is exactly the standing generalized affine one-phase architecture introduced.

If

$$\kappa^2 = C_1^2 + C_2^2 + C_3^2,$$

then for every one-phase scalar $F(s_N)$,

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha F(s_N) = \kappa^{2\alpha} (-\partial_{ss})^\alpha F(s_N).} \tag{5}$$

This is **exact**, not asymptotic. Therefore

$$\boxed{\mathcal{R}_{\mathrm{multi},N} = 0}$$

for every

$$\boxed{1 < \alpha < 1 + \delta.}$$

We establish this one-phase identity.

This is particularly valuable for the interval argument: we do not need to prove a three-dimensional fractional-symbol approximation uniformly in α. It is identically exact throughout the interval.

Use the distinguished phase for the first interval theorem

For the first proof I would make the stronger specialization

$$\boxed{C_1 = C_2 = 0, \qquad C_3 = C_4 = 1.}$$

Then

$$\boxed{s_N = t - z - \theta_{*,N}}$$

and

$$\kappa = 1.$$

Consequently,

$$(-\Delta_{\mathbb{T}^3})^{\alpha} F(s_N) = (-\partial_{ss})^{\alpha} F(s_N).$$

Moreover incompressibility gives

$$C_1 u_x + C_2 u_y + C_3 u_z = K_N,$$

so

$$\boxed{u_z = K_N}$$

is constant in phase. Therefore

$$C_s = 0,$$

and all the reconstruction terms in (4) containing C_s vanish.

Most importantly, along a characteristic,

$$\boxed{\dot{s}_N = 1 - b_N.} \tag{6}$$

This keeps the exact phase law used for the finite-time conversion.

Thus the interval proof should first be developed in this distinguished one-phase subclass. The more general affine coefficients can be reconsidered afterward.

The interval-dependent similarity exponent

For general

$$1 < \alpha < 1 + \delta,$$

the similarity balance gives

$$\boxed{r_\alpha = 2\alpha - 1.} \tag{7}$$

Therefore we should no longer normalize with

$$W_N = \frac{s^{5/4} b_N}{p_N},$$

because $5/4$ belongs specifically to $\alpha = 9/8$.

Instead define

$$\boxed{r_\alpha := 2\alpha - 1}$$

and

$$\boxed{W_{N,\alpha}(s) := \frac{s^{r_\alpha} b_N(s)}{p_{N,\alpha}}.} \tag{8}$$

Equivalently,

$$\boxed{b_N(s) = p_{N,\alpha} s^{-(2\alpha-1)} W_{N,\alpha}(s).} \tag{9}$$

Since

$$\alpha = 1 + \varepsilon,$$

we have

$$r_\alpha = 1 + 2\varepsilon.$$

Thus throughout the desired interval,

$$1 < r_\alpha < 1 + 2\delta.$$

This replaces every occurrence of the fixed exponent $5/4$ in the terminal normalization by the parameter-dependent exponent

$$\boxed{2\alpha - 1.}$$

The profile equation for the whole interval

The important finding is slightly different from my previous answer: **the equation is not merely postulated in the earlier work. There is an exact $H \to F$ derivation.** However, the files themselves carefully distinguish that exact derivation from the preceding reduction of the full 3D reconstructed b_3-PDE to the H-profile equation.

1. The actual starting equation

The terminal similarity ansatz is

$$\boxed{b(s,y) = s^{-p}B(Y), \qquad Y = \frac{y - y_*}{s^\beta}, \qquad p = 2\alpha - 1.}$$

Then introduce

$$\boxed{H(Y) = \sqrt{B(Y)}}.$$

The manuscript defines the longitudinal similarity operator through

$$(-\partial_{ss})^\alpha \left[s^{-p/2} H\left(\frac{y - y_*}{s^\beta} \right) \right]_{y - y_* = s^\beta Y} = s^{-p/2-2\alpha} \big(\mathscr{L}_{\alpha,\beta} H \big)(Y).$$

After the transverse/reconstruction remainders have been made lower order, the terminal similarity equation obtained from the b_3 balance is

$$\boxed{\beta Y B' + (p - \gamma)B = -2\nu B^{-1/2} \mathscr{L}_{\alpha,\beta} \sqrt{B}.} \tag{1}$$

This is explicitly described as the leading-order profile equation supplied by the multidimensional fractional b_3-balance.

Since

$$B = H^2, \qquad B' = 2HH', \qquad B^{-1/2} = \frac{1}{H},$$

(1) becomes

$$2\beta Y H H' + (p - \gamma)H^2 = -\frac{2\nu}{H} \mathscr{L}_{\alpha,\beta} H.$$

Multiplication by $H/2$ gives the crucial intermediate equation

$$\boxed{\nu \mathscr{L}_{\alpha,\beta} H + \beta Y H^2 H' + \frac{p - \gamma}{2} H^3 = 0.} \tag{2}$$

This part is exact wherever $H > 0$.

2. Now comes the exact transformation producing the RHS

Set

$$\boxed{a = \frac{p}{2} = \alpha - \frac{1}{2}}$$

and define

$$\boxed{H(Y) = s^a F(s), \qquad Y = s^{-\beta}.} \tag{3}$$

The similarity operator satisfies

$$\boxed{\mathscr{L}_{\alpha,\beta} H = s^{a+2\alpha}(-\partial_{ss})^\alpha F.} \tag{4}$$

This is the step that transforms the fractional term.
Now differentiate $Y = s^{-\beta}$:

$$\frac{dY}{ds} = -\beta s^{-\beta-1} = -\frac{\beta Y}{s}.$$

Also,

$$H = s^a F$$

gives

$$\frac{dH}{ds} = a s^{a-1} F + s^a F' = s^{a-1}(aF + sF').$$

Therefore, by the chain rule,

$$H'(Y) = \frac{dH/ds}{dY/ds} = -\frac{s^a}{\beta Y}(aF + sF').$$

Thus

$$\boxed{H'(Y) = -\frac{s^a}{\beta Y}(aF + sF').} \tag{5}$$

3. This produces F^2F'

Since

$$H^2 = s^{2a} F^2,$$

we have

$$\beta Y H^2 H' = \beta Y (s^{2a} F^2) \left[-\frac{s^a}{\beta Y} (aF + sF') \right]$$
$$= -s^{3a} F^2 (aF + sF').$$

Therefore

$$\boxed{\beta Y H^2 H' = -s^{3a} \left(aF^3 + sF^2 F' \right).} \tag{6}$$

Notice something important: **the factor β cancels exactly.** This explains why the transverse similarity exponent does not occur in the final one-dimensional F-equation.

4. The second nonlinear term produces the remaining cubic piece

Because

$$H^3 = s^{3a} F^3,$$

we have

$$\boxed{\frac{p-\gamma}{2} H^3 = \frac{p-\gamma}{2} s^{3a} F^3.} \tag{7}$$

Add (6) and (7):

$$\beta Y H^2 H' + \frac{p-\gamma}{2} H^3 = -s^{3a} s F^2 F'$$
$$+ s^{3a} \left[-a + \frac{p-\gamma}{2} \right] F^3.$$

But

$$a = \frac{p}{2},$$

so

$$-a + \frac{p-\gamma}{2} = -\frac{p}{2} + \frac{p}{2} - \frac{\gamma}{2} = -\frac{\gamma}{2}.$$

Hence

$$\boxed{\beta Y H^2 H' + \frac{p-\gamma}{2} H^3 = -s^{3a} \left[sF^2 F' + \frac{\gamma}{2} F^3 \right].} \tag{8}$$

5. Substitute into the H-profile equation

Equation (2) was

$$\nu \mathscr{L}_{\alpha,\beta} H + \beta Y H^2 H' + \frac{p-\gamma}{2} H^3 = 0.$$

Using (4) and (8),

$$\nu s^{a+2\alpha}(-\partial_{ss})^{\alpha} F - s^{3a}\left[sF^2F' + \frac{\gamma}{2}F^3\right] = 0.$$

Therefore

$$\nu s^{a+2\alpha}(-\partial_{ss})^{\alpha} F = s^{3a+1}\left[F^2F' + \frac{\gamma}{2s}F^3\right]. \tag{9}$$

Now use

$$a = \alpha - \frac{1}{2}.$$

Then

$$a + 2\alpha = 3\alpha - \frac{1}{2},$$

while

$$3a + 1 = 3\left(\alpha - \frac{1}{2}\right) + 1 = 3\alpha - \frac{1}{2}.$$

Hence

$$\boxed{a + 2\alpha = 3a + 1.}$$

The powers of s on the two sides are therefore **identical**, so they cancel. We identify this exponent identity as the reason for choosing $a = \alpha - \frac{1}{2}$.

We obtain

$$\boxed{\nu(-\partial_{ss})^{\alpha} F = F^2F' + \frac{\gamma}{2s}F^3, \qquad s > 0.} \tag{10}$$

So the RHS is now completely accounted for.

6. The divergence-form check

Since

$$(F^3)' = 3F^2F',$$

we have

$$s^{-3\gamma/2}\frac{d}{ds}\left(s^{3\gamma/2}F^3\right) = s^{-3\gamma/2}\left[\frac{3\gamma}{2}s^{3\gamma/2-1}F^3 + 3s^{3\gamma/2}F^2F'\right]$$
$$= 3F^2F' + \frac{3\gamma}{2s}F^3.$$

Thus equation (10) is equivalently

$$\boxed{3\nu(-\partial_{ss})^\alpha F = s^{-3\gamma/2}\frac{d}{ds}\left(s^{3\gamma/2}F^3\right).}$$

The manuscript explicitly gives this equivalent divergence form as a check.

7. Where does the $\kappa^{2\alpha}$ factor enter?

If

$$s_N = C_4t - C_3z - C_2y - C_1x - \theta_{*,N}$$

and

$$\kappa^2 = C_1^2 + C_2^2 + C_3^2,$$

then for a strict one-phase function,

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha F(s_N) = \kappa^{2\alpha}(-\partial_{ss})^\alpha F(s_N).}$$

The proof is Fourier-theoretic: the n-th Fourier mode of $F(s_N)$ has spatial wave vector

$$-n(C_1, C_2, C_3),$$

whose fractional-Laplacian multiplier is

$$|n(C_1, C_2, C_3)|^{2\alpha} = |n|^{2\alpha}\kappa^{2\alpha}.$$

That produces the generalized affine form

$$\boxed{\nu\kappa^{2\alpha}(-\partial_{ss})^\alpha F_\alpha = F_\alpha^2F_\alpha' + \frac{\gamma_\alpha}{2s}F_\alpha^3.} \tag{11}$$

Now $\kappa^{2\alpha}$ must either remain visible or be absorbed into an effective one-dimensional viscosity

$$\nu_{\text{eff}} = \nu\kappa^{2\alpha}.$$

The exact one-phase reduction leads to

$$\boxed{\nu\kappa^{2\alpha}(-\partial_{ss})^\alpha F_\alpha = F_\alpha^2F_\alpha' + \frac{\gamma_\alpha}{2s}F_\alpha^3.} \tag{10}$$

We derive this equation for general α.

In our distinguished phase, $\kappa = 1$, so

$$\boxed{\nu(-\partial_{ss})^{\alpha} F_{\alpha} = F_{\alpha}^{2} F_{\alpha}' + \frac{\gamma_{\alpha}}{2s} F_{\alpha}^{3}.} \tag{11}$$

Instead of fixing

$$\gamma = \frac{1}{2},$$

(when I chose $\alpha = 9/8$ for example) instead choose a continuation law

$$\boxed{\gamma_{\alpha} = \lambda(\alpha - 1), \qquad \lambda > 0.} \tag{12}$$

Then

$$\gamma_{\alpha} \to 0 \qquad (\alpha \downarrow 1).$$

At the endpoint,

$$(\alpha, \gamma) = (1, 0),$$

equation (11) becomes

$$-\nu F_1'' = F_1^2 F_1',$$

which integrates exactly to

$$-\nu F_1' = \frac{1}{3} F_1^3$$

and hence

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s + s_0)}}.} \tag{13}$$

We then identify this exact solution as the continuation anchor.

The theorem we should actually try to prove

We should **not yet begin by asserting blow-up for the interval**. We should formulate the following continuation theorem as the first major new target:

$$\boxed{\begin{array}{l}\text{There exists } \delta_0 > 0 \text{ such that, for every } 1 < \alpha < 1 + \delta_0 \text{ with } \gamma_{\alpha} = \lambda(\alpha - \\ 1), \text{ the phase-normalized profile equation (11) possesses a positive solution} \\ F_{\alpha} \text{ satisfying the required endpoint conditions, and } F_{\alpha} \to F_1 \text{ in a weighted} \\ \text{Banach space as } \alpha \downarrow 1.\end{array}} \tag{14}$$

We should aim for the stronger quantitative estimate

$$\boxed{\left\| \frac{F_\alpha - F_1}{F_1} \right\|_X \leq C(\alpha - 1).} \tag{15}$$

If

$$C\delta_0 < \frac{1}{2},$$

then

$$F_\alpha \geq F_1 - |F_\alpha - F_1| \geq \frac{1}{2} F_1 > 0,$$

so positivity follows uniformly:

$$\boxed{F_\alpha(s) > 0 \quad \forall s > 0, \quad 1 < \alpha < 1 + \delta_0.} \tag{16}$$

That would remove the need for the difficult general-α maximum-principle argument.

What finite-time blow-up would then look like

Once the profile, entry, bootstrap, coefficient gap, and limiting theorem have all been established uniformly, we would obtain

$$b_3(s) \asymp s^{-(2\alpha-1)}.$$

Using

$$\dot{s} = 1 - b_3,$$

the singular term dominates, so

$$-\dot{s} \asymp s^{-(2\alpha-1)}.$$

Hence

$$dt \asymp s^{2\alpha-1}\, ds,$$

and therefore

$$\boxed{T_* - t \asymp s^{2\alpha}.} \tag{17}$$

Consequently,

$$s \asymp (T_* - t)^{1/(2\alpha)}$$

and

$$\boxed{b_3(t) \asymp (T_* - t)^{-\frac{2\alpha-1}{2\alpha}}.} \tag{18}$$

Thus the eventual interval theorem would have the form

$$\boxed{\forall \alpha \in (1, 1+\delta_0): \qquad \lim_{t \uparrow T_{*,\alpha}} b_3(t) = +\infty, \qquad T_{*,\alpha} < \infty.} \tag{19}$$

But (19) is the **conclusion we are working toward**, not something we should state as proved before establishing the uniform continuation and closure estimates.

The next mathematical step should therefore be very focused: **linearize (11) at the explicit $(\alpha, \gamma, F) = (1, 0, F_1)$, fix the translation mode, construct an explicit inverse for the resulting second-order operator, and prove the operator estimates needed for an implicit-function theorem uniformly for $1 < \alpha < 1 + \delta$.** That is where we can try to prove the existence of a genuine $\delta > 0$, rather than merely assume one.

3 Part B

We can do the linearization explicitly, and it exposes both the promising part of the $\alpha \downarrow 1$ strategy and one point that has to be handled carefully.

The manuscript supports the starting point: for general α, the strict one-phase profile equation is

$$\nu_{\text{eff}}(-\partial_{ss})^{\alpha} F = F^2 F' + \frac{\gamma}{2s} F^3,$$

and in the distinguished phase $\kappa = 1$, $\nu_{\text{eff}} = \nu$.

1. Set up the continuation problem

Write

$$\alpha = 1 + \varepsilon, \qquad \gamma = \lambda \varepsilon, \qquad 0 \le \varepsilon < \delta.$$

Define

$$\boxed{\mathscr{G}(\varepsilon, F) = \nu(-\partial_{ss})^{1+\varepsilon} F - F^2 F' - \frac{\lambda \varepsilon}{2s} F^3.} \tag{20}$$

We seek

$$\mathscr{G}(\varepsilon, F_\varepsilon) = 0.$$

At $\varepsilon = 0$,

$$\nu(-\partial_{ss}) F_1 - F_1^2 F_1' = 0.$$

Using

$$(-\partial_{ss}) F = -F'',$$

this is

$$-\nu F_1'' - F_1^2 F_1' = 0. \tag{21}$$

Integrating,

$$-\nu F_1' - \frac{1}{3}F_1^3 = C.$$

For the decaying branch, $C = 0$, so

$$-\nu F_1' = \frac{1}{3}F_1^3. \tag{22}$$

Consequently

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}, \qquad s_0 > 0.} \tag{23}$$

This is precisely the $\alpha = 1, \gamma = 0$ anchor identified in the manuscript.

2. Linearize at F_1

Let

$$F = F_1 + h.$$

At $\varepsilon = 0$,

$$D_F \mathscr{G}(0, F_1)h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.$$

Therefore define

$$\boxed{L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.} \tag{24}$$

But

$$(F_1^2 h)' = F_1^2 h' + 2F_1 F_1' h,$$

so

$$\boxed{L_1 h = -\left(\nu h' + F_1^2 h\right)'.} \tag{25}$$

This factorization is the main advantage of continuing from $\alpha = 1$.
Since

$$F_1^2 = \frac{3\nu}{2(s+s_0)},$$

we obtain

$$\boxed{L_1 h = -\nu \left[h' + \frac{3}{2(s+s_0)} h \right]'.} \tag{26}$$

So inversion of the linearized profile operator reduces to **two first-order integrations**.

3. Determine the kernel exactly

Set

$$L_1 h = 0.$$

Then

$$\left[h' + \frac{3}{2(s+s_0)} h \right]' = 0.$$

Hence

$$h' + \frac{3}{2(s+s_0)} h = C_1. \tag{27}$$

The integrating factor is

$$\mu(s) = (s+s_0)^{3/2}.$$

Thus

$$\left[(s+s_0)^{3/2} h \right]' = C_1 (s+s_0)^{3/2}.$$

Integration gives

$$(s+s_0)^{3/2} h = \frac{2C_1}{5} (s+s_0)^{5/2} + C_2.$$

Therefore

$$\boxed{h(s) = \frac{2C_1}{5}(s+s_0) + C_2 (s+s_0)^{-3/2}.} \tag{28}$$

The first mode grows linearly and is excluded from any decaying profile space.
Thus the admissible kernel is

$$\boxed{\ker L_1 = \operatorname{span}\left\{ (s+s_0)^{-3/2} \right\}.} \tag{29}$$

And

$$\partial_{s_0} F_1 = -\frac{1}{2} \sqrt{\frac{3\nu}{2}} (s+s_0)^{-3/2}.$$

Hence

$$\boxed{\ker L_1 = \operatorname{span}\{\partial_{s_0} F_1\}.} \tag{30}$$

So the only admissible zero mode is exactly the one-parameter s_0-modulation of the explicit family.

This is excellent for continuation: it is not an unexplained degeneracy.

4. Fix the zero mode

We need one normalization.

A simple choice is to fix the value at $s = 0$:

$$\boxed{F_\varepsilon(0) = F_1(0).} \tag{31}$$

For the perturbation this means

$$\boxed{h(0) = 0.} \tag{32}$$

But the kernel function satisfies

$$(s_0)^{-3/2} \neq 0.$$

Therefore (32) removes the kernel.

Equivalently we could treat s_0 as a modulation parameter and impose an orthogonality condition. For a first proof, (31) is simpler.

5. Construct the inverse explicitly

Solve

$$L_1 h = f.$$

From (26),

$$-\nu \left[h' + \frac{3}{2(s+s_0)} h \right]' = f.$$

Define

$$g(s) = h' + \frac{3}{2(s+s_0)} h.$$

Then

$$g'(s) = -\frac{1}{\nu} f(s).$$

If the admissible solution satisfies $g(\infty) = 0$, then

$$\boxed{g(s) = \frac{1}{\nu}\int_s^\infty f(\tau)\,d\tau.} \tag{33}$$

Now solve

$$h' + \frac{3}{2(s+s_0)}h = g(s).$$

Multiplying by $(s+s_0)^{3/2}$,

$$\left[(s+s_0)^{3/2}h(s)\right]' = (s+s_0)^{3/2}g(s).$$

Using $h(0) = 0$,

$$\boxed{h(s) = \frac{1}{\nu(s+s_0)^{3/2}}\int_0^s (\sigma+s_0)^{3/2}\left[\int_\sigma^\infty f(\tau)\,d\tau\right]d\sigma.} \tag{34}$$

Thus we have an explicit candidate inverse

$$\boxed{(L_1^{-1}f)(s) = \frac{1}{\nu(s+s_0)^{3/2}}\int_0^s (\sigma+s_0)^{3/2}\int_\sigma^\infty f(\tau)\,d\tau\,d\sigma.} \tag{35}$$

This is considerably simpler than the Mellin inverse required at $\alpha = 9/8$.

6. A first weighted estimate

Suppose

$$|f(s)| \le M(s+s_0)^{-2-\eta}, \qquad \eta > 0. \tag{36}$$

Then

$$\left|\int_\sigma^\infty f(\tau)d\tau\right| \le \frac{M}{1+\eta}(\sigma+s_0)^{-1-\eta}.$$

Therefore

$$|h(s)| \le \frac{M}{\nu(1+\eta)}(s+s_0)^{-3/2}\int_0^s (\sigma+s_0)^{1/2-\eta}d\sigma.$$

For $\eta \neq 3/2$,

$$\int_0^s (\sigma+s_0)^{1/2-\eta}d\sigma = \frac{(s+s_0)^{3/2-\eta} - s_0^{3/2-\eta}}{3/2-\eta}.$$

Hence for

$$0 < \eta < \frac{3}{2},$$

$$\boxed{|h(s)| \leq \frac{M}{\nu(1+\eta)(3/2-\eta)}(s+s_0)^{-\eta}.} \tag{37}$$

Thus

$$\boxed{\sup_{s\geq 0}(s+s_0)^{\eta}|h(s)| \leq C_{\nu,\eta}M.} \tag{38}$$

This gives an explicit weighted L^∞ inverse estimate.
Differentiating

$$h' = g - \frac{3}{2(s+s_0)}h$$

also yields

$$\boxed{|h'(s)| \leq C_{\nu,\eta}M(s+s_0)^{-1-\eta}.} \tag{39}$$

And the equation itself gives the corresponding second derivative estimate.
So a natural first pair of spaces is schematically

$$X_\eta = \left\{h : \sup(s+s_0)^{\eta}|h| + \sup(s+s_0)^{1+\eta}|h'| + \sup(s+s_0)^{2+\eta}|h''| < \infty\right\},$$

with

$$Y_\eta = \left\{f : \sup(s+s_0)^{2+\eta}|f| < \infty\right\}.$$

Then, after the phase condition,

$$\boxed{L_1^{-1} : Y_\eta \longrightarrow X_\eta} \tag{40}$$

has an explicit bound.

7. Expand the fractional operator in ε

Now set

$$\alpha = 1 + \varepsilon.$$

On the Fourier side,

$$\widehat{(-\partial_{ss})^{1+\varepsilon}F}(\xi) = |\xi|^{2+2\varepsilon}\widehat{F}(\xi).$$

Since

$$|\xi|^{2+2\varepsilon} = |\xi|^2 e^{2\varepsilon \log|\xi|},$$

we obtain formally

$$|\xi|^{2+2\varepsilon} = |\xi|^2 + 2\varepsilon|\xi|^2 \log|\xi| + O\big(\varepsilon^2|\xi|^2 \log^2|\xi|\big)\,.$$

Therefore

$$\boxed{(-\partial_{ss})^{1+\varepsilon} = (-\partial_{ss}) + 2\varepsilon(-\partial_{ss}) \log|D| + O(\varepsilon^2).} \tag{41}$$

In particular,

$$\boxed{\partial_\alpha(-\partial_{ss})^\alpha\big|_{\alpha=1} = 2(-\partial_{ss}) \log|D|.} \tag{42}$$

This is the operator whose weighted mapping properties now need to be proved.

8. Compute the first continuation correction

Seek

$$\boxed{F_\varepsilon = F_1 + \varepsilon\Phi + O(\varepsilon^2).} \tag{43}$$

Insert this and

$$\gamma = \lambda\varepsilon$$

into

$$\mathscr{G}(\varepsilon, F_\varepsilon) = 0.$$

The zeroth-order equation vanishes.
At order ε,

$$L_1\Phi + 2\nu(-\partial_{ss}) \log|D| F_1 - \frac{\lambda}{2s} F_1^3 = 0.$$

Therefore

$$\boxed{L_1\Phi = -2\nu(-\partial_{ss}) \log|D| F_1 + \frac{\lambda}{2s} F_1^3.} \tag{44}$$

Using our explicit inverse,

$$\boxed{\Phi = L_1^{-1}\left[-2\nu(-\partial_{ss}) \log|D| F_1 + \frac{\lambda}{2s} F_1^3\right].} \tag{45}$$

So the first derivative of the profile branch with respect to α is, in principle, explicitly determined.

9. But equation (44) reveals a problem with $\gamma = \lambda(\alpha - 1)$

Because

$$F_1(0) > 0,$$

the term

$$\frac{F_1^3}{s}$$

behaves like

$$\frac{F_1(0)^3}{s} \qquad (s \downarrow 0).$$

That is singular.

This means that the simple continuation

$$\gamma = \lambda(\alpha - 1)$$

from the shifted profile

$$F_1(s) = C(s + s_0)^{-1/2}$$

does **not** automatically belong to the weighted spaces just proposed.

This is an important finding.

There are two clean possibilities:

$$\boxed{\text{A. Set } \gamma \equiv 0}$$

throughout the small interval, or reformulate the endpoint/profile space so that the $\gamma F^3/s$ term is compatible with the required behavior at $s = 0$.

For the first interval theorem, $\gamma = 0$ **is substantially cleaner**.

Then

$$\boxed{\nu(-\partial_{ss})^\alpha F_\alpha = F_\alpha^2 F_\alpha'.} \tag{46}$$

At $\alpha = 1$, this is exactly solved by F_1.

The first correction becomes simply

$$\boxed{L_1\Phi = -2\nu(-\partial_{ss}) \log|D|F_1.} \tag{47}$$

This removes the artificial $1/s$ singularity from the continuation problem.

10. The actual implicit-function theorem

Define, with $\gamma = 0$,

$$\boxed{\mathscr{G}(\alpha, F) = \nu(-\partial_{ss})^{\alpha} F - F^2 F'.} \tag{48}$$

We have

$$\mathscr{G}(1, F_1) = 0$$

and

$$D_F \mathscr{G}(1, F_1) = L_1.$$

After imposing

$$F(0) = F_1(0),$$

the admissible kernel of L_1 is removed.
Suppose we prove that for appropriate weighted spaces X, Y,

$$L_1 : X_0 \to Y$$

is a bounded isomorphism, and that

$$(\alpha, F) \mapsto (-\partial_{ss})^{\alpha} F$$

is C^1 from a neighborhood of

$$(1, F_1)$$

into Y.
Then the Banach-space implicit-function theorem gives

$$\boxed{\exists\, \delta > 0 \quad \exists!\, F_\alpha \quad \text{for } 1 < \alpha < 1 + \delta} \tag{49}$$

with

$$\mathscr{G}(\alpha, F_\alpha) = 0$$

and

$$\boxed{\|F_\alpha - F_1\|_X \leq C|\alpha - 1|.} \tag{50}$$

This would be an **analytic interval theorem**, not a numerical certificate at a single exponent.

11. Positivity then follows from a relative norm

We should choose X so that it controls

$$\frac{F - F_1}{F_1}.$$

Suppose the continuation estimate is strengthened to

$$\boxed{\sup_{s>0} \frac{|F_\alpha(s) - F_1(s)|}{F_1(s)} \le C_*(\alpha - 1).} \tag{51}$$

Choose

$$\boxed{\delta < \frac{1}{2C_*}.} \tag{52}$$

Then for every

$$1 < \alpha < 1 + \delta,$$

$$F_\alpha \ge F_1 - C_*(\alpha - 1)F_1 > \frac{1}{2}F_1.$$

Hence

$$\boxed{F_\alpha(s) > 0 \qquad \forall s > 0.} \tag{53}$$

This would solve the profile-positivity obligation for the entire interval at once.

12. What we have proved and what remains before invoking IFT

We have now proved algebraically:

$$\boxed{L_1 h = -\nu \left(h' + \frac{3}{2(s + s_0)} h \right)'}$$

and found its admissible kernel exactly,

$$\boxed{\ker L_1 = \operatorname{span}\{\partial_{s_0} F_1\}.}$$

After fixing $F(0)$, that kernel disappears. We have also obtained the explicit inverse formula (35) and elementary weighted estimates such as (38)–(39).

What is **not yet proved** is the crucial nonlocal functional-analysis statement

$$\boxed{(\alpha, F) \mapsto (-\partial_{ss})^\alpha F \quad \text{is } C^1 \text{ between the exact weighted spaces needed here}} \tag{54}$$

with constants uniform for

$$1 \le \alpha \le 1 + \delta.$$

There is also a domain issue we must settle carefully: the fractional profile operator is inherited from the whole-line one-phase fractional Laplacian, whereas $F_1(s) = C(s + s_0)^{-1/2}$ has so far been written only for $s > 0$. Before applying the whole-line Fourier multiplier in (41), we must specify the whole-line extension/profile formulation consistently. We cannot simply apply $\log |D|$ to an unspecified half-line extension.

So the next proof step is now sharply defined: **choose the whole-line weighted profile space and extension compatible with $(-\partial_{ss})^\alpha$, then prove the uniform multiplier estimate needed for (54).** Only after that would the implicit-function theorem genuinely produce the desired $\delta > 0$.

4 Part C

The next step is to make the continuation argument mathematically well-posed on the same whole-line fractional operator used. We continue from the $\gamma = 0$ interval formulation, since that avoids the F^3/s singularity we just uncovered.

13. Put the profile problem on the whole line

We want

$$\boxed{\nu(-\partial_{ss})^\alpha F_\alpha = F_\alpha^2 F_\alpha', \qquad 1 < \alpha < 1 + \delta.} \tag{55}$$

The difficulty is that our explicit anchor

$$F_1(s) = \sqrt{\frac{3\nu}{2(s + s_0)}}$$

was written for $s > 0$, whereas the Fourier definition

$$\widehat{(-\partial_{ss})^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi) \tag{56}$$

is naturally a whole-line operator.

We therefore should **not arbitrarily extend** F_1. Instead, it is better to use a coordinate in which the endpoint $s = 0$ is sent to infinity and the profile problem becomes a whole-line problem—the same basic idea already used by the Mellin/logarithmic formulation.

Set

$$\boxed{x = \log s, \qquad s = e^x.} \tag{57}$$

Then

$$s \downarrow 0 \iff x \to -\infty, \qquad s \to \infty \iff x \to +\infty.$$

This turns the half-line profile into a whole-line heteroclinic-type problem.

There is, however, an important distinction: the fractional Laplacian is **not translation invariant in** x after this change. It becomes the Mellin-conjugated operator used rather than simply $|D_x|^{2\alpha}$. That is precisely why its Mellin symbol is introduced.

14. Use the general α Mellin conjugation

For the terminal similarity problem define

$$\boxed{a_\alpha := \alpha - \frac{1}{2}.} \tag{58}$$

The natural conjugation is

$$\boxed{G_\alpha(x) = s^{a_\alpha} F_\alpha(s), \qquad s = e^x.} \tag{59}$$

Thus

$$F_\alpha(s) = s^{-a_\alpha} G_\alpha(\log s). \tag{60}$$

At $\alpha = 1$,

$$a_1 = \frac{1}{2}.$$

For the explicit anchor,

$$F_1(s) = c_\nu (s + s_0)^{-1/2}, \qquad c_\nu := \sqrt{\frac{3\nu}{2}},$$

we obtain

$$G_1(x) = e^{x/2} F_1(e^x) = c_\nu \left(\frac{e^x}{e^x + s_0} \right)^{1/2}.$$

Hence

$$\boxed{G_1(x) = \frac{c_\nu}{\sqrt{1 + s_0 e^{-x}}}.} \tag{61}$$

Now we have an explicit whole-line profile.

Its endpoints are

$$\boxed{G_1(-\infty) = 0, \qquad G_1(+\infty) = c_\nu.} \tag{62}$$

This is exactly the geometry we want: the explicit $\alpha = 1$ solution has become a smooth whole-line connector.

15. Normalize the right equilibrium

Define

$$\boxed{Z_1(x) = \frac{G_1(x)}{c_\nu}.} \tag{63}$$

Then

$$\boxed{Z_1(x) = \frac{1}{\sqrt{1 + s_0 e^{-x}}}.} \tag{64}$$

Consequently

$$Z_1(-\infty) = 0, \qquad Z_1(+\infty) = 1,$$

and

$$\boxed{0 < Z_1(x) < 1.} \tag{65}$$

This gives us an **explicit positive whole-line connector at the continuation endpoint** $\alpha = 1$.

Its derivative is

$$Z_1'(x) = \frac{s_0 e^{-x}}{2(1 + s_0 e^{-x})^{3/2}} > 0. \tag{66}$$

Thus Z_1 is strictly increasing.

16. Endpoint asymptotics are explicit

As $x \to -\infty$,

$$s_0 e^{-x} \to \infty.$$

Therefore

$$Z_1(x) = s_0^{-1/2} e^{x/2} \left(1 + s_0^{-1} e^{x}\right)^{-1/2}.$$

Using

$$(1 + y)^{-1/2} = 1 - \frac{1}{2}y + \frac{3}{8}y^2 + O(y^3),$$

we obtain

$$\boxed{Z_1(x) = s_0^{-1/2} e^{x/2} - \frac{1}{2} s_0^{-3/2} e^{3x/2} + \frac{3}{8} s_0^{-5/2} e^{5x/2} + O(e^{7x/2}).} \tag{67}$$

So the left indicial exponent at the anchor is

$$\boxed{q_-(1) = \frac{1}{2}.} \tag{68}$$

For general α, the formula is

$$q_-(\alpha) = 2\alpha - \frac{3}{2}.$$

Therefore

$$q_-(1+\varepsilon) = \frac{1}{2} + 2\varepsilon, \tag{69}$$

which converges continuously to the exact anchor exponent.
That is a strong consistency check on the proposed continuation.

17. Right endpoint

As $x \to +\infty$,

$$s_0 e^{-x} \to 0.$$

Hence

$$Z_1(x) = 1 - \frac{s_0}{2}e^{-x} + \frac{3s_0^2}{8}e^{-2x} + O(e^{-3x}). \tag{70}$$

Thus

$$\boxed{1 - Z_1(x) \sim \frac{s_0}{2}e^{-x}.} \tag{71}$$

The right stable exponent at the anchor is therefore

$$\boxed{r_+(1) = 1.} \tag{72}$$

This is useful because the continuation begins with strictly positive endpoint exponents:

$$q_-(1) = \frac{1}{2} > 0, \qquad r_+(1) = 1 > 0.$$

By continuity, we should expect both to remain positive for sufficiently small $\alpha - 1$, once the corresponding indicial roots are proved continuous.

18. Fix translation exactly

The parameter s_0 is just translation in x.
Indeed,

$$Z_1(x) = (1 + s_0 e^{-x})^{-1/2}.$$

Write

$$s_0 = e^{x_0}.$$

Then

$$Z_1(x) = \left(1 + e^{-(x-x_0)}\right)^{-1/2}.$$

Thus changing s_0 simply translates the profile.
Choose the normalization

$$\boxed{Z_\alpha(0) = \frac{1}{\sqrt{2}}.} \tag{73}$$

At $\alpha = 1$, this gives

$$s_0 = 1.$$

Therefore from now on we may take

$$\boxed{Z_1(x) = (1 + e^{-x})^{-1/2}.} \tag{74}$$

This removes the zero mode globally.

19. Work with the derivative

Define

$$\boxed{w_\alpha(x) = Z'_\alpha(x).} \tag{75}$$

At the anchor,

$$\boxed{w_1(x) = \frac{e^{-x}}{2(1 + e^{-x})^{3/2}}.} \tag{76}$$

This has exponential decay at both ends:

$$w_1(x) \sim \frac{1}{2} e^{x/2}, \qquad x \to -\infty,$$

and

$$w_1(x) \sim \frac{1}{2} e^{-x}, \qquad x \to +\infty.$$

Therefore

$$\boxed{w_1 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap H^m(\mathbb{R})} \tag{77}$$

for every finite m.
Moreover,

$$\int_{-\infty}^{\infty} w_1(x)\,dx = Z_1(+\infty) - Z_1(-\infty) = 1. \tag{78}$$

This is a much cleaner continuation variable than F_1 itself.

20. Choose a two-sided exponentially weighted space

Pick

$$\boxed{0 < \eta < \frac{1}{2}.} \tag{79}$$

Define a weight schematically by

$$\omega_\eta(x) = \begin{cases} e^{-\eta x}, & x \le 0, \\ e^{\eta x}, & x \ge 0. \end{cases} \tag{80}$$

Then

$$\omega_\eta w_1$$

still decays exponentially at both ends because

$$\eta < \frac{1}{2} < 1.$$

A natural space is therefore

$$\boxed{X_\eta = \left\{ h : \omega_\eta h \in H^{2+\sigma}(\mathbb{R}) \right\},} \tag{81}$$

where $\sigma > 0$ will provide the extra regularity needed to control variation of the fractional order.

Take

$$\boxed{Y_\eta = \left\{ f : \omega_\eta f \in L^2(\mathbb{R}) \right\}.} \tag{82}$$

The phase-fixed tangent space also imposes

$$\boxed{\int_{\mathbb{R}} h(x)\,dx = 0,} \tag{83}$$

because the total jump of Z must remain

$$Z(+\infty) - Z(-\infty) = 1.$$

The translation normalization supplies the second scalar condition needed to remove translation.

21. Why a small extra derivative is enough for varying α

The key multiplier difference is

$$|\xi|^{2+2\varepsilon} - |\xi|^2.$$

For any fixed

$$0 < \sigma < 1,$$

we can use

$$|\log r| \le C_\sigma (r^\sigma + r^{-\sigma})$$

away from the removable low-frequency point.
At high frequency,

$$|\xi|^{2+2\varepsilon} - |\xi|^2 = |\xi|^2 \left(e^{2\varepsilon \log |\xi|} - 1 \right).$$

By the mean-value theorem,

$$\left| e^{2\varepsilon \log |\xi|} - 1 \right| \le 2\varepsilon |\log |\xi|| \, |\xi|^{2\varepsilon}$$

for $|\xi| \ge 1$.
If

$$0 \le \varepsilon \le \frac{\sigma}{4},$$

then

$$|\xi|^{2\varepsilon} \le |\xi|^{\sigma/2}.$$

Also

$$\log |\xi| \le C_\sigma |\xi|^{\sigma/2}.$$

Hence

$$\boxed{\left| |\xi|^{2+2\varepsilon} - |\xi|^2 \right| \le C_\sigma \varepsilon |\xi|^{2+\sigma}, \qquad |\xi| \ge 1.} \tag{84}$$

Therefore

$$\boxed{\| [(-\partial_{ss})^{1+\varepsilon} - (-\partial_{ss})] f \|_{L^2} \le C_\sigma \varepsilon \| f \|_{H^{2+\sigma}},} \tag{85}$$

for the translation-invariant whole-line operator.

This is the elementary estimate we wanted.

It shows why we should not work merely in H^2: the logarithmic derivative of the fractional power costs an arbitrarily small amount of extra regularity.

22. But in Mellin coordinates the multiplier is different

For our actual profile problem, the relevant multiplier after logarithmic conjugation is not simply

$$|\xi|^{2\alpha}.$$

The Mellin calculation produces a symbol of the form

$$\boxed{m_\alpha(\xi) = K_\alpha^+\big(-a_\alpha + i\xi\big), \qquad a_\alpha = \alpha - \frac{1}{2}.} \tag{86}$$

At $\alpha = 9/8$ attempt, I used precisely this structure for the high-frequency inverse analysis. For the interval argument we need the **general-α** estimate

$$\boxed{m_\alpha(\xi) = m_1(\xi) + O\big((\alpha-1)\langle\xi\rangle^{2+\sigma}\big)} \tag{87}$$

uniformly for

$$1 \le \alpha \le 1 + \delta.$$

More precisely, what we want is

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \le C_{\sigma,\delta}|\alpha-1|\langle\xi\rangle^{2+\sigma}.} \tag{88}$$

If this holds, then

$$\boxed{\|(M_\alpha - M_1)h\|_{Y_\eta} \le C|\alpha-1|\|h\|_{X_\eta}.} \tag{89}$$

That is exactly the perturbative operator estimate required by the implicit-function theorem.

23. We can already see how invertibility persists

Let

$$\mathscr{H}_\alpha(w) = 0$$

denote the differentiated, phase-normalized Mellin profile equation.
At $\alpha = 1$,

$$\mathscr{H}_1(w_1) = 0.$$

Let

$$\boxed{\mathcal{L}_1 = D_w\mathscr{H}_1(w_1).} \tag{90}$$

After the mass and translation constraints, suppose

$$\boxed{\|\mathcal{L}_1^{-1}\|_{Y_\eta \to X_\eta} \le C_0.} \tag{91}$$

For nearby α,

$$\mathcal{L}_\alpha = \mathcal{L}_1 + \mathcal{E}_\alpha,$$

where the desired multiplier and coefficient estimates give

$$\boxed{\|\mathcal{E}_\alpha\|_{X_\eta \to Y_\eta} \le C_1|\alpha - 1|.} \tag{92}$$

Thus

$$\mathcal{L}_\alpha = \mathcal{L}_1 \left[I + \mathcal{L}_1^{-1}\mathcal{E}_\alpha\right].$$

If

$$C_0 C_1 |\alpha - 1| < 1,$$

the bracket is invertible by the Neumann series.
Therefore

$$\boxed{\mathcal{L}_\alpha^{-1} = \left[I + \mathcal{L}_1^{-1}\mathcal{E}_\alpha\right]^{-1} \mathcal{L}_1^{-1}.} \tag{93}$$

And

$$\boxed{\|\mathcal{L}_\alpha^{-1}\| \le \frac{C_0}{1 - C_0 C_1 |\alpha - 1|}.} \tag{94}$$

In particular, if

$$C_0 C_1 \delta \le \frac{1}{2},$$

then

$$\boxed{\sup_{1 \le \alpha \le 1+\delta} \|\mathcal{L}_\alpha^{-1}\| \le 2C_0.} \tag{95}$$

(This is exactly the kind of **uniform inverse bound** that was found to be difficult to establish independently at a jump of $\alpha = 9/8$).

24. Quantitative implicit-function argument

Let

$$e_\alpha = w_\alpha - w_1.$$

Expand:

$$0 = \mathscr{H}_\alpha(w_1 + e_\alpha).$$

Then

$$0 = \mathscr{H}_\alpha(w_1) + \mathcal{L}_1 e_\alpha + \mathcal{R}_\alpha e_\alpha + \mathcal{Q}_\alpha(e_\alpha),$$

where

$$\|\mathscr{H}_\alpha(w_1)\|_Y \leq C_2|\alpha - 1|, \tag{96}$$

$$\|\mathcal{R}_\alpha\|_{X\to Y} \leq C_1|\alpha - 1|, \tag{97}$$

and

$$\|\mathcal{Q}_\alpha(e)\|_Y \leq C_3\|e\|_X^2. \tag{98}$$

Apply $\mathcal{L}_1^{-1}$:

$$\|e_\alpha\|_X \leq C_0C_2|\alpha - 1| + C_0C_1|\alpha - 1|\|e_\alpha\|_X + C_0C_3\|e_\alpha\|_X^2. \tag{99}$$

Choose

$$\delta$$

so that

$$C_0C_1\delta \leq \frac{1}{4}.$$

Then

$$\frac{3}{4}\|e_\alpha\|_X \leq C_0C_2(\alpha - 1) + C_0C_3\|e_\alpha\|_X^2.$$

A standard bootstrap gives, after shrinking δ again,

$$\boxed{\|w_\alpha - w_1\|_{X_\eta} \leq 2C_0C_2(\alpha - 1).} \tag{100}$$

Integrating $w_\alpha - w_1$ and using the phase normalization gives

$$\boxed{\|Z_\alpha - Z_1\|_{X_Z} \leq C_4(\alpha - 1).} \tag{101}$$

This is the desired analytic continuation estimate.

25. Positivity now follows without a maximum principle

Because

$$Z_1(x) > 0$$

for every finite x, ordinary absolute L^∞ closeness is not sufficient near $x = -\infty$, where $Z_1 \to 0$.

We therefore need the continuation norm to preserve the **relative left-tail coefficient**. Since

$$Z_1(x) \sim e^{x/2} \qquad (x \to -\infty),$$

and

$$Z_\alpha(x) \sim A_-(\alpha) e^{q_-(\alpha)x},$$

with

$$q_-(\alpha) = 2\alpha - \frac{3}{2} = \frac{1}{2} + 2(\alpha - 1),$$

positivity follows if

$$\boxed{A_-(\alpha) > 0} \tag{102}$$

and the profile remains positive on a compact transition interval.

At $\alpha = 1$,

$$A_-(1) = 1.$$

If the implicit-function construction gives continuity,

$$|A_-(\alpha) - 1| \le C_A(\alpha - 1),$$

then choosing

$$\delta < \frac{1}{2C_A}$$

gives

$$\boxed{A_-(\alpha) \ge \frac{1}{2} > 0.} \tag{103}$$

On a fixed compact interval $[-R, R]$,

$$m_R := \min_{[-R,R]} Z_1 > 0.$$

If

$$\|Z_\alpha - Z_1\|_{L^\infty([-R,R])} < \frac{m_R}{2},$$

then

$$Z_\alpha \geq \frac{m_R}{2} > 0.$$

At the right endpoint,

$$Z_\alpha \to 1,$$

so positivity is again automatic for sufficiently small perturbations.

Thus the whole line is divided into

$$(-\infty, -R], \qquad [-R, R], \qquad [R, \infty),$$

and positivity follows separately from left-tail coefficient continuity, compact-interval closeness, and right-tail stability.

This is much stronger than trying to recover a maximum principle for an operator of order > 2.

26. We can now state the first interval profile theorem precisely

What we are aiming to prove is:

Small-supercritical profile theorem. There exists $\delta_{\text{prof}} > 0$ such that for every

$$1 < \alpha < 1 + \delta_{\text{prof}},$$

the phase-normalized one-phase profile equation

$$\nu(-\partial_{ss})^\alpha F_\alpha = F_\alpha^2 F_\alpha'$$

possesses a positive profile whose logarithmically conjugated normalization Z_α satisfies

$$Z_\alpha(-\infty) = 0, \qquad Z_\alpha(+\infty) = 1,$$

and

$$\|Z_\alpha - Z_1\|_{X_Z} + \|Z_\alpha' - Z_1'\|_{X_\eta} \leq C(\alpha - 1).$$

Moreover the constants are uniform for $1 < \alpha < 1 + \delta_{\text{prof}}$. (104)

If we prove (104), then we have an analytic continuation theorem on an interval.

Where the proof genuinely stands

We have now identified the correct whole-line anchor

$$\boxed{Z_1(x) = (1 + e^{-x})^{-1/2},}$$

its endpoint exponents

$$\boxed{q_-(1) = \frac{1}{2}, \qquad r_+(1) = 1,}$$

its exponentially localized derivative w_1, and the perturbative mechanism that would give a uniform inverse and an interval of profiles.

The one part I have **not yet proved** is (88), because to do it rigorously we need the explicit general-α Gamma-function formula for

$$K_\alpha^+(\kappa).$$

Differentiate it with respect to α. Then we can try to prove an explicit bound for

$$\partial_\alpha K_\alpha^+\left(-\alpha + \frac{1}{2} + i\xi\right)$$

and thereby establish (88) rather than leave it as an assumption.

5 Part D

Calculations shown contain the exact general-α Mellin symbol we need, and with it we can prove the perturbation estimate near $\alpha = 1$.

We have that

$$\boxed{K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi}\sin\big(\pi(\alpha-\lambda)\big)} \tag{105}$$

through

$$(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda-2\alpha}.$$

It also states that after

$$F(s) = s^{-a}G(\log s), \qquad a = \alpha - \frac{1}{2},$$

the logarithmic/Mellin operator acts on $e^{\lambda x}$ by

$$\mathcal{M}_a(e^{\lambda x}) = K_\alpha^+(-a+\lambda)e^{\lambda x}.$$

So we can now do the calculation exactly.

27. Exact Fourier-line Mellin symbol for general α

On the Fourier line put

$$\lambda = i\xi.$$

Since

$$a = \alpha - \frac{1}{2},$$

the relevant argument of K_α^+ is

$$z_\alpha(\xi) = -a + i\xi = -\alpha + \frac{1}{2} + i\xi.$$

Therefore

$$m_\alpha(\xi) := K_\alpha^+\left(-\alpha + \frac{1}{2} + i\xi\right). \tag{106}$$

Substitute this into (105).
First,

$$z_\alpha + 1 = \frac{3}{2} - \alpha + i\xi.$$

Second,

$$2\alpha - z_\alpha = 3\alpha - \frac{1}{2} - i\xi.$$

Third,

$$\alpha - z_\alpha = 2\alpha - \frac{1}{2} - i\xi.$$

Hence the exact symbol is

$$\boxed{m_\alpha(\xi) = \frac{\Gamma\left(\frac{3}{2} - \alpha + i\xi\right)\Gamma\left(3\alpha - \frac{1}{2} - i\xi\right)}{\pi} \sin\left[\pi\left(2\alpha - \frac{1}{2} - i\xi\right)\right].} \tag{107}$$

This is the formula we needed.
For

$$1 \le \alpha < \frac{5}{4},$$

the real parts of the Gamma arguments satisfy

$$\frac{3}{2} - \alpha > \frac{1}{4}, \qquad 3\alpha - \frac{1}{2} > \frac{5}{2},$$

so neither Gamma factor encounters a pole.
Moreover, for

$$1 \le \alpha < \frac{5}{4},$$

the sine factor has no zero for real ξ, except at the excluded endpoint $(\alpha, \xi) = (5/4, 0)$. Thus on every closed interval

$$1 \le \alpha \le 1 + \delta < \frac{5}{4}$$

the symbol is analytic in α for every $\xi \in \mathbb{R}$.

28. Check the anchor $\alpha = 1$

At $\alpha = 1$,

$$\boxed{m_1(\xi) = \frac{\Gamma(\frac{1}{2} + i\xi)\Gamma(\frac{5}{2} - i\xi)}{\pi} \sin\left[\pi\left(\frac{3}{2} - i\xi\right)\right].} \tag{108}$$

Now

$$\sin\left(\frac{3\pi}{2} - i\pi\xi\right) = -\cosh(\pi\xi).$$

Therefore

$$m_1(\xi) = -\frac{\cosh(\pi\xi)}{\pi}\Gamma\left(\frac{1}{2} + i\xi\right)\Gamma\left(\frac{5}{2} - i\xi\right). \tag{109}$$

Use

$$\Gamma\left(\frac{5}{2} - i\xi\right) = \left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right)\Gamma\left(\frac{1}{2} - i\xi\right)$$

and the classical identity

$$\Gamma\left(\frac{1}{2} + i\xi\right)\Gamma\left(\frac{1}{2} - i\xi\right) = \frac{\pi}{\cosh(\pi\xi)}.$$

Everything cancels, giving the remarkably simple formula

$$\boxed{m_1(\xi) = -\left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right).} \tag{110}$$

Expanding,

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.} \tag{111}$$

So at the continuation endpoint the apparently complicated Mellin operator becomes a **second-order constant-coefficient differential operator in logarithmic coordinates**.

That is exactly what we should expect from the local $\alpha = 1$ problem.

29. Differentiate m_α with respect to α

Set

$$A_\alpha(\xi) = \frac{3}{2} - \alpha + i\xi,$$

$$B_\alpha(\xi) = 3\alpha - \frac{1}{2} - i\xi,$$

and

$$C_\alpha(\xi) = \pi\left(2\alpha - \frac{1}{2} - i\xi\right).$$

Then

$$m_\alpha = \frac{1}{\pi}\Gamma(A_\alpha)\Gamma(B_\alpha)\sin C_\alpha.$$

Since

$$\frac{d}{dz}\log\Gamma(z) = \psi(z),$$

we obtain

$$\partial_\alpha \log m_\alpha = -\psi(A_\alpha) + 3\psi(B_\alpha) + 2\pi\cot C_\alpha.$$

Therefore

$$\boxed{\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi)\left[-\psi\left(\frac{3}{2} - \alpha + i\xi\right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi\right) + 2\pi\cot\big(\pi(2\alpha - \tfrac{1}{2} - i\xi)\big)\right].} \tag{112}$$

This is exact.

30. Uniform bound for the cotangent term

Let

$$z = x + iy.$$

One has

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

and

$$|\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Hence

$$|\cot z|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}. \tag{113}$$

Here

$$x = \pi\left(2\alpha - \frac{1}{2}\right), \qquad y = -\pi\xi.$$

Fix

$$0 < \delta < \frac{1}{4}.$$

Then

$$1 \le \alpha \le 1 + \delta$$

lies a positive distance from the zero at $\alpha = 5/4$.
Consequently there exists

$$c_\delta > 0$$

such that

$$\left|\sin \pi\left(2\alpha - \frac{1}{2}\right)\right| \ge c_\delta$$

uniformly in $\alpha \in [1, 1+\delta]$.
From (113),

$$\boxed{\sup_{\substack{1\le\alpha\le1+\delta\\ \xi\in\mathbb{R}}} \left|\cot\big(\pi(2\alpha - \tfrac{1}{2} - i\xi)\big)\right| \le C_\delta.} \tag{114}$$

So the cotangent term causes no high-frequency growth.

31. Uniform digamma estimate

On any fixed right half-plane

$$\Re z \ge c > 0,$$

the standard digamma asymptotic gives

$$\psi(z) = \log z - \frac{1}{2z} + O(|z|^{-2})$$

as $|z| \to \infty$.
In particular,

$$\boxed{|\psi(z)| \le C_c\,(1+\log\langle \Im z\rangle)\,, \qquad \Re z \ge c,} \tag{115}$$

where

$$\langle \xi\rangle = (1+\xi^2)^{1/2}.$$

For our two Gamma arguments,

$$\Re A_\alpha = \frac{3}{2} - \alpha \ge \frac{1}{2} - \delta > 0,$$

and

$$\Re B_\alpha = 3\alpha - \frac{1}{2} \ge \frac{5}{2}.$$

Therefore

$$|\psi(A_\alpha(\xi))| + |\psi(B_\alpha(\xi))| \le C_\delta\,(1+\log\langle\xi\rangle)\,. \tag{116}$$

Combining (112), (114), and (116),

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_\delta |m_\alpha(\xi)|\,(1+\log\langle\xi\rangle)\,.} \tag{117}$$

We now only need to bound m_α.

32. Uniform growth of m_α

Use Stirling's estimate in vertical strips:

$$|\Gamma(\sigma+i\xi)| \le C_{\sigma_0,\sigma_1} e^{-\pi|\xi|/2}\langle\xi\rangle^{\sigma-1/2}$$

for

$$\sigma_0 \le \sigma \le \sigma_1.$$

Therefore

$$|\Gamma(A_\alpha)| \le C_\delta e^{-\pi|\xi|/2}\langle\xi\rangle^{1-\alpha}, \tag{118}$$

because

$$\Re A_\alpha - \frac{1}{2} = 1-\alpha.$$

Likewise,

$$|\Gamma(B_\alpha)| \le C_\delta e^{-\pi|\xi|/2}\langle\xi\rangle^{3\alpha-1}. \tag{119}$$

Multiplication gives

$$|\Gamma(A_\alpha)\Gamma(B_\alpha)| \le C_\delta e^{-\pi|\xi|}\langle\xi\rangle^{2\alpha}. \tag{120}$$

Meanwhile

$$|\sin(x-iy)| \le Ce^{|y|}.$$

Thus

$$\left|\sin\pi\left(2\alpha-\frac{1}{2}-i\xi\right)\right| \le Ce^{\pi|\xi|}. \tag{121}$$

The exponentials cancel exactly. Hence

$$\boxed{|m_\alpha(\xi)| \le C_\delta\langle\xi\rangle^{2\alpha}.} \tag{122}$$

This is the expected order-2α Mellin growth.
Substituting into (117),

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_\delta\langle\xi\rangle^{2\alpha}\left(1+\log\langle\xi\rangle\right).} \tag{123}$$

This is the crucial derivative estimate.

33. Convert it to the desired $2+\sigma$ estimate

Fix an arbitrary

$$\boxed{0<\sigma<1.}$$

We will choose δ sufficiently small.
Take

$$\boxed{0<\delta\le\frac{\sigma}{4}.} \tag{124}$$

Then

$$1 \le \alpha \le 1+\delta$$

implies

$$2\alpha \le 2+2\delta \le 2+\frac{\sigma}{2}.$$

Therefore

$$\langle\xi\rangle^{2\alpha} \le \langle\xi\rangle^{2+\sigma/2}.$$

Also

$$1+\log\langle\xi\rangle \le C_\sigma\langle\xi\rangle^{\sigma/2}.$$

Consequently (123) gives

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.} \tag{125}$$

Now apply the mean-value theorem in α:

$$m_\alpha(\xi) - m_1(\xi) = (\alpha - 1)\partial_\beta m_\beta(\xi)$$

for some

$$1 < \beta < \alpha.$$

Hence

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \le C_{\delta,\sigma}(\alpha - 1)\langle\xi\rangle^{2+\sigma}.} \tag{126}$$

This is exactly estimate (88) that was left unproved in the previous step. So that part is now proved.

34. Operator-norm consequence

Let

$$\widehat{\mathcal{M}_\alpha h}(\xi) = m_\alpha(\xi)\widehat{h}(\xi).$$

Then by Plancherel,

$$\begin{aligned}\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2}^2 &= \int_{\mathbb{R}} |m_\alpha(\xi) - m_1(\xi)|^2 |\widehat{h}(\xi)|^2 \, d\xi \\ &\le C_{\delta,\sigma}^2(\alpha - 1)^2 \int_{\mathbb{R}} \langle\xi\rangle^{4+2\sigma} |\widehat{h}(\xi)|^2 \, d\xi.\end{aligned}$$

Therefore

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \le C_{\delta,\sigma}(\alpha - 1)\|h\|_{H^{2+\sigma}}.} \tag{127}$$

Equivalently,

$$\boxed{\|\mathcal{M}_\alpha - \mathcal{M}_1\|_{H^{2+\sigma}\to L^2} \le C_{\delta,\sigma}(\alpha - 1).} \tag{128}$$

This is already enough for an **unweighted** implicit-function formulation.

35. We can prove C^1 dependence as well

From (112),

$$\partial_\alpha m_\alpha$$

exists pointwise.
We have just obtained the uniform domination

$$|\partial_\alpha m_\alpha(\xi)| \le C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.$$

Hence, for

$$h \in H^{2+\sigma},$$

dominated convergence gives

$$\boxed{\partial_\alpha \mathcal{M}_\alpha h = \mathcal{F}^{-1}\left[(\partial_\alpha m_\alpha)\widehat{h}\right]} \tag{129}$$

in L^2.
Moreover,

$$\boxed{\alpha \mapsto \mathcal{M}_\alpha \in \mathcal{L}(H^{2+\sigma}, L^2)}$$

is C^1 on every sufficiently small closed interval

$$[1, 1+\delta]. \tag{130}$$

Thus the nonlocal operator dependence required by the implicit-function theorem is no longer merely formal.

36. A useful bonus: ellipticity is uniform near $\alpha = 1$

Stirling also gives the corresponding lower asymptotic

$$|m_\alpha(\xi)| \sim |\xi|^{2\alpha} \qquad (|\xi| \to \infty),$$

uniformly for α in a compact subinterval of $(1, 5/4)$.
At $\alpha = 1$, we know exactly

$$m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.$$

Therefore

$$|m_1(\xi)| \asymp \langle\xi\rangle^2$$

away from any low-frequency issue associated with the full nonlinear linearization.
Since

$$m_\alpha - m_1 = O_{H^{2+\sigma}\to L^2}(\alpha - 1),$$

high-frequency ellipticity persists uniformly for sufficiently small δ.

Thus there exist $R < \infty$, $c > 0$, and $\delta_1 > 0$ such that

$$\boxed{|m_\alpha(\xi)| \geq c|\xi|^{2\alpha}, \qquad |\xi| \geq R, \quad 1 \leq \alpha \leq 1 + \delta_1.} \tag{131}$$

This means that the difficult infinite-dimensional tail estimate that I calculated at 9/8 becomes a perturbation of an explicitly known second-order operator near $\alpha = 1$.

37. What about the exponentially weighted space?

There is one distinction we should preserve carefully.

Equation (127) proves

$$H^{2+\sigma} \to L^2$$

operator continuity directly.

For an exponential weight $e^{\eta x}$, Fourier conjugation shifts

$$\xi \mapsto \xi + i\eta.$$

Thus the weighted symbol is

$$\boxed{m_{\alpha,\eta}(\xi) = K_\alpha^+\left(-\alpha + \frac{1}{2} - \eta + i\xi\right)} \tag{132}$$

up to the sign convention for the chosen side of the weight.

The same Gamma/digamma argument works provided the shifted arguments remain a fixed distance from their Gamma poles and sine zeros.

Since at $\alpha = 1$ the left endpoint exponent is

$$q_-(1) = \frac{1}{2}$$

and the right exponent is

$$r_+(1) = 1,$$

we can choose a small fixed weight, for example

$$\boxed{0 < \eta < \frac{1}{4}.} \tag{133}$$

After shrinking δ, all shifted Gamma arguments remain inside fixed pole-free vertical strips. Repeating (112)–(126) then gives

$$\boxed{\|\mathcal{M}_{\alpha,\eta} - \mathcal{M}_{1,\eta}\|_{H^{2+\sigma}\to L^2} \leq C_{\eta,\sigma}(\alpha - 1).} \tag{134}$$

For a two-sided weight one performs the calculation separately on the two ends and patches with a compact transition region. That patching step is standard but is **not contained explicitly in the manuscript**, so I would state it as the next functional-analytic lemma rather than silently regard it as proved.

38. What this closes

It has been stated that the explicit $\alpha = 1$ solution is a continuation anchor, but that continuation requires both

1. phase-fixed invertibility, and
2. uniform control of the nonlocal operator.

We have now established the second requirement at the multiplier level:

$$\boxed{\|\mathcal{M}_\alpha - \mathcal{M}_1\|_{H^{2+\sigma}\to L^2} \le C(\alpha - 1).}$$

And we have an explicit formula for its derivative,

$$\boxed{\partial_\alpha m_\alpha = m_\alpha \left[-\psi(A_\alpha) + 3\psi(B_\alpha) + 2\pi \cot C_\alpha\right].}$$

So the nonlocal operator is genuinely a small perturbation of the $\alpha = 1$ operator.

There is also an important consistency point: the general endpoint calculation gives

$$F(s) \sim A s^{\alpha-1} \qquad (s \downarrow 0)$$

and

$$K_\alpha^+(\alpha - 1) = 0.$$

Thus as $\alpha \downarrow 1$, the left power tends to s^0, exactly matching the nonzero left value of the explicit $\alpha = 1$ anchor. So the endpoint limit itself is consistent.

The remaining continuation obstruction is now narrower

We should **not yet claim the interval profile theorem**, because the remaining issue is the **full phase-fixed linearized operator**, not merely its Mellin principal part.

We identify that operator schematically as

$$\mathscr{L}_1 h = \nu D\mathcal{M}_a[G_1]h - 2G_1G_1'h - G_1^2h' + 3cG_1^2h,$$

and the missing theorem is the phase-fixed invertibility/spectral gap.

That is now the natural next calculation. Because

$$\boxed{Z_1(x) = (1 + e^{-x})^{-1/2}}$$

is explicit and

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}}$$

is also explicit, we can write the **entire $\alpha = 1$ phase-fixed linearized ODE in closed form**, solve its homogeneous equation, identify all kernel modes, and attempt to construct its Green function. If that succeeds with a bounded inverse, the actual implicit-function theorem producing an existing $\delta > 0$ becomes very close.

6 Part E

The $\alpha = 1$ linearization is unusually favorable: it reduces to an explicit second-order ODE, and its two homogeneous solutions can be written in closed form. This gives the nondegeneracy needed for the continuation argument.

The $\alpha = 1$ anchor is

$$-\nu F'' = F^2 F' = \frac{1}{3}(F^3)',$$

hence

$$-\nu F' = \frac{1}{3}F^3, \qquad F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}.$$

This paper identifies this solution as the continuation anchor while saying that phase-fixed invertibility is the missing theorem.

39. Put the exact $\alpha = 1$ profile into logarithmic variables

At $\alpha = 1$,

$$a = \alpha - \frac{1}{2} = \frac{1}{2}, \qquad c = \frac{2\alpha - 1 - \gamma}{2}.$$

For the classical anchor used here, $\gamma = 0$, so

$$\boxed{c = \frac{1}{2}.}$$

Set

$$s = e^x$$

and

$$G_1(x) = s^{1/2} F_1(s).$$

Then

$$G_1(x) = \sqrt{\frac{3\nu}{2}}\sqrt{\frac{e^x}{e^x + s_0}}.$$

Introduce

$$H_* := \sqrt{\frac{3\nu}{2}}$$

and

$$Z(x) := \frac{G_1(x)}{H_*}.$$

Thus

$$\boxed{Z(x) = \left(1 + s_0 e^{-x}\right)^{-1/2}.} \tag{135}$$

Translation in x changes s_0, so without loss of generality for the spectral calculation we take

$$s_0 = 1.$$

Therefore

$$\boxed{Z(x) = \frac{1}{\sqrt{1 + e^{-x}}}.} \tag{136}$$

It is convenient to introduce

$$t(x) := Z(x)^2 = \frac{1}{1 + e^{-x}}. \tag{137}$$

Then

$$t' = t(1 - t)$$

and

$$\boxed{Z' = \frac{1}{2}Z(1 - Z^2) = \frac{1}{2}\sqrt{t}(1 - t).} \tag{138}$$

40. The Mellin operator becomes an explicit differential operator

From the previous calculation,

$$m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.$$

Since $D = \partial_x$ has Fourier multiplier $i\xi$,

$$-D^2 + 2D - \frac{3}{4}$$

has multiplier

$$\xi^2 + 2i\xi - \frac{3}{4}.$$

Consequently

$$\boxed{\mathcal{M}_{1/2} = -\partial_x^2 + 2\partial_x - \frac{3}{4}.} \tag{139}$$

Thus the apparently nonlocal Mellin problem has become a completely local second-order ODE at the continuation anchor.

41. Exact full linearization

The autonomous profile equation is

$$\nu \mathcal{M}_{1/2} G = G^2 G' - \frac{1}{2} G^3. \tag{140}$$

Define

$$\mathscr{F}_1(G) = \nu \mathcal{M}_{1/2} G - G^2 G' + \frac{1}{2} G^3.$$

Then

$$\mathscr{F}_1(G_1) = 0.$$

Linearizing $G = G_1 + \varepsilon h$,

$$D(G^2 G')[h] = 2G_1 G_1' h + G_1^2 h',$$

while

$$D\left(\frac{1}{2} G^3\right)[h] = \frac{3}{2} G_1^2 h.$$

Therefore

$$\boxed{\mathscr{L}_1 h = \nu\left(-h'' + 2h' - \frac{3}{4}h\right) - 2G_1 G_1' h - G_1^2 h' + \frac{3}{2} G_1^2 h.} \tag{141}$$

There exists a corresponding linearization.
Because

$$G_1 = H_* Z, \qquad H_*^2 = \frac{3\nu}{2},$$

we have

$$\frac{G_1^2}{\nu} = \frac{3}{2}Z^2 = \frac{3}{2}t$$

and

$$\frac{2G_1G_1'}{\nu} = 3ZZ'.$$

Divide (141) by ν:

$$\boxed{\frac{1}{\nu}\mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2}t\right)h' + \left(-\frac{3}{4} - 3ZZ' + \frac{9}{4}t\right)h.} \tag{142}$$

Since

$$ZZ' = \frac{1}{2}t(1-t),$$

the zeroth-order coefficient becomes

$$-\frac{3}{4} - \frac{3}{2}t(1-t) + \frac{9}{4}t = -\frac{3}{4} + \frac{3}{4}t + \frac{3}{2}t^2.$$

Hence

$$\boxed{\frac{1}{\nu}\mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2}t\right)h' + \left(-\frac{3}{4} + \frac{3}{4}t + \frac{3}{2}t^2\right)h.} \tag{143}$$

This is the exact phase-linearized differential operator.

42. Translation gives one kernel element automatically

Because the profile equation is invariant under

$$x \mapsto x + \tau,$$

differentiate

$$\mathscr{F}_1(G_1(\,\cdot + \tau)) = 0$$

at $\tau = 0$. We obtain

$$\boxed{\mathscr{L}_1 G_1' = 0.} \tag{144}$$

Thus

$$h_1(x) = G_1'(x)$$

is a homogeneous solution.
Ignoring the harmless constant H_*,

$$\boxed{h_1(x) = Z'(x) = \frac{1}{2}\sqrt{t}(1-t).} \tag{145}$$

Equivalently,

$$\boxed{h_1(x) = \frac{1}{2}\frac{e^{x/2}}{(1+e^x)^{3/2}}}$$

up to the equivalent translation convention.
Its endpoint behavior is

$$h_1(x) \sim \frac{1}{2}e^{x/2}, \qquad x \to -\infty, \tag{146}$$

and

$$h_1(x) \sim \frac{1}{2}e^{-x}, \qquad x \to +\infty. \tag{147}$$

Thus the translation mode is admissible at **both** ends.

43. Construct the second homogeneous solution exactly

Multiply (143) by -1. The homogeneous equation takes the standard form

$$h'' + P(x)h' + Q(x)h = 0$$

with

$$P(x) = -2 + \frac{3}{2}t(x). \tag{148}$$

Given h_1, reduction of order gives

$$h_2 = h_1 \int \frac{e^{-\int P(x)\,dx}}{h_1(x)^2}\,dx. \tag{149}$$

Because

$$t' = t(1-t),$$

we have

$$dx = \frac{dt}{t(1-t)}.$$

Now

$$\int P\,dx = \int \frac{-2 + \frac{3}{2}t}{t(1-t)}\,dt.$$

Partial fractions give

$$\frac{-2+\frac{3}{2}t}{t(1-t)} = -\frac{2}{t} - \frac{1}{2(1-t)}.$$

Therefore

$$\int P\,dx = -2\log t + \frac{1}{2}\log(1-t),$$

so

$$\boxed{e^{-\int P dx} = t^2(1-t)^{-1/2}.} \tag{150}$$

Also

$$h_1^2 = \frac{1}{4}t(1-t)^2.$$

Consequently

$$\frac{e^{-\int P dx}}{h_1^2} = 4t(1-t)^{-5/2}.$$

Including

$$dx = \frac{dt}{t(1-t)}$$

gives

$$\int \frac{e^{-\int P dx}}{h_1^2}\,dx = 4\int (1-t)^{-7/2}\,dt.$$

Hence

$$= \frac{8}{5}(1-t)^{-5/2} + C. \tag{151}$$

Thus one may take

$$\boxed{h_2(x) = h_1(x)(1-t(x))^{-5/2}.} \tag{152}$$

Apart from an irrelevant constant,

$$\boxed{h_2(x) = \sqrt{t(x)}(1-t(x))^{-3/2}.} \tag{153}$$

This is an exact second homogeneous solution.

44. Endpoint behavior of the second mode

As

$$x \to +\infty,$$

we have

$$1 - t \sim e^{-x}.$$

Therefore

$$\boxed{h_2(x) \sim e^{3x/2}.} \tag{154}$$

So h_2 grows exponentially at the right endpoint.
It is therefore excluded from every natural heteroclinic space imposing decay at $+\infty$.
At the left endpoint, it is useful to subtract its component parallel to h_1. Define

$$\widetilde{h}_2 = h_1 \left[(1-t)^{-5/2} - 1\right]. \tag{155}$$

Since

$$(1-t)^{-5/2} = 1 + \frac{5}{2}t + O(t^2)$$

and

$$t \sim e^x,$$

we obtain

$$\widetilde{h}_2 \sim Ce^{3x/2} \qquad (x \to -\infty). \tag{156}$$

Thus the two left indicial rates are

$$\boxed{\lambda_-^{(1)} = \frac{1}{2}, \qquad \lambda_-^{(2)} = \frac{3}{2}.} \tag{157}$$

At the right endpoint the admissible mode has rate

$$\boxed{r_+(1) = 1,} \tag{158}$$

while the other solution grows like $e^{3x/2}$.

45. The kernel is exactly one-dimensional before phase fixing

Every solution of

$$\mathscr{L}_1 h = 0$$

is

$$h = C_1 h_1 + C_2 h_2.$$

If h belongs to the heteroclinic perturbation space, it must decay at $+\infty$.
But

$$h_2(x) \sim e^{3x/2}.$$

Therefore necessarily

$$C_2 = 0.$$

Hence

$$h = C_1 G_1'.$$

We have therefore proved

$$\boxed{\ker \mathscr{L}_1 = \operatorname{span}\{G_1'\}} \tag{159}$$

in the natural two-ended heteroclinic space.

This is stronger than merely identifying a translation mode: there is **no second admissible zero mode**.

That is precisely the nondegeneracy statement we had left open at the $\alpha = 1$ anchor.

46. Phase fixing removes the kernel completely

Choose any finite phase point x_0. Since

$$G_1'(x) > 0$$

for every finite x,

$$G_1'(x_0) \neq 0.$$

Impose

$$\boxed{h(x_0) = 0.} \tag{160}$$

If

$$h \in \ker \mathscr{L}_1,$$

then by (159),

$$h = C\, G_1'.$$

Thus

$$0 = h(x_0) = C\, G_1'(x_0).$$

Since $G_1'(x_0) \neq 0$,

$$C = 0.$$

Therefore

$$\boxed{\ker\left(\mathscr{L}_1\big|_{\{h(x_0)=0\}}\right) = \{0\}.} \tag{161}$$

So the phase-fixed linearization is injective.
This is the first major continuation result.

47. Exact Wronskian

For

$$h'' + Ph' + Qh = 0,$$

Abel's identity gives

$$W'(x) = -P(x)W(x).$$

Hence

$$W(x) = W_0 e^{-\int P dx}.$$

Using (150),

$$\boxed{W(h_1, h_2)(x) = C_W t(x)^2 (1 - t(x))^{-1/2}.} \tag{162}$$

In particular,

$$\boxed{W(h_1, h_2)(x) \neq 0 \qquad (-\infty < x < \infty).} \tag{163}$$

There is therefore no interior degeneracy whatsoever.

48. Variation-of-parameters Green representation

Write

$$\mathscr{L}_1 h = f.$$

After division by $-\nu$, this is

$$h'' + Ph' + Qh = -\frac{f}{\nu}. \tag{164}$$

Variation of parameters gives

$$h_p(x) = -h_1(x) \int^x \frac{h_2(y)g(y)}{W(y)}\,dy + h_2(x) \int^x \frac{h_1(y)g(y)}{W(y)}\,dy, \tag{165}$$

where

$$g = -\frac{f}{\nu}.$$

Equivalently,

$$\boxed{h_p(x) = \frac{1}{\nu} h_1(x) \int^x \frac{h_2(y)f(y)}{W(y)}\,dy - \frac{1}{\nu} h_2(x) \int^x \frac{h_1(y)f(y)}{W(y)}\,dy.} \tag{166}$$

Because h_2 grows at $+\infty$, the coefficient multiplying it must be chosen so that its growing component cancels. This gives the right-decaying solution

$$\boxed{h(x) = Ch_1(x) + \frac{h_1(x)}{\nu} \int_{x_0}^x \frac{h_2(y)f(y)}{W(y)}\,dy + \frac{h_2(x)}{\nu} \int_x^\infty \frac{h_1(y)f(y)}{W(y)}\,dy.} \tag{167}$$

The remaining constant C is exactly the translation freedom.
Phase fixing $h(x_0) = 0$ determines it uniquely:

$$C = -\frac{h_2(x_0)}{\nu h_1(x_0)} \int_{x_0}^\infty \frac{h_1(y)f(y)}{W(y)}\,dy. \tag{168}$$

Therefore the phase-fixed inverse is explicitly

$$\boxed{\begin{aligned}(\mathscr{L}_{1,\mathrm{ph}}^{-1} f)(x) = {} & \frac{h_1(x)}{\nu} \int_{x_0}^x \frac{h_2(y)f(y)}{W(y)}\,dy \\ & + \frac{h_2(x)}{\nu} \int_x^\infty \frac{h_1(y)f(y)}{W(y)}\,dy \\ & - \frac{h_1(x)h_2(x_0)}{\nu h_1(x_0)} \int_{x_0}^\infty \frac{h_1(y)f(y)}{W(y)}\,dy.\end{aligned}} \tag{169}$$

So we have an **explicit Green operator**.

49. The Green kernel itself

Equation (169) may be written

$$h(x) = \int_{\mathbb{R}} \mathcal{G}_1(x,y) f(y)\, dy.$$

For $x \geq x_0$, its principal two pieces have the standard form

$$\boxed{\mathcal{G}_1(x,y) = \frac{1}{\nu W(y)} \begin{cases} h_1(x)h_2(y), & x_0 \leq y \leq x, \\ h_2(x)h_1(y), & y \geq x, \end{cases}} \tag{170}$$

together with the rank-one phase correction from (169).

The important fact is that every factor here is explicit:

$$h_1 = \frac{1}{2}\sqrt{t}(1-t),$$

$$h_2 = \sqrt{t}(1-t)^{-3/2},$$

$$W = C_W t^2 (1-t)^{-1/2},$$

$$t = \frac{1}{1+e^{-x}}.$$

Thus there is no numerical spectral problem at $\alpha = 1$.

50. What has actually been proved

We can now upgrade the statement at the classical anchor substantially.

At $\alpha = 1$:

$$\boxed{\begin{gathered} G_1(x) = \sqrt{\frac{3\nu}{2}}\,(1+e^{-x})^{-1/2}, \\ \mathcal{M}_{1/2} = -\partial_x^2 + 2\partial_x - \frac{3}{4}, \\ \ker \mathscr{L}_1 = \operatorname{span}\{G_1'\}, \\ \ker \mathscr{L}_{1,\mathrm{ph}} = \{0\}, \\ \mathscr{L}_{1,\mathrm{ph}}^{-1} \text{ has the explicit Green representation (169).} \end{gathered}} \tag{171}$$

This proves the **kernel/nondegeneracy part** of the missing phase-fixed invertibility statement.

There is one functional-analytic step I would keep separate rather than overstate: to claim

$$\boxed{\mathscr{L}_{1,\mathrm{ph}} : X_\eta \longrightarrow Y_\eta \quad \text{is a bounded isomorphism}}$$

for the exact weighted spaces to be used, we still have to insert the endpoint asymptotics of (169) into those particular norms and prove the corresponding Green-kernel estimate. The explicit formulas make that a tractable estimate, but injectivity alone does not prove surjectivity in an unspecified weighted space.

51. Why this is enough to make the interval strategy much stronger

Previously we said that continuation from $\alpha = 1$ required an invertibility theorem and uniform nonlocal control.

We now have:

$$\boxed{\text{nonlocal perturbation: } \|\mathcal{M}_\alpha - \mathcal{M}_1\| \le C(\alpha - 1),}$$

from the preceding proof, and

$$\boxed{\text{phase-fixed kernel at } \alpha = 1 : \quad \ker \mathscr{L}_{1,\mathrm{ph}} = \{0\},}$$

from the exact calculation above.

The general-α logarithmic profile already has

$$G(-\infty) = 0, \qquad G(+\infty) = H_*,$$

with left exponential rate

$$q(\alpha) = 2\alpha - \frac{3}{2} > 0,$$

so the endpoint structure varies continuously for α near 1.

Thus the route to

$$\boxed{\exists\, \delta > 0 \quad \text{such that the positive profile exists for every} \quad 1 < \alpha < 1 + \delta}$$

is now very concrete: prove the weighted Green estimate from (169), formulate the phase-fixed nonlinear map on fixed weighted spaces, and apply the Banach-space implicit-function theorem using the $O(\alpha - 1)$ multiplier estimate already proved.

That is the next piece I would prove, because it would turn the explicit calculation above into the actual **open interval in** α that you wanted, rather than a single-α certification.

7 Part F

Let us proceed, but the Green-function calculation reveals an important structural point that improves the continuation argument: after removing translation, we must also account for the **adjoint solvability condition**. Injectivity of the phase-fixed operator by itself is not yet surjectivity.

52. Choose weights compatible with both endpoint rates

At $\alpha = 1$, we found

$$h_1(x) = Z_1'(x) = \frac{1}{2}\sqrt{t(x)}(1 - t(x)), \qquad t(x) = \frac{1}{1 + e^{-x}},$$

with

$$h_1(x) \sim \frac{1}{2}e^{x/2} \quad (x \to -\infty), \qquad h_1(x) \sim \frac{1}{2}e^{-x} \quad (x \to +\infty).$$

Choose

$$\boxed{0 < \eta < \frac{1}{2}.} \tag{172}$$

Let

$$\omega_\eta(x) \asymp \begin{cases} e^{-\eta x}, & x \le -1, \\ 1, & |x| \le 1, \\ e^{\eta x}, & x \ge 1. \end{cases}$$

Define

$$X_\eta = \left\{h : \omega_\eta h \in H^2(\mathbb{R})\right\}, \qquad Y_\eta = \left\{f : \omega_\eta f \in L^2(\mathbb{R})\right\}. \tag{173}$$

Then $h_1 \in X_\eta$ because

$$e^{-\eta x}h_1(x) = O(e^{(1/2-\eta)x}) \quad (x \to -\infty)$$

and

$$e^{\eta x}h_1(x) = O(e^{-(1-\eta)x}) \quad (x \to +\infty).$$

Thus translation is genuinely an element of the weighted space.

53. Recheck the two fundamental solutions

Ignoring constant normalizations,

$$\boxed{h_1 = \sqrt{t}(1 - t)} \tag{174}$$

and reduction of order gives

$$\boxed{h_2 = \sqrt{t}(1 - t)^{-3/2}.} \tag{175}$$

The Wronskian satisfies

$$\boxed{W(t) = C_W t^2 (1 - t)^{-1/2}.} \tag{176}$$

Consequently

$$h_2(x) \sim e^{x/2} \quad (x \to -\infty),$$

whereas

$$h_2(x) \sim e^{3x/2} \quad (x \to +\infty).$$

A more useful independent left solution is obtained by subtracting the common $e^{x/2}$ component:

$$\boxed{h_L = h_2 - Ch_1,}$$

with C chosen so that

$$\boxed{h_L(x) \asymp e^{3x/2}, \qquad x \to -\infty.} \tag{177}$$

Thus the indicial roots are exactly

	first rate	second rate
$x \to -\infty$	$1/2$	$3/2$
$x \to +\infty$	-1	$3/2$.

(178)

The spectral gap around the translation mode is therefore explicit.

54. A correction to the naive inverse argument

Here is the point we must not skip.

Because

$$h_1 = Z_1'$$

decays at **both** ends, the whole-line operator has a nontrivial kernel. For a Fredholm operator of index zero, that normally comes with a one-dimensional cokernel.

Therefore imposing merely

$$h(0) = 0$$

removes the kernel but does **not automatically make**

$$\mathscr{L}_1 : X_\eta \to Y_\eta$$

onto.

This means that the previous Green formula cannot be interpreted as a solution for completely arbitrary $f \in Y_\eta$ without checking a compatibility condition.

We can determine that compatibility condition explicitly.

55. Compute the adjoint zero mode

Divide by $-\nu$ and write the homogeneous equation as

$$Ah = h'' + P(x)h' + Q(x)h = 0,$$

where

$$P(x) = -2 + \frac{3}{2}t(x). \tag{179}$$

The formal adjoint is

$$A^*\psi = \psi'' - (P\psi)' + Q\psi. \tag{180}$$

A standard identity says that if $Ah = 0$, then

$$\psi = e^{\int P dx} h$$

solves $A^*\psi = 0$.
We already calculated

$$e^{\int P dx} = t^{-2}(1-t)^{1/2}.$$

Taking $h = h_1 = \sqrt{t}(1-t)$, we obtain

$$\boxed{\psi_1(t) = t^{-3/2}(1-t)^{3/2}} \tag{181}$$

up to an irrelevant positive constant.
Since

$$\frac{1-t}{t} = e^{-x},$$

this simplifies dramatically:

$$\boxed{\psi_1(x) = e^{-3x/2}.} \tag{182}$$

So the adjoint neutral mode is completely explicit.
This is a very useful discovery.

56. The exact solvability condition

If

$$\mathscr{L}_1 h = f$$

and the boundary terms vanish in the chosen weighted pairing, then

$$\langle \psi_1, \mathscr{L}_1 h \rangle = \langle \mathscr{L}_1^* \psi_1, h \rangle = 0.$$

Therefore necessarily

$$\boxed{\int_{\mathbb{R}} e^{-3x/2} f(x)\, dx = 0,} \tag{183}$$

whenever this integral is the appropriate convergent pairing.

Thus

$$\boxed{\operatorname{Ran} \mathscr{L}_1 \subset \{f : \langle \psi_1, f \rangle = 0\}\,.} \tag{184}$$

Under the Fredholm realization with the corresponding endpoint conditions, this is the expected codimension-one range.

So the continuation problem has:

$$\boxed{\dim \ker \mathscr{L}_1 = 1, \qquad \dim \ker \mathscr{L}_1^* = 1.} \tag{185}$$

This is the usual translation-degenerate heteroclinic situation.

57. Why this does not obstruct continuation

It tells us that we need one additional scalar degree of freedom.

Fortunately the profile problem already has natural scalar parameters.

For example, instead of freezing the right amplitude H_* from the beginning, write

$$\boxed{G(x) = H\, Z(x),} \tag{186}$$

and treat H as an unknown.

Alternatively one may retain a coefficient such as c, γ, or another normalization parameter generated by the similarity reduction.

The correct implicit-function formulation is therefore not simply

$$\mathscr{F}(\alpha, Z) = 0.$$

It should be

$$\boxed{\mathscr{F}(\alpha, Z, H) = 0, \qquad \Phi(Z) = 0,} \tag{187}$$

where Φ fixes translation.

The derivative at the anchor has block form

$$\boxed{D_{(Z,H)} \begin{pmatrix} \mathscr{F} \\ \Phi \end{pmatrix} = \begin{pmatrix} \mathscr{L}_1 & q \\ \Phi' & 0 \end{pmatrix},} \tag{188}$$

where

$$q = \partial_H \mathscr{F}(1, Z_1, H_*).$$

Now there are two scalar degeneracies to handle:

- Φ' removes the translation kernel;
- q must span the missing cokernel direction.

The required transversality condition is simply

$$\boxed{\langle \psi_1, q \rangle \neq 0.} \tag{189}$$

If (189) holds, the bordered operator (188) is invertible.

This is the standard Lyapunov–Schmidt mechanism, but here every ingredient at $\alpha = 1$ is explicit.

58. Abstract bordered-inverse proof

Suppose

$$\mathscr{L}_1 h + aq = f, \tag{190}$$

together with

$$\Phi(h) = r. \tag{191}$$

Pair (190) with ψ_1. Since

$$\langle \psi_1, \mathscr{L}_1 h \rangle = 0,$$

we get

$$a\langle \psi_1, q \rangle = \langle \psi_1, f \rangle.$$

Hence, provided (189),

$$\boxed{a = \frac{\langle \psi_1, f \rangle}{\langle \psi_1, q \rangle}.} \tag{192}$$

Define

$$f_0 = f - aq.$$

Then

$$\langle \psi_1, f_0 \rangle = 0.$$

Thus f_0 belongs to the range of $\mathscr{L}_1$, and we solve

$$\mathscr{L}_1 h_0 = f_0.$$

The general solution is

$$h = h_0 + bh_1.$$

Finally choose b from

$$\Phi(h) = r :$$

$$b = \frac{r - \Phi(h_0)}{\Phi(h_1)}. \tag{193}$$

Since we choose the phase functional so that

$$\boxed{\Phi(h_1) \neq 0,} \tag{194}$$

b is unique.
Therefore the bordered operator is bijective.

59. Quantitative inverse estimate

Let

$$d_{\text{cok}} := |\langle \psi_1, q \rangle| > 0$$

and

$$d_{\text{ph}} := |\Phi(h_1)| > 0.$$

Then

$$|a| \leq d_{\text{cok}}^{-1} \|\psi_1\|_{Y_\eta^*} \|f\|_{Y_\eta}. \tag{195}$$

Consequently

$$\|f_0\|_{Y_\eta} \leq \left(1 + \frac{\|\psi_1\|_{Y_\eta^*} \|q\|_{Y_\eta}}{d_{\text{cok}}}\right) \|f\|_{Y_\eta}. \tag{196}$$

If

$$\mathscr{L}_1^{-1} : Y_{\eta,0} \to X_\eta / \operatorname{span}\{h_1\}$$

has Green bound C_G, where

$$Y_{\eta,0} = \{f : \langle \psi_1, f \rangle = 0\},$$

then

$$\|h_0\|_{X_\eta} \leq C_G \|f_0\|_{Y_\eta}. \tag{197}$$

And

$$|b| \leq d_{\text{ph}}^{-1} \left(|r| + \|\Phi\| \, \|h_0\|_{X_\eta}\right). \tag{198}$$

Thus

$$\boxed{\|h\|_{X_\eta} + |a| \le C_{\mathrm{bord}} \left(\|f\|_{Y_\eta} + |r| \right).} \tag{199}$$

This is the precise inverse estimate needed by the implicit-function theorem.

60. Green estimate on the range subspace

For

$$f \in Y_{\eta,0},$$

the growing coefficient cancels by (183). Variation of parameters then gives a solution whose endpoint behavior is governed by the decaying fundamental modes.

Schematically,

$$h(x) = h_L(x) \int_{-\infty}^{x} K_L(y) f(y)\, dy + h_R(x) \int_{x}^{\infty} K_R(y) f(y)\, dy, \tag{200}$$

after the solvability cancellation, where the kernels are explicit combinations of

$$h_1, \quad h_2, \quad W^{-1}.$$

The endpoint exponents give, for $0 < \eta < 1/2$,

$$|\omega_\eta(x) h(x)| \le C_\eta \int_{\mathbb{R}} e^{-c_\eta |x-y|} |\omega_\eta(y) f(y)|\, dy, \tag{201}$$

where one may take $c_\eta > 0$ depending on the distances

$$\frac{1}{2} - \eta, \qquad 1 - \eta.$$

Young's inequality therefore gives

$$\boxed{\|\omega_\eta h\|_{L^2} \le C_\eta \|\omega_\eta f\|_{L^2}.} \tag{202}$$

Using

$$\mathscr{L}_1 h = f$$

to solve for h'', together with the bounded coefficients in (143), yields

$$\boxed{\|\omega_\eta h\|_{H^2} \le C_\eta \|\omega_\eta f\|_{L^2}.} \tag{203}$$

Thus, on the solvability subspace,

$$\boxed{\|h\|_{X_\eta} \le C_G(\eta, \nu) \|f\|_{Y_\eta}.} \tag{204}$$

This completes the functional mechanism modulo the precise choice of weighted dual pairing at the left endpoint. For the final proof, I would formulate the weights using separate exponential dichotomy spaces at the two ends so that (183) is interpreted rigorously; the raw factor $e^{-3x/2}$ itself is not in the ordinary unweighted L^2 dual.

61. Now perturb to $1 < \alpha < 1 + \delta$

Let the full bordered derivative be

$$\mathbb{L}_\alpha = \begin{pmatrix} \mathscr{L}_\alpha & q_\alpha \\ \Phi'_\alpha & 0 \end{pmatrix}.$$

At $\alpha = 1$,

$$\mathbb{L}_1^{-1}$$

exists by the bordered construction above, provided the transversality condition (189) is verified for the particular scalar parameter retained in the profile equation.

From our previous Mellin calculation,

$$\|\mathcal{M}_\alpha - \mathcal{M}_1\| \le C_M(\alpha - 1).$$

Smooth dependence of the remaining coefficients gives

$$\boxed{\|\mathbb{L}_\alpha - \mathbb{L}_1\| \le C_{\mathrm{pert}}(\alpha - 1).} \tag{205}$$

Set

$$C_0 := \|\mathbb{L}_1^{-1}\|.$$

If

$$\boxed{C_0 C_{\mathrm{pert}} \delta < 1,} \tag{206}$$

then

$$\mathbb{L}_\alpha = \mathbb{L}_1 \left[I + \mathbb{L}_1^{-1}(\mathbb{L}_\alpha - \mathbb{L}_1) \right]$$

is invertible for every

$$1 \le \alpha \le 1 + \delta.$$

Moreover,

$$\boxed{\|\mathbb{L}_\alpha^{-1}\| \le \frac{C_0}{1 - C_0 C_{\mathrm{pert}} \delta}.} \tag{207}$$

Choosing

$$C_0 C_{\mathrm{pert}} \delta \le \frac{1}{2}$$

gives

$$\boxed{\sup_{1 \le \alpha \le 1+\delta} \|\mathbb{L}_\alpha^{-1}\| \le 2C_0.} \tag{208}$$

This is precisely the uniform inverse estimate we wanted.

62. The implicit-function theorem now has the correct form

Define

$$\mathbb{F}(\alpha, Z, H) = \begin{pmatrix} \mathscr{F}(\alpha, Z, H) \\ \Phi(Z) \end{pmatrix}.$$

At the anchor,

$$\mathbb{F}(1, Z_1, H_*) = 0.$$

If the transversality condition

$$\boxed{\langle \psi_1, \partial_H \mathscr{F}(1, Z_1, H_*) \rangle \neq 0} \tag{209}$$

holds, then

$$D_{(Z,H)} \mathbb{F}(1, Z_1, H_*)$$

is an isomorphism.
The Banach implicit-function theorem then gives

$$\boxed{\exists\, \delta_{\text{prof}} > 0}$$

and a unique phase-normalized branch

$$\boxed{\alpha \longmapsto (Z_\alpha, H_\alpha)}$$

for

$$\boxed{1 < \alpha < 1 + \delta_{\text{prof}},} \tag{210}$$

with

$$\mathbb{F}(\alpha, Z_\alpha, H_\alpha) = 0$$

and

$$\boxed{\|Z_\alpha - Z_1\|_{X_\eta} + |H_\alpha - H_*| \leq C(\alpha - 1).} \tag{211}$$

The left and right endpoint coefficients then vary continuously, so after reducing δ_{prof},

$$\boxed{Z_\alpha(x) > 0} \tag{212}$$

throughout the whole line.

The decisive remaining calculation

We have narrowed the profile continuation problem further than before. The multiplier dependence, explicit anchor, kernel, adjoint mode, Green construction, and bordered mechanism are now available.

But there is one quantity we must **actually compute rather than assume**:

$$\boxed{\left\langle e^{-3x/2}, \partial_H \mathscr{F}(1, Z_1, H_*) \right\rangle .} \tag{213}$$

If it is nonzero for the correct amplitude or, if amplitude is not an independent parameter in the normalized profile problem, for the correct retained scalar parameter—then the bordered derivative is genuinely invertible and the local interval profile theorem follows by IFT.

So the next step is to go back to the exact autonomous G-equation, retain its amplitude/coefficient parameter **before normalization**, compute $q = \partial_H \mathscr{F}$ explicitly at

$$Z_1 = (1 + e^{-x})^{-1/2},$$

and evaluate (213). That calculation will tell us whether H is the right unfolding parameter or whether we should instead retain c (or another similarity coefficient).

8 Part G

The endpoint index count shows that the adjoint mode is not in the dual of the weighted range space. Thus the previous cokernel detour is not needed: for the small two-sided weights considered here, the unphased linearized operator has Fredholm index +1, with the one-dimensional kernel generated by translation, and imposing a single phase condition produces the correct index-zero operator. In particular, no additional amplitude parameter is required merely to restore surjectivity.

Let us make this precise, first at the explicit anchor

$$\alpha = 1.$$

For $\gamma = 0$, the profile equation is

$$-\nu F'' = F^2 F' = \frac{1}{3}(F^3)'.$$

After one integration, using the decaying condition at $+\infty$,

$$-\nu F' = \frac{1}{3}F^3.$$

Hence

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s + s_0)}}}$$

for $s_0 > 0$.

Write

$$c_\nu = \sqrt{\frac{3\nu}{2}},$$

so that

$$F_1(s) = c_\nu (s + s_0)^{-1/2}.$$

The linearization of

$$\mathcal{N}_1(F) = -\nu F'' - F^2 F'$$

at F_1 is

$$L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.$$

Since

$$F_1^2 = \frac{3\nu}{2(s + s_0)}$$

and

$$2F_1 F_1' = -\frac{3\nu}{2(s + s_0)^2},$$

we obtain

$$L_1 h = -\nu h'' - \frac{3\nu}{2(s + s_0)} h' + \frac{3\nu}{2(s + s_0)^2} h.$$

This factorizes exactly as

$$\boxed{L_1 h = -\nu \left[h' + \frac{3}{2(s + s_0)} h \right]'.}$$

Indeed,

$$\left[h' + \frac{3}{2(s + s_0)} h \right]' = h'' + \frac{3}{2(s + s_0)} h' - \frac{3}{2(s + s_0)^2} h.$$

Therefore the homogeneous equation

$$L_1 h = 0$$

reduces to

$$h' + \frac{3}{2(s + s_0)} h = C_1.$$

Multiplying by the integrating factor

$$(s + s_0)^{3/2},$$

we obtain

$$\frac{d}{ds} \left[(s + s_0)^{3/2} h(s) \right] = C_1 (s + s_0)^{3/2}.$$

Consequently,

$$(s + s_0)^{3/2} h(s) = \frac{2C_1}{5} (s + s_0)^{5/2} + C_2,$$

and hence

$$\boxed{h(s) = \frac{2C_1}{5}(s+s_0) + C_2(s+s_0)^{-3/2}.}$$

The first mode grows linearly as $s \to \infty$, while the second mode decays:

$$(s+s_0)^{-3/2} \longrightarrow 0.$$

Moreover,

$$F_1'(s) = -\frac{c_\nu}{2}(s+s_0)^{-3/2},$$

so the decaying homogeneous mode is precisely the translation mode:

$$\boxed{\ker L_1 = \operatorname{span}\{F_1'\}}$$

in a decaying function space.

Thus the only kernel degeneracy is the expected translation invariance.

We now impose one phase condition. For example, after fixing s_0, we may require

$$\boxed{h(0) = 0.}$$

Since

$$F_1'(0) \neq 0,$$

the translation mode does not satisfy this phase condition. Therefore

$$\boxed{\ker L_1\big|_{\{h(0)=0\}} = \{0\}.}$$

We can do considerably better: the inverse can be written explicitly.

Suppose

$$L_1 h = f.$$

Define

$$g(s) = h'(s) + \frac{3}{2(s+s_0)}h(s).$$

Then

$$-\nu g'(s) = f(s),$$

or

$$g'(s) = -\frac{1}{\nu}f(s).$$

Imposing

$$g(\infty) = 0$$

gives

$$\boxed{g(s) = \frac{1}{\nu}\int_s^\infty f(\tau)\,d\tau.}$$

Hence

$$h' + \frac{3}{2(s+s_0)}h = \frac{1}{\nu}\int_s^\infty f(\tau)\,d\tau.$$

Multiplication by $(s+s_0)^{3/2}$ yields

$$\frac{d}{ds}\left[(s+s_0)^{3/2}h(s)\right] = \frac{(s+s_0)^{3/2}}{\nu}\int_s^\infty f(\tau)\,d\tau.$$

Using the phase condition

$$h(0) = 0,$$

we therefore obtain

$$\boxed{(L_1^{-1}f)(s) = \frac{1}{\nu(s+s_0)^{3/2}}\int_0^s (\sigma+s_0)^{3/2}\left(\int_\sigma^\infty f(\tau)\,d\tau\right)d\sigma.}$$

This formula directly proves surjectivity in suitable weighted spaces and shows why an additional cokernel condition is not required.

For example, fix

$$0 < \eta < \frac{3}{2}$$

and define

$$\boxed{Y_\eta = \left\{ f : \sup_{s\geq 0}(s+s_0)^{2+\eta}|f(s)| < \infty \right\}.}$$

Let

$$M = \|f\|_{Y_\eta}.$$

Then

$$|f(\tau)| \leq M(\tau+s_0)^{-2-\eta}.$$

Therefore

$$\left|\int_s^\infty f(\tau)\,d\tau\right| \leq M\int_s^\infty (\tau+s_0)^{-2-\eta}\,d\tau,$$

and hence

$$\boxed{\left|\int_s^\infty f(\tau)\,d\tau\right| \leq \frac{M}{1+\eta}(s+s_0)^{-1-\eta}.}$$

Substituting this estimate into the inverse formula gives

$$|h(s)| \leq \frac{M}{\nu(1+\eta)}(s+s_0)^{-3/2}\int_0^s (\sigma+s_0)^{1/2-\eta}\,d\sigma.$$

For

$$0 < \eta < \frac{3}{2},$$

we have

$$\int_0^s (\sigma+s_0)^{1/2-\eta}\,d\sigma = \frac{(s+s_0)^{3/2-\eta} - s_0^{3/2-\eta}}{3/2-\eta}.$$

Consequently,

$$\boxed{|h(s)| \leq \frac{M}{\nu(1+\eta)(3/2-\eta)}(s+s_0)^{-\eta}.}$$

Since

$$h' = g - \frac{3}{2(s+s_0)}h,$$

we also obtain

$$|h'(s)| \le \frac{M}{\nu(1+\eta)}(s+s_0)^{-1-\eta} + \frac{3}{2(s+s_0)}|h(s)|.$$

Thus

$$\boxed{|h'(s)| \le C_{\nu,\eta} M (s+s_0)^{-1-\eta}.}$$

Differentiating

$$g = h' + \frac{3}{2(s+s_0)}h$$

gives

$$-\frac{1}{\nu}f = h'' + \frac{3}{2(s+s_0)}h' - \frac{3}{2(s+s_0)^2}h.$$

Hence

$$h'' = -\frac{1}{\nu}f - \frac{3}{2(s+s_0)}h' + \frac{3}{2(s+s_0)^2}h.$$

It follows that

$$\boxed{|h''(s)| \le C_{\nu,\eta} M (s+s_0)^{-2-\eta}.}$$

Define

$$\boxed{X_\eta = \left\{ h : h(0) = 0, \quad \sup_{s\ge 0}(s+s_0)^{\eta}|h(s)| + \sup_{s\ge 0}(s+s_0)^{1+\eta}|h'(s)| + \sup_{s\ge 0}(s+s_0)^{2+\eta}|h''(s)| < \infty \right\}.}$$

Then the preceding estimates prove

$$\boxed{\|L_1^{-1} f\|_{X_\eta} \le C_{\nu,\eta}\|f\|_{Y_\eta}.}$$

Therefore

$$\boxed{L_1 : X_\eta \longrightarrow Y_\eta}$$

is a bounded isomorphism after the translation phase has been fixed.

This agrees with the endpoint Fredholm count in logarithmic variables.

Indeed, introduce

$$x = \log s$$

and

$$F(s) = s^{-1/2} G(x).$$

Then

$$F'' = s^{-5/2}\left(G'' - 2G' + \frac{3}{4}G\right),$$

so

$$-\partial_{ss} F = s^{-5/2}\left(-G'' + 2G' - \frac{3}{4}G\right).$$

Thus

$$\boxed{\mathcal{M}_{1/2} = -D^2 + 2D - \frac{3}{4}.}$$

Its Fourier symbol is

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

This is exactly what follows from the Mellin Gamma symbol

$$K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi} \sin\pi(\alpha-\lambda).$$

Setting

$$a_\alpha = \alpha - \frac{1}{2}$$

and

$$z_\alpha(\xi) = -a_\alpha + i\xi = -\alpha + \frac{1}{2} + i\xi,$$

we have

$$m_\alpha(\xi) = K_\alpha^+\left(-\alpha + \frac{1}{2} + i\xi\right).$$

Therefore

$$\boxed{m_\alpha(\xi) = \frac{\Gamma\left(\frac{3}{2} - \alpha + i\xi\right)\Gamma\left(3\alpha - \frac{1}{2} - i\xi\right)}{\pi} \sin\pi\left(2\alpha - \frac{1}{2} - i\xi\right).}$$

At $\alpha = 1$,

$$m_1(\xi) = \frac{\Gamma(\frac{1}{2} + i\xi)\Gamma(\frac{5}{2} - i\xi)}{\pi} \sin\pi\left(\frac{3}{2} - i\xi\right).$$

Using the Gamma recurrence and reflection identities gives

$$m_1(\xi) = -\left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right),$$

and therefore

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

Differentiating the exact symbol with respect to α gives

$$\boxed{\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi)\left[-\psi\left(\frac{3}{2} - \alpha + i\xi\right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi\right) + 2\pi\cot\pi\left(2\alpha - \frac{1}{2} - i\xi\right)\right].}$$

Fix

$$0 < \delta < \frac{1}{4}.$$

For

$$1 \le \alpha \le 1 + \delta,$$

the Gamma arguments remain in pole-free right half-planes. Standard Stirling estimates imply

$$\boxed{|m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha}.}$$

The corresponding digamma estimates imply

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha} \left(1 + \log\langle \xi \rangle\right).}$$

For every

$$0 < \sigma < 1,$$

after choosing

$$0 < \delta \le \frac{\sigma}{4},$$

the logarithmic factor can be absorbed into the additional Sobolev power, giving

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_{\delta,\sigma} \langle \xi \rangle^{2+\sigma}.}$$

The mean-value theorem then yields

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \le C_{\delta,\sigma} |\alpha - 1| \langle \xi \rangle^{2+\sigma}.}$$

By Plancherel,

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \le C_{\delta,\sigma} |\alpha - 1| \|h\|_{H^{2+\sigma}}.}$$

Thus

$$\boxed{\alpha \longmapsto \mathcal{M}_\alpha \in \mathcal{L}\left(H^{2+\sigma}, L^2\right)}$$

is C^1 near $\alpha = 1$.

We now check the Fredholm count directly in logarithmic variables.

Normalize

$$H_*(1) = \sqrt{\frac{3\nu}{2}}$$

and write

$$Z_1(x) = \frac{G_1(x)}{H_*(1)} = (1 + e^{-x})^{-1/2}.$$

Set

$$t(x) = Z_1(x)^2 = \frac{1}{1 + e^{-x}}.$$

Then

$$t' = t(1 - t)$$

and

$$Z_1' = \frac{1}{2} Z_1 (1 - Z_1^2) = \frac{1}{2} \sqrt{t} (1 - t).$$

The linearization in G-variables is

$$\boxed{\frac{1}{\nu} \mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2} t\right) h' + \left(-\frac{3}{4} + \frac{3}{4} t + \frac{3}{2} t^2\right) h.}$$

As

$$x \to -\infty, \qquad t \to 0,$$

the limiting homogeneous equation is

$$-h'' + 2h' - \frac{3}{4}h = 0,$$

or

$$h'' - 2h' + \frac{3}{4}h = 0.$$

Its characteristic roots are

$$\boxed{\lambda_-^{(1)} = \frac{1}{2}, \qquad \lambda_-^{(2)} = \frac{3}{2}.}$$

Both modes decay as

$$x \to -\infty.$$

At the other endpoint,

$$x \to +\infty, \qquad t \to 1.$$

The limiting equation becomes

$$-h'' + \frac{1}{2}h' + \frac{3}{2}h = 0,$$

or

$$h'' - \frac{1}{2}h' - \frac{3}{2}h = 0.$$

Its characteristic roots are

$$\boxed{\lambda_+^{(1)} = \frac{3}{2}, \qquad \lambda_+^{(2)} = -1.}$$

Exactly one mode decays as

$$x \to +\infty.$$

Consequently, for a sufficiently small two-sided exponential weight

$$0 < \eta < \frac{1}{2},$$

the dimension of the admissible subspace at the left endpoint is 2, while the dimension of the admissible subspace at the right endpoint is 1.

For the associated first-order 2×2 system, the Fredholm index is therefore

$$\boxed{\operatorname{ind} \mathscr{L}_1 = 2 + 1 - 2 = 1.}$$

This is exactly consistent with the translation kernel

$$\ker \mathscr{L}_1 = \operatorname{span}\{G_1'\}.$$

Hence

$$\boxed{\dim \ker \mathscr{L}_1 = 1, \qquad \operatorname{codim} \operatorname{Ran} \mathscr{L}_1 = 0.}$$

In particular, the formal adjoint solution previously obtained does not generate a cokernel element in these weighted spaces. The reason is that its growth at the left endpoint excludes it from the relevant dual weighted space.

After imposing one phase condition transverse to translation, the kernel dimension decreases by one. Hence the phase-fixed operator has

$$\boxed{\operatorname{ind} \mathscr{L}_1^{\mathrm{ph}} = 0.}$$

Since its kernel is trivial,

$$\boxed{\ker \mathscr{L}_1^{\mathrm{ph}} = \{0\},}$$

and the range already has codimension zero, we obtain

$$\boxed{\mathscr{L}_1^{\mathrm{ph}} : X_\eta^{\mathrm{ph}} \longrightarrow Y_\eta \quad \text{is an isomorphism.}}$$

Thus the correct continuation picture at the anchor is

$$\boxed{\text{one translation kernel} \quad + \quad \text{zero cokernel} \quad \xrightarrow{\text{one phase condition}} \quad \text{bounded isomorphism.}}$$

There is therefore no need, at this stage, to introduce an artificial amplitude variable solely to compensate for a supposed one-dimensional cokernel.

The next issue is the genuinely nonlocal one.

For

$$1 < \alpha < 1 + \delta,$$

set

$$a_\alpha = \alpha - \frac{1}{2},$$

and

$$F_\alpha(s) = s^{-a_\alpha} G_\alpha(\log s).$$

The autonomous equation is

$$\boxed{\nu \mathcal{M}_{a_\alpha} G_\alpha = G_\alpha^2 G_\alpha' - c_\alpha G_\alpha^3,}$$

where, for $\gamma = 0$,

$$\boxed{c_\alpha = \frac{2\alpha - 1}{2}.}$$

The equilibrium amplitude is not a free parameter. At

$$G_\alpha = H_*(\alpha),$$

we have

$$\nu K_\alpha^+(-a_\alpha) H_*(\alpha) = -c_\alpha H_*(\alpha)^3.$$

Hence

$$\boxed{H_*(\alpha)^2 = -\frac{\nu K_\alpha^+(-a_\alpha)}{c_\alpha}.}$$

At $\alpha = 1$,

$$K_1^+\left(-\frac{1}{2}\right) = -\frac{3}{4}$$

and

$$c_1 = \frac{1}{2},$$

so

$$H_*(1)^2 = \frac{3}{2}\nu,$$

as required.

The desired heteroclinic satisfies

$$\boxed{G_\alpha(-\infty) = 0, \qquad G_\alpha(+\infty) = H_*(\alpha).}$$

The left asymptotic exponent is determined by

$$K_\alpha^+(-a_\alpha + q_\alpha) = 0.$$

Since

$$-a_\alpha + q_\alpha = \alpha - 1,$$

we obtain

$$\boxed{q_\alpha = 2\alpha - \frac{3}{2}.}$$

Thus

$$G_\alpha(x) \sim A_\alpha e^{q_\alpha x} \qquad (x \to -\infty).$$

At the anchor,

$$q_1 = \frac{1}{2},$$

which agrees with

$$Z_1(x) = (1 + e^{-x})^{-1/2} = e^{x/2} - \frac{1}{2}e^{3x/2} + \frac{3}{8}e^{5x/2} + O(e^{7x/2}).$$

At the right endpoint,

$$Z_1(x) = 1 - \frac{1}{2}e^{-x} + \frac{3}{8}e^{-2x} + O(e^{-3x}).$$

Therefore the explicit $\alpha = 1$ profile has precisely the two asymptotic structures required for a local continuation argument.

The remaining analytical point before invoking the implicit-function theorem is to verify that the particular one-sided realization of

$$(-\partial_{ss})^\alpha$$

used in the profile equation has an operator domain that varies continuously through

$$\alpha = 1.$$

This point cannot be skipped.

The Mellin identity is written using

$$s_+^\lambda,$$

namely

$$\boxed{(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda-2\alpha}.}$$

For

$$\alpha > 1,$$

the desired profile satisfies

$$F_\alpha(s) \sim A_\alpha s^{\alpha-1} \qquad (s \downarrow 0),$$

so

$$F_\alpha(0) = 0.$$

At the anchor,

$$\alpha = 1,$$

however,

$$F_1(s) = c_\nu (s + s_0)^{-1/2}$$

has

$$F_1(0) = c_\nu s_0^{-1/2} > 0.$$

Thus, if the notation s_+^λ corresponds to literal zero extension to the negative half-line, the trace of the extended profile changes qualitatively at $\alpha = 1$. In that realization, the local differential equation for $s > 0$ does not by itself establish continuity of the fractional operator domain at the anchor.

Consequently, the correct continuation theorem has the following logical structure:

explicit profile at $\alpha = 1$

$\Downarrow$

phase-fixed linearization is invertible

$\Downarrow$

Mellin multiplier is C^1 in α

$\Downarrow$

verify compatibility of the one-sided realization/domain at $\alpha = 1$

$\Downarrow$

apply the implicit-function theorem

$\Downarrow$

$\exists\, \delta > 0 : \quad 1 < \alpha < 1 + \delta$ admits a nearby profile.

Thus the phase-fixed linearization itself is not the obstruction. At the explicit $\alpha = 1$ anchor it has the correct invertibility structure. The next decisive step is to identify precisely the extension convention and operator domain underlying the manuscript's one-sided fractional operator and prove that this realization is compatible with continuation through $\alpha = 1$.

9 Part H

Continuation of the One-Phase Profile from $\alpha = 1$

We now continue the one-phase profile analysis from the explicit $\alpha = 1$ solution. The objective is to replace the isolated value $\alpha = 1$ by a genuine interval

$$\boxed{1 < \alpha < 1 + \delta}$$

for some $\delta > 0$.
For the first continuation theorem we take

$$\boxed{\gamma = 0.}$$

The profile equation is therefore

$$\boxed{\nu(-\partial_{ss})^{\alpha} F_{\alpha} = F_{\alpha}^{2} F_{\alpha}', \qquad 1 < \alpha < 1 + \delta.}$$

Introduce

$$\boxed{a_{\alpha} = \alpha - \frac{1}{2},}$$

and make the logarithmic change of variables

$$\boxed{x = \log s, \qquad G_{\alpha}(x) = s^{a_{\alpha}} F_{\alpha}(s).}$$

Equivalently,

$$F_{\alpha}(s) = s^{-a_{\alpha}} G_{\alpha}(\log s).$$

The one-sided Mellin realization used is

$$(-\partial_{ss})^{\alpha} s_{+}^{\lambda} = K_{\alpha}^{+}(\lambda) s^{\lambda - 2\alpha},$$

where

$$\boxed{K_{\alpha}^{+}(\lambda) = \frac{\Gamma(\lambda + 1)\Gamma(2\alpha - \lambda)}{\pi} \sin \pi(\alpha - \lambda).}$$

Consequently,

$$(-\partial_{ss})^{\alpha} F = s^{-a_{\alpha} - 2\alpha} \mathcal{M}_{a_{\alpha}} G,$$

where

$$\boxed{\mathcal{M}_{a_{\alpha}}(e^{\lambda x}) = K_{\alpha}^{+}(-a_{\alpha} + \lambda) e^{\lambda x}.}$$

The autonomous profile equation becomes

$$\boxed{\nu \mathcal{M}_{a_\alpha} G = G^2 G' - c_\alpha G^3,}$$

with

$$\boxed{c_\alpha = \frac{2\alpha - 1}{2}.}$$

The desired heteroclinic satisfies

$$\boxed{G(-\infty) = 0, \qquad G(+\infty) = H_*(\alpha).}$$

The right-hand equilibrium is not a free amplitude. Substitution of $G = H_*$ into the autonomous equation gives

$$\nu K_\alpha^+(-a_\alpha) H_* = -c_\alpha H_*^3,$$

and hence

$$\boxed{H_*^2(\alpha) = -\frac{\nu K_\alpha^+(-a_\alpha)}{c_\alpha}.}$$

At $\alpha = 1$,

$$a_1 = \frac{1}{2}, \qquad c_1 = \frac{1}{2},$$

and

$$K_1^+\left(-\frac{1}{2}\right) = -\frac{3}{4}.$$

Therefore

$$\boxed{H_*^2(1) = \frac{3}{2}\nu,}$$

so that

$$\boxed{H_*(1) = \sqrt{\frac{3\nu}{2}}.}$$

The Explicit Anchor at $\alpha = 1$

When

$$\alpha = 1, \qquad \gamma = 0,$$

the profile equation reduces to

$$-\nu F'' = F^2 F' = \frac{1}{3}(F^3)'.$$

Integrating and choosing the decaying branch gives

$$-\nu F' = \frac{1}{3}F^3.$$

Hence

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}.}$$

Using the scaling symmetry, we fix the translation/scale phase by choosing

$$\boxed{s_0 = 1.}$$

Thus

$$F_1(s) = \sqrt{\frac{3\nu}{2}}\,(1+s)^{-1/2}.$$

Since

$$G_1(x) = s^{1/2}F_1(s), \qquad s = e^x,$$

we obtain

$$G_1(x) = \sqrt{\frac{3\nu}{2}}\left(\frac{e^x}{1+e^x}\right)^{1/2}.$$

Define

$$\boxed{H_1 = \sqrt{\frac{3\nu}{2}}}$$

and normalize by

$$Z_1(x) = \frac{G_1(x)}{H_1}.$$

Then

$$\boxed{Z_1(x) = (1+e^{-x})^{-1/2}.}$$

Introduce

$$\boxed{t(x) = Z_1^2(x) = \frac{1}{1+e^{-x}}.}$$

Then

$$t' = t(1-t)$$

and

$$\boxed{Z_1' = \frac{1}{2} Z_1(1 - Z_1^2) = \frac{1}{2}\sqrt{t}(1-t).}$$

Equivalently,

$$\boxed{Z_1'(x) = \frac{1}{2} \frac{e^{x/2}}{(1+e^x)^{3/2}}.}$$

The endpoint expansions are

$$\boxed{Z_1(x) = e^{x/2} - \frac{1}{2}e^{3x/2} + \frac{3}{8}e^{5x/2} + O(e^{7x/2}), \qquad x \to -\infty,}$$

and

$$\boxed{Z_1(x) = 1 - \frac{1}{2}e^{-x} + \frac{3}{8}e^{-2x} + O(e^{-3x}), \qquad x \to +\infty.}$$

Thus the left decay exponent at the anchor is

$$\boxed{q_-(1) = \frac{1}{2},}$$

whereas the right decay exponent is

$$\boxed{r_+(1) = 1.}$$

For general α, the Mellin balance gives

$$-a_\alpha + \lambda = \alpha - 1,$$

and therefore

$$\boxed{q_-(\alpha) = 2\alpha - \frac{3}{2}.}$$

In particular,

$$q_-(\alpha) \longrightarrow \frac{1}{2} \qquad \text{as } \alpha \downarrow 1.$$

Exact Mellin Symbol Near $\alpha = 1$

Set

$$\boxed{z_\alpha(\xi) = -a_\alpha + i\xi = -\alpha + \frac{1}{2} + i\xi.}$$

The Fourier–Mellin symbol is

$$m_\alpha(\xi) = K_\alpha^+\left(-\alpha + \frac{1}{2} + i\xi\right).$$

Using the exact Gamma-function representation,

$$\boxed{m_\alpha(\xi) = \frac{\Gamma\left(\frac{3}{2} - \alpha + i\xi\right)\Gamma\left(3\alpha - \frac{1}{2} - i\xi\right)}{\pi} \sin\pi\left(2\alpha - \frac{1}{2} - i\xi\right).}$$

At $\alpha = 1$,

$$m_1(\xi) = \frac{\Gamma\left(\frac{1}{2} + i\xi\right)\Gamma\left(\frac{5}{2} - i\xi\right)}{\pi} \sin\pi\left(\frac{3}{2} - i\xi\right).$$

Using the Gamma recurrence relation,

$$\Gamma\left(\frac{5}{2} - i\xi\right) = \left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right)\Gamma\left(\frac{1}{2} - i\xi\right),$$

together with

$$\Gamma\left(\frac{1}{2} + i\xi\right)\Gamma\left(\frac{1}{2} - i\xi\right) = \frac{\pi}{\cosh\pi\xi},$$

and

$$\sin\left(\frac{3\pi}{2} - i\pi\xi\right) = -\cosh\pi\xi,$$

we obtain

$$m_1(\xi) = -\left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right).$$

Therefore

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

Consequently,

$$\boxed{\mathcal{M}_{1/2} = -D^2 + 2D - \frac{3}{4}, \qquad D = \frac{d}{dx}.}$$

This agrees directly with the conjugation formula

$$F(s) = s^{-1/2} G(\log s).$$

Indeed,

$$F''(s) = s^{-5/2} \left(G'' - 2G' + \frac{3}{4} G \right),$$

so

$$\boxed{-\partial_{ss} F = s^{-5/2} \left(-G'' + 2G' - \frac{3}{4} G \right).}$$

Perturbation of the Mellin Symbol

Differentiate the exact symbol with respect to α. Writing $\psi = \Gamma'/\Gamma$ for the digamma function gives

$$\boxed{\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi) \left[-\psi\left(\frac{3}{2} - \alpha + i\xi \right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi \right) + 2\pi \cot \pi \left(2\alpha - \frac{1}{2} - i\xi \right) \right].}$$

Fix

$$0 < \delta < \frac{1}{4}.$$

For

$$1 \le \alpha \le 1 + \delta,$$

the Gamma-function arguments remain in pole-free right half-planes.
Stirling asymptotics imply

$$\boxed{|m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha}.}$$

The standard digamma estimate gives

$$|\psi(z)| \le C(1 + \log\langle \Im z \rangle)$$

uniformly in the relevant vertical strips. Hence

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha} \left(1 + \log\langle \xi \rangle\right).}$$

For every

$$0 < \sigma < 1,$$

choose

$$0 < \delta \leq \frac{\sigma}{4}.$$

Then

$$2\alpha \leq 2 + 2\delta \leq 2 + \frac{\sigma}{2},$$

and the logarithmic factor may be absorbed into the remaining power. Therefore

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \leq C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.}$$

The mean-value theorem now gives

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \leq C_{\delta,\sigma}|\alpha - 1|\langle\xi\rangle^{2+\sigma}.}$$

By Plancherel,

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \leq C_{\delta,\sigma}|\alpha - 1|\|h\|_{H^{2+\sigma}}.}$$

Thus

$$\boxed{\mathcal{M}_\alpha \longrightarrow \mathcal{M}_1 \quad \text{in } \mathcal{L}(H^{2+\sigma}, L^2) \quad \text{as } \alpha \downarrow 1.}$$

Moreover,

$$\boxed{\alpha \longmapsto \mathcal{M}_\alpha \in \mathcal{L}(H^{2+\sigma}, L^2)}$$

is C^1 in a sufficiently small neighborhood of $\alpha = 1$.

Linearization at the Explicit Anchor

At

$$\alpha = 1, \qquad c_1 = \frac{1}{2},$$

the autonomous equation is

$$\nu\mathcal{M}_{1/2}G = G^2G' - \frac{1}{2}G^3.$$

Define

$$\mathcal{F}(1, G) = \nu\mathcal{M}_{1/2}G - G^2G' + \frac{1}{2}G^3.$$

The linearization at G_1 is

$$\boxed{\mathscr{L}_1 h = \nu\left(-h'' + 2h' - \frac{3}{4}h\right) - 2G_1G_1'h - G_1^2h' + \frac{3}{2}G_1^2h.}$$

Since

$$G_1^2 = \frac{3\nu}{2}t$$

and

$$\frac{G_1 G_1'}{\nu} = \frac{3}{4}t(1-t),$$

we obtain

$$\boxed{\frac{1}{\nu}\mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2}t\right)h' + \left(-\frac{3}{4} + \frac{3}{4}t + \frac{3}{2}t^2\right)h.}$$

Translation invariance implies

$$\boxed{\mathscr{L}_1 G_1' = 0.}$$

Thus the translation mode is

$$h_1(x) = G_1'(x).$$

Up to the constant factor H_1,

$$\boxed{h_1(x) = \frac{1}{2}\sqrt{t}(1-t).}$$

Its asymptotic behavior is

$$\boxed{h_1(x) \sim \frac{1}{2}e^{x/2}, \qquad x \to -\infty,}$$

and

$$\boxed{h_1(x) \sim \frac{1}{2}e^{-x}, \qquad x \to +\infty.}$$

Second Homogeneous Solution

Multiply the homogeneous equation by -1 and write it in standard form

$$h'' + P(x)h' + Q(x)h = 0,$$

where

$$P(x) = -2 + \frac{3}{2}t(x).$$

Reduction of order gives

$$h_2 = h_1 \int \frac{e^{-\int P(x)\,dx}}{h_1^2}\,dx.$$

Using

$$t' = t(1-t),$$

we obtain

$$\boxed{e^{-\int P\,dx} = t^2(1-t)^{-1/2}}$$

up to an irrelevant nonzero multiplicative constant.
Since

$$h_1^2 \propto t(1-t)^2,$$

we find

$$\frac{e^{-\int P dx}}{h_1^2} \propto t(1-t)^{-5/2}.$$

Using

$$dx = \frac{dt}{t(1-t)},$$

the integral becomes

$$\int (1-t)^{-7/2}\,dt \propto (1-t)^{-5/2}.$$

Hence one may choose

$$\boxed{h_2(x) = \sqrt{t}\,(1-t)^{-3/2}.}$$

At the right endpoint,

$$1 - t \sim e^{-x},$$

and therefore

$$\boxed{h_2(x) \sim e^{3x/2}, \qquad x \to +\infty.}$$

At the left endpoint,

$$t \sim e^{x},$$

so the displayed h_2 shares the leading $e^{x/2}$ behavior of the translation mode. Subtracting the appropriate multiple of h_1 produces the second left mode with behavior

$$\boxed{\widetilde{h}_2(x) \sim e^{3x/2}, \qquad x \to -\infty.}$$

Thus the asymptotic characteristic exponents are

$$\boxed{\lambda_-^{(1)} = \frac{1}{2}, \qquad \lambda_-^{(2)} = \frac{3}{2}}$$

at $x = -\infty$, while at $x = +\infty$ they are

$$\boxed{\lambda_+^{(1)} = -1, \qquad \lambda_+^{(2)} = \frac{3}{2}.}$$

Indeed, as $x \to -\infty$,

$$t \to 0,$$

and the limiting equation is

$$h'' - 2h' + \frac{3}{4}h = 0,$$

whose roots are

$$\boxed{\lambda = \frac{1}{2}, \frac{3}{2}.}$$

As $x \to +\infty$,

$$t \to 1,$$

and the limiting equation becomes

$$h'' - \frac{1}{2}h' - \frac{3}{2}h = 0,$$

whose roots are

$$\boxed{\lambda = -1, \frac{3}{2}.}$$

Correct Fredholm Index Count

Choose a two-sided exponential weight satisfying

$$\boxed{0 < \eta < \frac{1}{2}.}$$

Let

$$\omega_\eta(x) \asymp \begin{cases} e^{-\eta x}, & x \ll 0, \\ e^{\eta x}, & x \gg 0. \end{cases}$$

Consider

$$X_\eta = \left\{ h : \omega_\eta h \in H^2(\mathbb{R}) \right\},$$

and

$$Y_\eta = \left\{ f : \omega_\eta f \in L^2(\mathbb{R}) \right\}.$$

At $x = -\infty$, both characteristic modes

$$e^{x/2}, \qquad e^{3x/2}$$

belong to the weighted space because

$$0 < \eta < \frac{1}{2}.$$

Thus the dimension of the admissible subspace at the left endpoint is

$$\boxed{d_- = 2.}$$

At $x = +\infty$, only

$$e^{-x}$$

belongs to the weighted space, whereas

$$e^{3x/2}$$

does not. Hence

$$\boxed{d_+ = 1.}$$

For a second-order scalar equation, the Fredholm index is therefore

$$\boxed{\operatorname{ind} \mathscr{L}_1 = d_- + d_+ - 2 = 2 + 1 - 2 = 1.}$$

The kernel contains exactly the translation mode,

$$\boxed{\ker \mathscr{L}_1 = \operatorname{span}\{G_1'\}.}$$

Consequently,

$$\dim \ker \mathscr{L}_1 = 1$$

and

$$\operatorname{ind} \mathscr{L}_1 = 1$$

imply

$$\boxed{\dim \operatorname{coker} \mathscr{L}_1 = 0.}$$

Therefore there is no additional cokernel compatibility condition in these weighted spaces.

In particular, the formal adjoint solution that grows strongly at $x = -\infty$ does not belong to the appropriate dual weighted space and does not define a cokernel functional.

Thus an additional free amplitude parameter is not required merely to remove a cokernel.

The only degeneracy is the expected translation symmetry.

Phase Fixing

Fix the translation by imposing, for example,

$$\boxed{Z(0) = \frac{1}{\sqrt{2}}.}$$

At the anchor,

$$Z_1(0) = \frac{1}{\sqrt{2}}.$$

Since

$$Z_1'(0) = \frac{1}{4\sqrt{2}} \neq 0,$$

the translation orbit intersects this phase condition transversally.

Equivalently, for perturbations h,

$$\boxed{h(0) = 0.}$$

Since

$$G_1'(0) \neq 0,$$

the translation mode does not belong to the phase-fixed domain.

Thus

$$\boxed{\ker\left(\mathscr{L}_1\big|_{\{h(0)=0\}}\right) = \{0\}.}$$

Because the unrestricted operator has Fredholm index 1, imposing one independent scalar phase condition lowers the index by one. Therefore

$$\boxed{\operatorname{ind}\left(\mathscr{L}_1\big|_{\{h(0)=0\}}\right) = 0.}$$

Since the phase-fixed kernel is zero, its cokernel is also zero.

Hence the phase-fixed operator is the correct candidate for an isomorphism.

Direct Verification in the Original s-Variable

The invertibility can be seen particularly transparently at $\alpha = 1$ before passing to logarithmic coordinates.

Recall

$$F_1(s) = \sqrt{\frac{3\nu}{2}}\,(s + s_0)^{-1/2}.$$

The linearization of

$$-\nu F'' - F^2 F' = 0$$

at F_1 is

$$L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.$$

Since

$$F_1^2 = \frac{3\nu}{2(s + s_0)},$$

we obtain

$$\boxed{L_1 h = -\left(\nu h' + F_1^2 h\right)'.}$$

Equivalently,

$$\boxed{L_1 h = -\nu \left[h' + \frac{3}{2(s + s_0)} h\right]'.}$$

The homogeneous equation

$$L_1 h = 0$$

integrates once to

$$h' + \frac{3}{2(s + s_0)} h = C_1.$$

Multiplication by the integrating factor

$$(s + s_0)^{3/2}$$

gives

$$\frac{d}{ds}\left[(s + s_0)^{3/2} h\right] = C_1 (s + s_0)^{3/2}.$$

Therefore

$$\boxed{h(s) = \frac{2C_1}{5}(s+s_0) + C_2(s+s_0)^{-3/2}.}$$

The growing solution

$$(s+s_0)$$

is excluded by the decay condition.
The remaining homogeneous solution is

$$(s+s_0)^{-3/2},$$

which is precisely proportional to the translation derivative

$$F_1'(s).$$

Thus the decaying kernel is exactly one-dimensional.
Imposing

$$h(0) = 0$$

eliminates this mode.

Explicit Phase-Fixed Inverse

Let

$$L_1 h = f.$$

Set

$$g(s) = h'(s) + \frac{3}{2(s+s_0)}h(s).$$

Then

$$-\nu g'(s) = f(s).$$

Impose the decaying condition

$$g(\infty) = 0.$$

It follows that

$$\boxed{g(s) = \frac{1}{\nu}\int_s^\infty f(\tau)\,d\tau.}$$

Now solve

$$h' + \frac{3}{2(s+s_0)}h = g(s)$$

subject to the phase condition

$$h(0) = 0.$$

Multiplication by $(s+s_0)^{3/2}$ gives

$$\frac{d}{ds}\left[(s+s_0)^{3/2}h(s)\right] = (s+s_0)^{3/2}g(s).$$

Hence

$$(s+s_0)^{3/2}h(s) = \int_0^s (\sigma+s_0)^{3/2}g(\sigma)\,d\sigma.$$

Substituting the expression for g,

$$\boxed{(L_1^{-1}f)(s) = \frac{1}{\nu(s+s_0)^{3/2}}\int_0^s (\sigma+s_0)^{3/2}\int_\sigma^\infty f(\tau)\,d\tau\,d\sigma.}$$

This gives an explicit inverse for the phase-fixed linearized operator.

A Concrete Weighted Estimate

Let

$$r = s + s_0.$$

For

$$0 < \eta < \frac{3}{2},$$

define

$$\boxed{\|f\|_{Y_\eta} = \sup_{s\geq 0} r^{2+\eta}|f(s)|.}$$

Let

$$M = \|f\|_{Y_\eta}.$$

Then

$$|f(\tau)| \leq M(\tau+s_0)^{-2-\eta}.$$

Therefore

$$\int_s^\infty |f(\tau)|\,d\tau \leq M\int_s^\infty (\tau+s_0)^{-2-\eta}\,d\tau,$$

and hence

$$\boxed{\int_s^\infty |f(\tau)|\,d\tau \le \frac{M}{1+\eta}(s+s_0)^{-1-\eta}.}$$

Consequently,

$$|g(s)| \le \frac{M}{\nu(1+\eta)} r^{-1-\eta}.$$

Using the explicit inverse,

$$|h(s)| \le \frac{M}{\nu(1+\eta)r^{3/2}} \int_0^s (\sigma+s_0)^{1/2-\eta}\,d\sigma.$$

For

$$0 < \eta < \frac{3}{2},$$

we obtain

$$\int_0^s (\sigma+s_0)^{1/2-\eta}\,d\sigma \le \frac{r^{3/2-\eta}}{3/2-\eta}.$$

Thus

$$\boxed{|h(s)| \le \frac{M}{\nu(1+\eta)(3/2-\eta)} r^{-\eta}.}$$

Furthermore,

$$h' = g - \frac{3}{2r}h,$$

so

$$\boxed{|h'(s)| \le C_{\nu,\eta} M r^{-1-\eta}.}$$

Differentiating

$$g = h' + \frac{3}{2r}h$$

gives

$$g' = h'' + \frac{3}{2r}h' - \frac{3}{2r^2}h.$$

Since

$$g' = -\frac{1}{\nu}f,$$

we obtain

$$h'' = -\frac{1}{\nu} f - \frac{3}{2r} h' + \frac{3}{2r^2} h.$$

Therefore

$$\boxed{|h''(s)| \le C_{\nu,\eta} M r^{-2-\eta}.}$$

Define

$$\boxed{\|h\|_{X_\eta} = \sup_{s\ge 0} r^{\eta} |h(s)| + \sup_{s\ge 0} r^{1+\eta} |h'(s)| + \sup_{s\ge 0} r^{2+\eta} |h''(s)|.}$$

Then

$$\boxed{\|L_1^{-1} f\|_{X_\eta} \le C_{\nu,\eta} \|f\|_{Y_\eta}.}$$

Thus, on these phase-fixed weighted spaces,

$$\boxed{L_1 : X_\eta \longrightarrow Y_\eta}$$

has a bounded inverse.

This explicitly confirms the Fredholm conclusion: the only noninvertibility of the unrestricted anchor linearization is the translation mode, and after phase fixing no additional amplitude parameter or adjoint compatibility condition is required.

The Remaining Continuation Issue

The preceding argument establishes the two principal ingredients at the formal continuation anchor:

$$\boxed{\text{phase-fixed invertibility at } \alpha = 1,}$$

and

$$\boxed{\mathcal{M}_\alpha - \mathcal{M}_1 = O(|\alpha - 1|) : H^{2+\sigma} \to L^2.}$$

However, one further structural point must be verified before applying the Banach-space implicit-function theorem without qualification.

The Mellin formula used above is a one-sided realization involving

$$s_+^{\lambda}.$$

For $\alpha > 1$, the desired left endpoint behavior is

$$\boxed{F_\alpha(s) \sim A_\alpha s^{\alpha-1}, \qquad s \downarrow 0,}$$

and therefore

$$F_\alpha(0) = 0$$

for every $\alpha > 1$.

At the anchor $\alpha = 1$,

$$F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}},$$

so

$$\boxed{F_1(0) = \sqrt{\frac{3\nu}{2s_0}} \neq 0.}$$

Thus the boundary trace of the F-profile changes at the endpoint $\alpha = 1$.

This does not appear as a discontinuity in the logarithmic variable, because

$$G_\alpha(x) = s^{a_\alpha} F_\alpha(s)$$

satisfies

$$G_\alpha(-\infty) = 0$$

for all α near 1, including $\alpha = 1$.

Indeed,

$$G_1(x) \sim H_1 e^{x/2}, \qquad x \to -\infty.$$

Nevertheless, the underlying fractional realization acts on F, and therefore the precise exterior-extension/domain convention at $s = 0$ must be included in the continuation theorem.

In particular, before asserting

$$\boxed{\exists\, \delta > 0 \quad \text{such that profiles exist for every} \quad 1 < \alpha < 1 + \delta,}$$

one must verify that the one-sided fractional operator used forms a C^1 family on a common Banach domain after the Mellin conjugation, including the endpoint/exterior-data convention.

This is the remaining functional-analytic point in the local continuation argument.

Corrected Status of the Continuation Argument

The analysis above gives the following hierarchy.

First, the explicit anchor is exact:

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}.}$$

Second, the Mellin symbol is exact:

$$\boxed{m_\alpha(\xi) = \frac{\Gamma(\frac{3}{2} - \alpha + i\xi)\Gamma(3\alpha - \frac{1}{2} - i\xi)}{\pi} \sin \pi \left(2\alpha - \frac{1}{2} - i\xi\right).}$$

Third,

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

Fourth, for every sufficiently small fixed $\sigma > 0$,

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \le C|\alpha - 1|\|h\|_{H^{2+\sigma}}.}$$

Fifth, the unrestricted anchor linearization has exactly the translation kernel and Fredholm index 1:

$$\boxed{\ker \mathscr{L}_1 = \operatorname{span}\{G_1'\}, \qquad \operatorname{ind} \mathscr{L}_1 = 1.}$$

Sixth, the phase condition removes this kernel:

$$\boxed{h(0) = 0.}$$

The phase-fixed operator has index zero and trivial kernel, and the direct s-variable calculation provides an explicit bounded inverse.

Therefore the earlier introduction of a separate adjoint compatibility condition and a free amplitude parameter is not required by the anchor linearization in these small-weight spaces.

The remaining issue is not an algebraic degeneracy of the $\alpha = 1$ linearization. It is the precise compatibility of the one-sided fractional operator, its exterior extension, and its domain as $\alpha \downarrow 1$.

Once that common-domain C^1 property is established, the natural next step is to define

$$\boxed{\mathscr{F}(\alpha, G) = 0}$$

on the phase-fixed weighted Banach spaces and apply the implicit function theorem at

$$\boxed{(\alpha, G) = (1, G_1).}$$

That would yield a genuine local branch

$$\boxed{\alpha \longmapsto G_\alpha}$$

for

$$\boxed{1 < \alpha < 1 + \delta,}$$

with

$$\boxed{G_\alpha(-\infty) = 0, \qquad G_\alpha(+\infty) = H_*(\alpha),}$$

and hence

$$\boxed{F_\alpha(s) = s^{-(\alpha-1/2)} G_\alpha(\log s).}$$

The corresponding endpoint behavior would be

$$\boxed{F_\alpha(s) \sim A_\alpha s^{\alpha-1}, \qquad s \downarrow 0,}$$

and

$$\boxed{F_\alpha(s) \sim H_*(\alpha) s^{-(\alpha-1/2)}, \qquad s \to \infty.}$$

Thus the central remaining task for the interval theorem is to close the operator-domain/exterior-extension step and thereby place the exact anchor inverse and the Mellin-symbol perturbation estimate into one common Banach-space implicit-function-theorem framework.

10 Part I

Continuation Audit Near $\alpha = 1$

The endpoint index count shows that the previous cokernel detour must be reconsidered. In the small two-sided exponentially weighted spaces, the formal adjoint mode does not belong to the relevant dual space. The phase-fixed linearized operator is therefore the correct object to study, and there is no reason at this stage to introduce an additional amplitude parameter.

We begin again from the exact $\alpha = 1$ profile and prove the phase-fixed invertibility directly.

1. Exact profile at $\alpha = 1$

For

$$\alpha = 1, \qquad \gamma = 0,$$

the profile equation becomes

$$-\nu F'' = F^2 F' = \frac{1}{3}(F^3)'.$$

After one integration, with the integration constant chosen to vanish at infinity,

$$-\nu F' = \frac{1}{3} F^3.$$

Hence

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}}$$

for $s_0 > 0$.

Write

$$c_\nu = \sqrt{\frac{3\nu}{2}},$$

so that

$$F_1(s) = c_\nu (s + s_0)^{-1/2}.$$

The translation parameter s_0 is the one-dimensional symmetry which must be removed by a phase condition.

2. Direct linearization in the s-variable

Define

$$\mathcal{F}_1(F) = -\nu F'' - F^2 F'.$$

Linearizing about F_1 gives

$$L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.$$

Since

$$F_1^2 = \frac{3\nu}{2(s + s_0)},$$

we have

$$F_1 F_1' = -\frac{3\nu}{4(s + s_0)^2}.$$

Therefore

$$L_1 h = -\nu h'' - \frac{3\nu}{2(s + s_0)} h' + \frac{3\nu}{2(s + s_0)^2} h.$$

Equivalently,

$$\boxed{L_1 h = -\nu \left[h' + \frac{3}{2(s + s_0)} h \right]'.}$$

This factorization makes the kernel and inverse completely explicit.

3. Homogeneous equation

Suppose

$$L_1 h = 0.$$

Then

$$\left[h' + \frac{3}{2(s + s_0)} h \right]' = 0,$$

and hence

$$h' + \frac{3}{2(s + s_0)} h = C_1.$$

Multiplication by the integrating factor

$$(s + s_0)^{3/2}$$

gives

$$\frac{d}{ds}\left[(s+s_0)^{3/2}h(s)\right] = C_1(s+s_0)^{3/2}.$$

Consequently,

$$(s+s_0)^{3/2}h(s) = \frac{2C_1}{5}(s+s_0)^{5/2} + C_2,$$

and therefore

$$\boxed{h(s) = \frac{2C_1}{5}(s+s_0) + C_2(s+s_0)^{-3/2}.}$$

The first solution grows linearly as $s \to \infty$, whereas the second solution decays.

Moreover,

$$\partial_{s_0} F_1 = -\frac{c_\nu}{2}(s+s_0)^{-3/2},$$

so the decaying homogeneous mode is precisely the translation mode.

Thus, before imposing a phase condition,

$$\ker L_1 = \operatorname{span}\left\{(s+s_0)^{-3/2}\right\}$$

in the natural decaying class.

A phase condition removes this one-dimensional kernel.

4. Explicit phase-fixed inverse

Consider

$$L_1 h = f.$$

Define

$$g(s) = h'(s) + \frac{3}{2(s+s_0)}h(s).$$

Then

$$-\nu g'(s) = f(s).$$

If the admissible solution satisfies

$$g(\infty) = 0,$$

then

$$\boxed{g(s) = \frac{1}{\nu}\int_s^\infty f(\tau)\,d\tau.}$$

Now solve

$$h' + \frac{3}{2(s+s_0)}h = g.$$

Multiplying by $(s+s_0)^{3/2}$,

$$\frac{d}{ds}\left[(s+s_0)^{3/2}h(s)\right] = (s+s_0)^{3/2}g(s).$$

Choose the phase condition

$$\boxed{h(0) = 0.}$$

Integration from 0 to s gives

$$(s+s_0)^{3/2}h(s) = \int_0^s (\sigma+s_0)^{3/2} g(\sigma)\, d\sigma.$$

Substituting the formula for g,

$$\boxed{(L_1^{-1}f)(s) = \frac{1}{\nu(s+s_0)^{3/2}} \int_0^s (\sigma+s_0)^{3/2} \int_\sigma^\infty f(\tau)\, d\tau\, d\sigma.}$$

Thus the phase-fixed inverse is not merely abstract: it is given by an explicit integral operator.

5. Weighted estimate

Let

$$r = s + s_0.$$

For

$$0 < \eta < \frac{3}{2},$$

define

$$\boxed{\|f\|_{Y_\eta} = \sup_{s\geq 0} (s+s_0)^{2+\eta} |f(s)|.}$$

Let

$$M = \|f\|_{Y_\eta}.$$

Then

$$|f(\tau)| \leq M(\tau+s_0)^{-2-\eta}.$$

Therefore

$$\left| \int_s^\infty f(\tau)\, d\tau \right| \leq M \int_s^\infty (\tau+s_0)^{-2-\eta}\, d\tau.$$

Hence

$$\boxed{\left| \int_s^\infty f(\tau)\, d\tau \right| \leq \frac{M}{1+\eta} (s+s_0)^{-1-\eta}.}$$

It follows that

$$|g(s)| \leq \frac{M}{\nu(1+\eta)} (s+s_0)^{-1-\eta}.$$

Using the explicit inverse,

$$|h(s)| \leq \frac{M}{\nu(1+\eta)} (s+s_0)^{-3/2} \int_0^s (\sigma+s_0)^{1/2-\eta}\, d\sigma.$$

Since

$$0 < \eta < \frac{3}{2},$$

we obtain

$$\int_0^s (\sigma + s_0)^{1/2-\eta}\, d\sigma \le \frac{1}{\frac{3}{2}-\eta}(s+s_0)^{3/2-\eta}.$$

Consequently,

$$\boxed{|h(s)| \le \frac{M}{\nu(1+\eta)(\frac{3}{2}-\eta)}(s+s_0)^{-\eta}.}$$

Next,

$$h' = g - \frac{3}{2(s+s_0)}h,$$

so

$$|h'(s)| \le C_{\nu,\eta} M (s+s_0)^{-1-\eta}.$$

Finally,

$$g' = h'' + \frac{3}{2(s+s_0)}h' - \frac{3}{2(s+s_0)^2}h = -\frac{1}{\nu}f.$$

Thus

$$h'' = -\frac{1}{\nu}f - \frac{3}{2(s+s_0)}h' + \frac{3}{2(s+s_0)^2}h,$$

and hence

$$\boxed{|h''(s)| \le C_{\nu,\eta} M (s+s_0)^{-2-\eta}.}$$

Define

$$\boxed{X_\eta = \left\{ h : h(0) = 0, \quad \sup_{s\ge 0} \left[r^\eta |h| + r^{1+\eta}|h'| + r^{2+\eta}|h''| \right] < \infty \right\}.}$$

Then

$$\boxed{L_1^{-1} : Y_\eta \longrightarrow X_\eta}$$

is bounded.

In particular,

$$\boxed{\|L_1^{-1} f\|_{X_\eta} \le C_{\nu,\eta} \|f\|_{Y_\eta}.}$$

Therefore the phase-fixed linearization at the explicit $\alpha = 1$ profile is an isomorphism in these weighted spaces.

6. Why the previous cokernel conclusion does not follow

The same conclusion is visible from the endpoint index count in logarithmic coordinates.

Set

$$x = \log s, \qquad t(x) = \frac{1}{1+e^{-x}}.$$

After normalization,

$$Z_1(x) = (1+e^{-x})^{-1/2} = \sqrt{t(x)}.$$

The linearized equation is

$$\boxed{\frac{1}{\nu}\mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2}t\right)h' + \left(-\frac{3}{4} + \frac{3}{4}t + \frac{3}{2}t^2\right)h.}$$

As

$$x \to -\infty, \qquad t \to 0,$$

the limiting homogeneous equation is

$$-h'' + 2h' - \frac{3}{4}h = 0.$$

Equivalently,

$$h'' - 2h' + \frac{3}{4}h = 0.$$

Its characteristic roots are

$$\boxed{\lambda_-^{(1)} = \frac{1}{2}, \qquad \lambda_-^{(2)} = \frac{3}{2}.}$$

Both modes decay as $x \to -\infty$.

As

$$x \to +\infty, \qquad t \to 1,$$

the limiting equation becomes

$$-h'' + \frac{1}{2}h' + \frac{3}{2}h = 0.$$

Equivalently,

$$h'' - \frac{1}{2}h' - \frac{3}{2}h = 0,$$

whose roots are

$$\boxed{\lambda_+^{(1)} = -1, \qquad \lambda_+^{(2)} = \frac{3}{2}.}$$

Thus exactly one mode decays as $x \to +\infty$.

For a sufficiently small two-sided exponential weight

$$0 < \eta < \frac{1}{2},$$

the admissible dimension at the left endpoint is therefore 2, whereas the admissible dimension at the right endpoint is 1.

For a second-order problem this gives the index count

$$2 + 1 - 2 = 1.$$

Thus the unphased operator naturally has Fredholm index $+1$, consistent with its one-dimensional translation kernel and zero cokernel.

After imposing one scalar phase condition, the index becomes

$$\boxed{0.}$$

Since the translation kernel is removed, the phase-fixed operator is the candidate isomorphism.

This agrees exactly with the explicit inverse obtained above.

In particular, the formal adjoint solution which grows strongly at $x \to -\infty$ cannot automatically be interpreted as a cokernel functional. Membership in the dual weighted space is essential.

Hence a condition of the form

$$\int_{\mathbb{R}} e^{-3x/2} f(x)\, dx = 0$$

is not justified merely by the existence of that formal adjoint solution.

No additional amplitude parameter is therefore required on the basis of that argument.

7. Mellin symbol for the fractional continuation

For

$$a = \alpha - \frac{1}{2},$$

the manuscript uses

$$F(s) = s^{-a} G(\log s)$$

and

$$(-\partial_{ss})^{\alpha} F = s^{-a-2\alpha} \mathcal{M}_a G.$$

The Mellin multiplier is

$$\boxed{K_{\alpha}^{+}(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi} \sin \pi(\alpha-\lambda).}$$

Consequently,

$$\mathcal{M}_a(e^{\lambda x}) = K_{\alpha}^{+}(-a+\lambda) e^{\lambda x}.$$

For Fourier frequency ξ, define

$$z_{\alpha}(\xi) = -a + i\xi = -\alpha + \frac{1}{2} + i\xi.$$

Then

$$m_{\alpha}(\xi) = K_{\alpha}^{+}\left(-\alpha + \frac{1}{2} + i\xi\right).$$

Therefore

$$\boxed{m_{\alpha}(\xi) = \frac{\Gamma(\frac{3}{2} - \alpha + i\xi)\Gamma(3\alpha - \frac{1}{2} - i\xi)}{\pi} \sin \pi\left(2\alpha - \frac{1}{2} - i\xi\right).}$$

At

$$\alpha = 1,$$

this becomes

$$m_1(\xi) = \frac{\Gamma(\frac{1}{2} + i\xi)\Gamma(\frac{5}{2} - i\xi)}{\pi} \sin \pi\left(\frac{3}{2} - i\xi\right).$$

Using the Gamma recurrence and reflection identities,

$$\boxed{m_1(\xi) = -\left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

Thus

$$\boxed{\mathcal{M}_{1/2} = -D^2 + 2D - \frac{3}{4}, \qquad D = \frac{d}{dx}.}$$

This also follows directly from

$$F(s) = s^{-1/2} G(\log s).$$

Indeed,

$$F'' = s^{-5/2}\left(G'' - 2G' + \frac{3}{4}G\right),$$

and therefore

$$\boxed{-\partial_{ss} F = s^{-5/2}\left(-G'' + 2G' - \frac{3}{4}G\right).}$$

8. Differentiation with respect to α

Differentiating the exact Gamma-function multiplier gives

$$\boxed{\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi)\left[-\psi\left(\frac{3}{2} - \alpha + i\xi\right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi\right) + 2\pi\cot\pi\left(2\alpha - \frac{1}{2} - i\xi\right)\right],}$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the digamma function.

Fix

$$0 < \delta < \frac{1}{4}.$$

For

$$1 \le \alpha \le 1 + \delta,$$

the Gamma arguments remain in pole-free right half-planes, and the sine factor remains away from its zeros for real ξ.

Stirling asymptotics imply

$$|m_\alpha(\xi)| \le C_\delta \langle\xi\rangle^{2\alpha}.$$

The standard digamma asymptotic

$$\psi(z) = \log z + O(|z|^{-1})$$

gives

$$|\partial_\alpha m_\alpha(\xi)| \le C_\delta \langle\xi\rangle^{2\alpha}\left(1 + \log\langle\xi\rangle\right).$$

Given

$$0 < \sigma < 1,$$

choose

$$0 < \delta \leq \frac{\sigma}{4}.$$

Then

$$2\alpha \leq 2 + 2\delta \leq 2 + \frac{\sigma}{2}.$$

Since

$$1 + \log\langle\xi\rangle \leq C_\sigma \langle\xi\rangle^{\sigma/2},$$

we obtain

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \leq C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.}$$

By the mean-value theorem,

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \leq C_{\delta,\sigma}|\alpha - 1|\langle\xi\rangle^{2+\sigma}.}$$

Hence Plancherel's theorem gives

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \leq C_{\delta,\sigma}|\alpha - 1|\|h\|_{H^{2+\sigma}}.}$$

Thus

$$\boxed{\alpha \longmapsto \mathcal{M}_\alpha \in \mathcal{L}\left(H^{2+\sigma}, L^2\right)}$$

is C^1 in a neighborhood of $\alpha = 1$.

9. Autonomous profile equation

For

$$a = \alpha - \frac{1}{2},$$

the logarithmic profile equation is

$$\boxed{\nu\mathcal{M}_a G = G^2 G' - cG^3,}$$

where

$$\boxed{c = \frac{2\alpha - 1 - \gamma}{2}.}$$

For the first continuation theorem we take

$$\gamma = 0,$$

so

$$\boxed{c_\alpha = \frac{2\alpha - 1}{2}.}$$

The desired heteroclinic satisfies

$$\boxed{G(-\infty) = 0, \qquad G(+\infty) = H_*(\alpha).}$$

The equilibrium value is not a free amplitude. Substituting $G = H_*$ into the autonomous equation gives

$$\nu K_\alpha^+(-a) H_* = -c_\alpha H_*^3.$$

For $H_* > 0$,

$$\boxed{H_*^2(\alpha) = -\frac{\nu K_\alpha^+(-a)}{c_\alpha}.}$$

At $\alpha = 1$,

$$a = \frac{1}{2}, \qquad K_1^+\left(-\frac{1}{2}\right) = -\frac{3}{4}, \qquad c_1 = \frac{1}{2}.$$

Therefore

$$\boxed{H_*^2(1) = \frac{3}{2}\nu,}$$

which agrees with

$$H_*(1) = c_\nu.$$

Thus $H_*(\alpha)$ is determined by the equation itself and, provided the Gamma factors remain regular, varies smoothly with α near 1.

10. Endpoint rates

The left Mellin root is determined by

$$-a + \lambda = \alpha - 1.$$

Since

$$a = \alpha - \frac{1}{2},$$

we obtain

$$\boxed{\lambda = q_-(\alpha) = 2\alpha - \frac{3}{2}.}$$

At $\alpha = 1$,

$$q_-(1) = \frac{1}{2}.$$

Indeed,

$$Z_1(x) = (1 + e^{-x})^{-1/2}$$

has the expansion

$$\boxed{Z_1(x) = e^{x/2} - \frac{1}{2}e^{3x/2} + \frac{3}{8}e^{5x/2} + O(e^{7x/2}), \qquad x \to -\infty.}$$

At the right endpoint,

$$\boxed{Z_1(x) = 1 - \frac{1}{2}e^{-x} + \frac{3}{8}e^{-2x} + O(e^{-3x}), \qquad x \to +\infty.}$$

Thus the anchor profile has left decay exponent

$$\frac{1}{2}$$

and right decay exponent

$$1.$$

These are exactly compatible with choosing a sufficiently small exponential weight

$$\boxed{0 < \eta < \frac{1}{2}.}$$

11. What is now established at the continuation anchor

The explicit $\alpha = 1$ calculation establishes the following facts:

(i) The profile $F_1 = \sqrt{\dfrac{3\nu}{2(s+s_0)}}$ is explicit.

(ii) The unphased linearization has exactly the translation mode $(s+s_0)^{-3/2}$ in the natural decaying class.

(iii) A phase condition removes this kernel.

(iv) The phase-fixed inverse is explicitly

$$(L_1^{-1}f)(s) = \frac{1}{\nu(s+s_0)^{3/2}} \int_0^s (\sigma+s_0)^{3/2} \int_\sigma^\infty f(\tau)\, d\tau\, d\sigma.$$

(v) The inverse satisfies weighted estimates.

(vi) The Mellin multiplier depends continuously and C^1-smoothly on α in stronger Sobolev topology.

Therefore the earlier proposed cokernel obstruction is not supported by the correct endpoint index count.

The remaining issue for a rigorous implicit-function continuation is not an additional scalar solvability condition. The decisive remaining issue is to put the nonlocal one-sided fractional operator and the $\alpha = 1$ anchor into one common Banach-space framework in which

$$\boxed{D_G\mathcal{F}(1, G_1)}$$

is the phase-fixed isomorphism above and

$$\boxed{(\alpha, G) \longmapsto \mathcal{F}(\alpha, G)}$$

is C^1.

12. The remaining operator-domain question

There is one structural point which must be checked before invoking the implicit function theorem.

The Mellin formula is written using

$$s_+^\lambda$$

and a one-sided realization:

$$(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda-2\alpha}.$$

For

$$\alpha > 1,$$

the desired profile has

$$F_\alpha(s) \sim A_\alpha s^{\alpha-1}, \qquad s \downarrow 0,$$

so

$$F_\alpha(0) = 0.$$

At the anchor

$$\alpha = 1,$$

however,

$$F_1(s) = c_\nu (s + s_0)^{-1/2}$$

satisfies

$$F_1(0) = c_\nu s_0^{-1/2} > 0.$$

Thus, if the one-sided fractional operator is defined through zero extension to $s < 0$, the boundary trace of the profile changes at $\alpha = 1$. In that realization, the operator domain itself may not vary continuously through the anchor.

This point cannot be removed merely by the Mellin multiplier estimate.

The logarithmic variable

$$G_\alpha(x) = s^{a_\alpha} F_\alpha(s)$$

does improve the endpoint geometry because

$$G_\alpha(-\infty) = 0$$

for all α near 1, including the anchor:

$$G_1(x) = c_\nu (1 + e^{-x})^{-1/2}.$$

Nevertheless, the underlying realization of

$$(-\partial_{ss})^\alpha$$

must still be specified consistently.

Therefore the next rigorous step is:

identify the exact one-sided extension/domain convention and prove that the transformed operator is C^1

Once that is established, the phase-fixed invertibility calculation above provides precisely the linear ingredient needed for the Banach-space implicit function theorem.

13. Target continuation theorem

The desired theorem can then be formulated as follows.

Target Theorem. There exists

$$\delta > 0$$

such that for every

$$1 \le \alpha < 1 + \delta$$

there exists a phase-normalized positive profile

$$G_\alpha$$

solving

$$\boxed{\nu \mathcal{M}_{a_\alpha} G_\alpha = G_\alpha^2 G_\alpha' - \frac{2\alpha - 1}{2} G_\alpha^3, \qquad a_\alpha = \alpha - \frac{1}{2},}$$

with

$$\boxed{G_\alpha(-\infty) = 0, \qquad G_\alpha(+\infty) = H_*(\alpha),}$$

and

$$\boxed{G_\alpha \longrightarrow G_1}$$

in the selected weighted profile space as

$$\alpha \downarrow 1.$$

Moreover,

$$\boxed{G_\alpha(x) \sim A_\alpha e^{(2\alpha - \frac{3}{2})x}, \qquad x \to -\infty,}$$

with

$$A_\alpha > 0,$$

and

$$\boxed{G_\alpha(x) \to H_*(\alpha) > 0, \qquad x \to +\infty.}$$

In the original similarity variable,

$$F_\alpha(s) = s^{-a_\alpha} G_\alpha(\log s),$$

so the left endpoint satisfies

$$\boxed{F_\alpha(s) \sim A_\alpha s^{\alpha - 1}, \qquad s \downarrow 0,}$$

while

$$\boxed{F_\alpha(s) \sim H_*(\alpha) s^{-(\alpha - \frac{1}{2})}, \qquad s \to \infty.}$$

The explicit $\alpha = 1$ phase-fixed inverse and the Mellin-symbol perturbation estimate establish the two principal local ingredients for this theorem. What remains is to verify that the precise one-sided fractional realization supplies a common C^1 operator framework across the endpoint $\alpha = 1$.

11 Part J

12 Correction of the Fredholm Analysis at the Explicit $\alpha = 1$ Anchor

We now return to the explicit anchor at

$$\boxed{\alpha = 1, \qquad \gamma = 0.}$$

The purpose of this calculation is to determine the precise invertibility properties of the linearized profile operator. In particular, the previous appearance of an adjoint mode must be interpreted in the correct weighted space before it can be used to infer the existence of a cokernel.

The conclusion will be that, in the natural small exponentially weighted spaces, the unphased operator has Fredholm index 1, with the one-dimensional kernel generated by translation. After fixing the translation phase, the operator has index zero and is invertible. No additional amplitude parameter is required at the $\alpha = 1$ anchor.

12.1 The explicit profile

For

$$\alpha = 1, \qquad \gamma = 0,$$

the scalar profile equation becomes

$$-\nu F'' = F^2 F' = \frac{1}{3}(F^3)'.$$

After one integration, using the decaying condition at infinity,

$$-\nu F' = \frac{1}{3}F^3.$$

Hence

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}.}$$

Choosing the translation phase

$$s_0 = 1,$$

we obtain

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2}}\,(1+s)^{-1/2}.}$$

Let

$$c_\nu = \sqrt{\frac{3\nu}{2}}.$$

Then

$$F_1(s) = c_\nu (1+s)^{-1/2}.$$

The logarithmic variable is

$$x = \log s,$$

and for $\alpha = 1$,

$$a = \alpha - \frac{1}{2} = \frac{1}{2}.$$

Therefore

$$G_1(x) = s^{1/2} F_1(s).$$

Consequently,

$$G_1(x) = c_\nu \frac{e^{x/2}}{\sqrt{1+e^x}} = c_\nu (1+e^{-x})^{-1/2}.$$

Defining

$$Z_1(x) = \frac{G_1(x)}{c_\nu},$$

we obtain the explicit heteroclinic

$$\boxed{Z_1(x) = (1+e^{-x})^{-1/2}.}$$

Introduce

$$\boxed{t(x) = Z_1(x)^2 = \frac{1}{1+e^{-x}}.}$$

Then

$$t' = t(1-t),$$

and

$$Z_1 = \sqrt{t}.$$

Therefore

$$Z_1' = \frac{1}{2\sqrt{t}} t' = \frac{1}{2}\sqrt{t}(1-t).$$

Thus

$$\boxed{Z_1' = \frac{1}{2} Z_1 (1 - Z_1^2).}$$

Equivalently,

$$\boxed{Z_1'(x) = \frac{1}{2} \frac{e^{x/2}}{(1+e^x)^{3/2}}.}$$

12.2 Endpoint asymptotics

As

$$x \to -\infty,$$

we have

$$e^x \to 0.$$

Therefore

$$Z_1(x) = e^{x/2}(1+e^x)^{-1/2}.$$

Expanding,

$$(1+e^x)^{-1/2} = 1 - \frac{1}{2}e^x + \frac{3}{8}e^{2x} + O(e^{3x}),$$

so

$$\boxed{Z_1(x) = e^{x/2} - \frac{1}{2}e^{3x/2} + \frac{3}{8}e^{5x/2} + O(e^{7x/2}).}$$

Similarly,

$$\boxed{Z_1'(x) = \frac{1}{2}e^{x/2} - \frac{3}{4}e^{3x/2} + \frac{15}{16}e^{5x/2} + O(e^{7x/2}).}$$

At the other endpoint,

$$x \to +\infty,$$

we have

$$Z_1(x) = (1+e^{-x})^{-1/2},$$

and hence

$$\boxed{Z_1(x) = 1 - \frac{1}{2}e^{-x} + \frac{3}{8}e^{-2x} + O(e^{-3x}).}$$

Therefore

$$\boxed{Z_1'(x) = \frac{1}{2}e^{-x} - \frac{3}{4}e^{-2x} + O(e^{-3x}).}$$

Thus the translation mode decays at both ends.

13 The Exact Mellin Multiplier

The one-sided Mellin multiplier is

$$\boxed{K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi}\sin\pi(\alpha-\lambda).}$$

Set

$$a_\alpha = \alpha - \frac{1}{2}.$$

For

$$F(s) = s^{-a_\alpha}G(\log s),$$

the fractional operator takes the form

$$(-\partial_{ss})^\alpha F = s^{-a_\alpha-2\alpha}\mathcal{M}_{a_\alpha}G,$$

where

$$\mathcal{M}_{a_\alpha}(e^{\lambda x}) = K_\alpha^+(-a_\alpha+\lambda)e^{\lambda x}.$$

On Fourier modes

$$e^{i\xi x},$$

define

$$z_\alpha(\xi) = -a_\alpha + i\xi = -\alpha + \frac{1}{2} + i\xi.$$

Then

$$m_\alpha(\xi) = K_\alpha^+(z_\alpha(\xi)).$$

Substitution gives

$$\boxed{m_\alpha(\xi) = \frac{\Gamma\left(\frac{3}{2}-\alpha+i\xi\right)\Gamma\left(3\alpha-\frac{1}{2}-i\xi\right)}{\pi}\sin\pi\left(2\alpha-\frac{1}{2}-i\xi\right).}$$

13.1 The multiplier at $\alpha = 1$

At

$$\alpha = 1,$$

we have

$$m_1(\xi) = \frac{\Gamma(\frac{1}{2} + i\xi)\Gamma(\frac{5}{2} - i\xi)}{\pi} \sin \pi \left(\frac{3}{2} - i\xi\right).$$

Using the Gamma recursion and reflection identities,

$$m_1(\xi) = -\left(\frac{3}{2} - i\xi\right)\left(\frac{1}{2} - i\xi\right).$$

Therefore

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

Consequently,

$$\boxed{\mathcal{M}_{1/2} = -D^2 + 2D - \frac{3}{4}, \qquad D = \frac{d}{dx}.}$$

This agrees directly with the conjugation identity

$$F(s) = s^{-1/2} G(\log s).$$

Indeed,

$$F'' = s^{-5/2}\left(G'' - 2G' + \frac{3}{4}G\right),$$

so

$$\boxed{-\partial_{ss} F = s^{-5/2}\left(-G'' + 2G' - \frac{3}{4}G\right).}$$

Thus the Mellin calculation and the direct differential calculation agree exactly.

14 Perturbation of the Mellin Symbol for α Near One

Differentiating the exact multiplier gives

$$\boxed{\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi)\left[-\psi\left(\frac{3}{2} - \alpha + i\xi\right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi\right) + 2\pi \cot \pi \left(2\alpha - \frac{1}{2} - i\xi\right)\right],}$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the digamma function.
Fix

$$0 < \delta < \frac{1}{4}.$$

For

$$1 \le \alpha \le 1 + \delta,$$

the arguments

$$\frac{3}{2} - \alpha + i\xi$$

and

$$3\alpha - \frac{1}{2} - i\xi$$

remain in pole-free right half-planes.
Moreover, the sine factor stays away from its zeros for real ξ.
Stirling asymptotics give

$$\boxed{|m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha}.}$$

The standard digamma estimate gives

$$|\psi(z)| \le C(1 + \log\langle z \rangle)$$

uniformly in the relevant vertical strips.
Consequently,

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha} \left(1 + \log\langle \xi \rangle\right).}$$

Let

$$0 < \sigma < 1.$$

If

$$0 < \delta \le \frac{\sigma}{4},$$

then

$$2\alpha \le 2 + 2\delta \le 2 + \frac{\sigma}{2}.$$

Using

$$1+\log\langle\xi\rangle \le C_\sigma \langle\xi\rangle^{\sigma/2},$$

we obtain

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.}$$

The mean-value theorem therefore gives

$$\boxed{|m_\alpha(\xi)-m_1(\xi)| \le C_{\delta,\sigma}|\alpha-1|\langle\xi\rangle^{2+\sigma}.}$$

By Plancherel,

$$\boxed{\|(\mathcal{M}_\alpha-\mathcal{M}_1)h\|_{L^2} \le C_{\delta,\sigma}|\alpha-1|\|h\|_{H^{2+\sigma}}.}$$

In particular,

$$\boxed{\alpha \longmapsto \mathcal{M}_\alpha \in \mathcal{L}\left(H^{2+\sigma},L^2\right)}$$

is C^1 in a sufficiently small interval around $\alpha=1$.

15 Linearization at the Explicit Anchor

For

$$\gamma=0,$$

the autonomous logarithmic profile equation is

$$\boxed{\nu\mathcal{M}_a G = G^2G' - cG^3, \qquad c=\frac{2\alpha-1}{2}.}$$

At

$$\alpha=1,$$

we have

$$a=\frac{1}{2}, \qquad c=\frac{1}{2}.$$

Hence

$$\nu\mathcal{M}_{1/2}G = G^2G' - \frac{1}{2}G^3.$$

Linearizing at G_1 gives

$$\boxed{\mathscr{L}_1 h = \nu\left(-h''+2h'-\frac{3}{4}h\right) - 2G_1G_1'h - G_1^2h' + \frac{3}{2}G_1^2h.}$$

Since

$$G_1^2 = \frac{3\nu}{2}t,$$

we have

$$\frac{G_1^2}{\nu} = \frac{3}{2}t.$$

Also,

$$\frac{2G_1G_1'}{\nu} = \frac{(G_1^2)'}{\nu} = \frac{3}{2}t' = \frac{3}{2}t(1-t).$$

Therefore

$$\frac{1}{\nu}\mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2}t\right)h' + \left[-\frac{3}{4} - \frac{3}{2}t(1-t) + \frac{9}{4}t\right]h.$$

The zeroth-order coefficient simplifies to

$$-\frac{3}{4} - \frac{3}{2}t + \frac{3}{2}t^2 + \frac{9}{4}t = -\frac{3}{4} + \frac{3}{4}t + \frac{3}{2}t^2.$$

Thus

$$\boxed{\frac{1}{\nu}\mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2}t\right)h' + \left(-\frac{3}{4} + \frac{3}{4}t + \frac{3}{2}t^2\right)h.}$$

16 Translation Mode

Because the profile equation is translation invariant in x,

$$G_1(x) \longmapsto G_1(x + \xi_0),$$

differentiation with respect to ξ_0 gives

$$\boxed{\mathscr{L}_1 G_1' = 0.}$$

Hence

$$h_1(x) = G_1'(x)$$

is a homogeneous solution.
Ignoring the constant c_ν, this is

$$\boxed{h_1(x) = \frac{1}{2}\sqrt{t}(1-t).}$$

Its asymptotics are

$$\boxed{h_1(x) \sim \frac{1}{2} e^{x/2}, \qquad x \to -\infty,}$$

and

$$\boxed{h_1(x) \sim \frac{1}{2} e^{-x}, \qquad x \to +\infty.}$$

Thus the translation mode belongs to every sufficiently weak two-sided exponentially weighted space.

17 The Second Homogeneous Solution

Multiply the homogeneous equation by -1. It takes the standard form

$$h'' + P(x)h' + Q(x)h = 0,$$

where

$$P(x) = -2 + \frac{3}{2} t(x).$$

Reduction of order gives

$$h_2 = h_1 \int \frac{e^{-\int P(x)\,dx}}{h_1(x)^2}\, dx.$$

Since

$$t' = t(1-t),$$

we have

$$dx = \frac{dt}{t(1-t)}.$$

Furthermore,

$$\int P\, dx = \int \left(-2 + \frac{3}{2} t \right) \frac{dt}{t(1-t)}.$$

Using

$$\frac{-2 + \frac{3}{2}t}{t(1-t)} = -\frac{2}{t} - \frac{1}{2(1-t)},$$

we obtain

$$\int P\, dx = -2 \log t + \frac{1}{2} \log(1-t) + \text{const.}$$

Hence

$$\boxed{e^{-\int P\,dx} = t^2(1-t)^{-1/2},}$$

up to an irrelevant multiplicative constant.
Since

$$h_1^2 = \frac{1}{4}t(1-t)^2,$$

we obtain

$$\frac{e^{-\int P dx}}{h_1^2} \asymp t(1-t)^{-5/2}.$$

After multiplication by

$$dx = \frac{dt}{t(1-t)},$$

the reduction integral becomes

$$\int (1-t)^{-7/2}\,dt \asymp (1-t)^{-5/2}.$$

Therefore one may take

$$\boxed{h_2(x) = \sqrt{t}\,(1-t)^{-3/2}.}$$

At

$$x \to +\infty,$$

we have

$$1 - t \sim e^{-x},$$

and hence

$$\boxed{h_2(x) \sim e^{3x/2}.}$$

Thus the second mode grows exponentially at the right endpoint.
At the left endpoint,

$$t \sim e^{x},$$

so

$$h_2(x) \sim e^{x/2}.$$

The leading $e^{x/2}$ contribution can be removed by subtracting an appropriate multiple of h_1. The resulting independent solution has the asymptotic behavior

$$\boxed{\widetilde{h}_2(x) \sim e^{3x/2}, \qquad x \to -\infty.}$$

Thus the two characteristic rates at the left endpoint are

$$\boxed{\lambda_-^{(1)} = \frac{1}{2}, \qquad \lambda_-^{(2)} = \frac{3}{2}.}$$

At the right endpoint the characteristic rates are

$$\boxed{\lambda_+^{(1)} = -1, \qquad \lambda_+^{(2)} = \frac{3}{2}.}$$

18 Correct Fredholm Index Count

We now choose a two-sided exponential weight satisfying

$$\boxed{0 < \eta < \frac{1}{2}.}$$

Let

$$\omega_\eta(x) \asymp \begin{cases} e^{-\eta x}, & x \ll 0, \\ e^{\eta x}, & x \gg 0. \end{cases}$$

Define

$$X_\eta = \left\{ h : \omega_\eta h \in H^2(\mathbb{R}) \right\},$$

and

$$Y_\eta = \left\{ f : \omega_\eta f \in L^2(\mathbb{R}) \right\}.$$

At

$$x \to -\infty,$$

the limiting equation is

$$h'' - 2h' + \frac{3}{4}h = 0.$$

Its characteristic polynomial is

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0,$$

with roots

$$\boxed{\lambda = \frac{1}{2}, \frac{3}{2}.}$$

Both modes decay as

$$x \to -\infty.$$

Because

$$\eta < \frac{1}{2},$$

both remain admissible in X_η.
Hence the dimension of the admissible space at the left endpoint is

$$\boxed{d_- = 2.}$$

At

$$x \to +\infty,$$

the limiting equation is

$$h'' - \frac{1}{2}h' - \frac{3}{2}h = 0.$$

Its characteristic polynomial is

$$\lambda^2 - \frac{1}{2}\lambda - \frac{3}{2} = 0.$$

The roots are

$$\boxed{\lambda = -1, \frac{3}{2}.}$$

Only

$$e^{-x}$$

decays as

$$x \to +\infty.$$

Thus

$$\boxed{d_+ = 1.}$$

For a second-order scalar equation, the exponential-dichotomy index count therefore gives

$$\boxed{\operatorname{ind} \mathscr{L}_1 = d_- + d_+ - 2 = 2 + 1 - 2 = 1.}$$

Thus the unphased operator has Fredholm index

$$\boxed{\operatorname{ind} \mathscr{L}_1 = 1.}$$

Since the translation mode gives a one-dimensional kernel,

$$\boxed{\dim\ker \mathscr{L}_1 = 1,}$$

we obtain

$$\boxed{\dim\operatorname{coker} \mathscr{L}_1 = 0.}$$

Therefore the unphased operator is surjective.

This is the crucial correction: there is no additional cokernel condition in these weighted spaces.

19 Why the Formal Adjoint Mode Does Not Produce a Cokernel

A formal adjoint solution can be constructed from the translation mode. For an equation

$$Ah = h'' + Ph' + Qh,$$

the corresponding formal adjoint is

$$A^*\psi = \psi'' - (P\psi)' + Q\psi.$$

If h solves

$$Ah = 0,$$

then the associated adjoint solution is proportional to

$$\psi = e^{\int P dx} h.$$

Using

$$e^{\int P dx} = t^{-2}(1-t)^{1/2}$$

and

$$h_1 = \frac{1}{2}\sqrt{t}(1-t),$$

we obtain

$$\psi_1 \asymp t^{-3/2}(1-t)^{3/2}.$$

Since

$$\frac{1-t}{t} = e^{-x},$$

this gives

$$\boxed{\psi_1(x) \asymp e^{-3x/2}.}$$

Formally,

$$\mathscr{L}_1^* \psi_1 = 0.$$

However,

$$\psi_1(x) \sim e^{-3x/2}$$

grows strongly as

$$x \to -\infty.$$

For

$$0 < \eta < \frac{1}{2},$$

this function does not define an admissible dual mode for the weighted range Y_η.
Therefore it does not represent a cokernel element of

$$\mathscr{L}_1 : X_\eta \to Y_\eta.$$

In particular, one must not impose the spurious condition

$$\int_{\mathbb{R}} e^{-3x/2} f(x)\, dx = 0.$$

The formal adjoint calculation is algebraically correct, but the mode is not in the relevant dual weighted space.
Hence

$$\boxed{\operatorname{coker} \mathscr{L}_1 = \{0\}.}$$

20 Phase Fixing Removes the Only Kernel

The remaining kernel is generated by translation:

$$\ker \mathscr{L}_1 = \operatorname{span}\{G_1'\}.$$

Choose a phase functional

$$\Phi(h)$$

such that

$$\boxed{\Phi(G_1') \neq 0.}$$

For example, one may fix the location of the midpoint by requiring

$$G(0) = \frac{H_*}{\sqrt{2}}.$$

At the linearized level this gives

$$h(0) = 0.$$

Since

$$G_1'(0) \neq 0,$$

the translation mode is excluded.
Define

$$X_\eta^{\mathrm{ph}} = \{h \in X_\eta : h(0) = 0\}.$$

The restriction

$$\boxed{\mathscr{L}_1 : X_\eta^{\mathrm{ph}} \longrightarrow Y_\eta}$$

therefore has

$$\dim \ker = 0.$$

The phase condition removes one dimension from the domain, so its Fredholm index becomes

$$1 - 1 = 0.$$

Since the original operator is surjective, the phase-fixed operator remains surjective.
Consequently,

$$\boxed{\mathscr{L}_1 : X_\eta^{\mathrm{ph}} \longrightarrow Y_\eta \quad \text{is an isomorphism.}}$$

Thus there exists

$$C_\eta < \infty$$

such that

$$\boxed{\|h\|_{X_\eta} \leq C_\eta \|\mathscr{L}_1 h\|_{Y_\eta}, \qquad h \in X_\eta^{\mathrm{ph}}.}$$

No additional amplitude parameter is required.

21 Direct Half-Line Verification of the Inverse

The preceding Fredholm calculation can be checked directly in the original s-variable.
At

$$\alpha = 1,$$

the profile is

$$F_1(s) = c_\nu (s + s_0)^{-1/2}.$$

The linearization of

$$-\nu F'' - F^2 F' = 0$$

is

$$L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.$$

Since

$$F_1^2 = \frac{3\nu}{2(s + s_0)},$$

we have

$$2F_1 F_1' = (F_1^2)' = -\frac{3\nu}{2(s + s_0)^2}.$$

Hence

$$L_1 h = -\nu h'' - \frac{3\nu}{2(s + s_0)} h' + \frac{3\nu}{2(s + s_0)^2} h.$$

This factors exactly as

$$\boxed{L_1 h = -\nu \left[h' + \frac{3}{2(s + s_0)} h \right]'.}$$

Consider

$$L_1 h = f.$$

Define

$$g(s) = h'(s) + \frac{3}{2(s + s_0)} h(s).$$

Then

$$-\nu g' = f.$$

Imposing

$$g(\infty) = 0$$

gives

$$\boxed{g(s) = \frac{1}{\nu} \int_s^\infty f(\tau)\, d\tau.}$$

The first-order equation

$$h' + \frac{3}{2(s+s_0)}h = g$$

has integrating factor

$$(s+s_0)^{3/2}.$$

Fix the translation phase by imposing

$$\boxed{h(0) = 0.}$$

Then

$$\left[(s+s_0)^{3/2}h(s)\right]' = (s+s_0)^{3/2}g(s).$$

Therefore

$$(s+s_0)^{3/2}h(s) = \int_0^s (\sigma+s_0)^{3/2}g(\sigma)\,d\sigma.$$

Substitution of g gives the exact inverse formula

$$\boxed{h(s) = \frac{1}{\nu(s+s_0)^{3/2}} \int_0^s (\sigma+s_0)^{3/2} \left[\int_\sigma^\infty f(\tau)\,d\tau\right] d\sigma.}$$

This proves uniqueness immediately.
Indeed, if

$$f = 0,$$

then

$$g = 0,$$

so

$$h(s) = C(s+s_0)^{-3/2}.$$

The phase condition

$$h(0) = 0$$

forces

$$C = 0.$$

Hence

$$\boxed{\ker L_1|_{\text{phase}} = \{0\}.}$$

22 Explicit Weighted Estimate

Let

$$0 < \eta < \frac{3}{2}.$$

Define

$$\boxed{\|f\|_{Y_\eta} = \sup_{s\geq 0}(s+s_0)^{2+\eta}|f(s)|.}$$

Let

$$M = \|f\|_{Y_\eta}.$$

Then

$$|f(\tau)| \leq M(\tau+s_0)^{-2-\eta}.$$

Therefore

$$\int_s^\infty |f(\tau)|\,d\tau \leq M\int_s^\infty (\tau+s_0)^{-2-\eta}\,d\tau.$$

Thus

$$\boxed{\int_s^\infty |f(\tau)|\,d\tau \leq \frac{M}{1+\eta}(s+s_0)^{-1-\eta}.}$$

Consequently,

$$|g(s)| \leq \frac{M}{\nu(1+\eta)}(s+s_0)^{-1-\eta}.$$

Using the inverse formula,

$$|h(s)| \leq \frac{M}{\nu(1+\eta)(s+s_0)^{3/2}}\int_0^s (\sigma+s_0)^{1/2-\eta}\,d\sigma.$$

For

$$0 < \eta < \frac{3}{2},$$

we obtain

$$\int_0^s (\sigma+s_0)^{1/2-\eta}\,d\sigma \leq \frac{(s+s_0)^{3/2-\eta}}{\frac{3}{2}-\eta}.$$

Hence

$$\boxed{|h(s)| \leq \frac{M}{\nu(1+\eta)(\frac{3}{2}-\eta)}(s+s_0)^{-\eta}.}$$

Furthermore,

$$h' = g - \frac{3}{2(s+s_0)}h,$$

so

$$\boxed{|h'(s)| \leq C_{\nu,\eta} M (s+s_0)^{-1-\eta}.}$$

Differentiating

$$g = h' + \frac{3}{2(s+s_0)}h$$

gives

$$g' = h'' + \frac{3}{2(s+s_0)}h' - \frac{3}{2(s+s_0)^2}h.$$

Since

$$g' = -\frac{1}{\nu}f,$$

we obtain

$$h'' = -\frac{1}{\nu}f - \frac{3}{2(s+s_0)}h' + \frac{3}{2(s+s_0)^2}h.$$

Therefore

$$\boxed{|h''(s)| \leq C_{\nu,\eta} M (s+s_0)^{-2-\eta}.}$$

Define

$$\boxed{\begin{aligned}\|h\|_{X_\eta} = & \sup_{s\geq 0}(s+s_0)^{\eta}|h(s)| \\ & + \sup_{s\geq 0}(s+s_0)^{1+\eta}|h'(s)| \\ & + \sup_{s\geq 0}(s+s_0)^{2+\eta}|h''(s)|.\end{aligned}}$$

Then

$$\boxed{\|h\|_{X_\eta} \leq C_{\nu,\eta}\|f\|_{Y_\eta}.}$$

Thus

$$\boxed{L_1 : X_\eta^{\mathrm{ph}} \longrightarrow Y_\eta}$$

has a bounded inverse.
This direct calculation independently confirms the Fredholm index calculation above.

23 Consequences for Continuation in α

The two decisive anchor properties are now separated cleanly.

First, the fractional Mellin operator satisfies

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \le C_{\delta,\sigma}|\alpha - 1|\|h\|_{H^{2+\sigma}}.}$$

Second, the phase-fixed linearization at the explicit anchor satisfies

$$\boxed{\mathscr{L}_1^{-1} : Y_\eta \longrightarrow X_\eta^{\mathrm{ph}}}$$

with

$$\boxed{\|\mathscr{L}_1^{-1} f\|_{X_\eta} \le C_\eta \|f\|_{Y_\eta}.}$$

Thus the translation degeneracy itself is completely removed by a single phase condition.

There is no additional Fredholm solvability condition and no additional amplitude parameter forced by the linearized problem.

The next issue is therefore not a cokernel problem.

The remaining issue is to place the nonlinear fractional profile map

$$\mathcal{F}(\alpha, G) = \nu \mathcal{M}_{a_\alpha} G - G^2 G' + c_\alpha G^3$$

on a common pair of Banach spaces for

$$1 \le \alpha \le 1 + \delta,$$

and to verify that the one-sided fractional realization has the required domain continuity at the endpoint

$$\alpha = 1.$$

This point is important because for

$$\alpha > 1$$

the desired left endpoint behavior is

$$\boxed{F_\alpha(s) \sim A_\alpha s^{\alpha-1}, \qquad s \downarrow 0,}$$

whereas at the explicit anchor

$$\boxed{F_1(0) = \sqrt{\frac{3\nu}{2s_0}} \neq 0.}$$

Thus the boundary trace of F_α changes as

$$\alpha \downarrow 1.$$

In the logarithmic variable this is considerably better behaved because

$$G_\alpha(x) = s^{a_\alpha} F_\alpha(s)$$

satisfies

$$\boxed{G_\alpha(-\infty) = 0}$$

for all α near 1.

Its expected left decay rate is

$$\boxed{q_-(\alpha) = 2\alpha - \frac{3}{2},}$$

which varies continuously from

$$q_-(1) = \frac{1}{2}.$$

This strongly favors carrying out the continuation theorem in the logarithmic Mellin variable G, while using the explicit s-variable inverse derived above to establish the anchor invertibility.

However, before applying the implicit-function theorem, one must verify that the precise one-sided realization of

$$(-\partial_{ss})^\alpha$$

used in the construction is compatible with this continuation through

$$\alpha = 1.$$

In particular, if the Mellin formula is implemented through the whole-line action on

$$s_+^\lambda,$$

then the exterior extension and the behavior at

$$s = 0$$

must be included explicitly in the operator domain.

Therefore the correct continuation program is

explicit anchor at $\alpha = 1$

$\Downarrow$

phase-fixed bounded inverse

$\Downarrow$

Mellin-symbol C^1 dependence on α

$\Downarrow$

common weighted operator domain

$\Downarrow$

implicit-function continuation

$\Downarrow$

$F_\alpha > 0$ and the required endpoint asymptotics

$\Downarrow$

profile existence for $1 < \alpha < 1 + \delta$.

The Fredholm obstruction is therefore removed.

The decisive remaining calculation is the common-domain and one-sided-extension analysis for the fractional operator at $\alpha = 1$. Once that is established, the phase-fixed implicit-function theorem can be applied to produce an actual interval

$$\boxed{1 < \alpha < 1 + \delta}$$

rather than a theorem at only one isolated exponent.

24 Part 7

The endpoint index count shows that the adjoint mode is not in the dual of the weighted range space. Consequently, the previous cokernel detour must be corrected: in the small-weight spaces considered below, the unphased linearized operator has Fredholm index $+1$, with the one-dimensional translation mode accounting for the kernel, while the cokernel is trivial. After fixing translation by a phase condition, the linearized operator has index zero and is the correct candidate for an isomorphism. No additional amplitude parameter is required for this purpose.

We now establish this directly at the explicit anchor $\alpha = 1$.

At $\alpha = 1$, with $\gamma = 0$, the profile equation is

$$-\nu F'' = F^2 F' = \frac{1}{3}(F^3)'.$$

Integrating with the decaying boundary condition at infinity gives

$$-\nu F' = \frac{1}{3}F^3.$$

Hence the explicit family of profiles is

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}}$$

with $s_0 > 0$.
Let

$$c_\nu = \sqrt{\frac{3\nu}{2}},$$

so that

$$F_1(s) = c_\nu (s+s_0)^{-1/2}.$$

The linearization of

$$-\nu F'' - F^2 F' = 0$$

at F_1 is

$$L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h.$$

Since

$$F_1^2 = \frac{3\nu}{2(s+s_0)},$$

and

$$(F_1^2)' = 2F_1 F_1' = -\frac{3\nu}{2(s+s_0)^2},$$

we obtain the exact divergence-form identity

$$\boxed{L_1 h = -\left(\nu h' + F_1^2 h\right)'}$$

or equivalently

$$\boxed{L_1 h = -\nu \left[h' + \frac{3}{2(s+s_0)} h\right]'}.$$

This identity makes the kernel and inverse completely explicit.
Indeed, the homogeneous equation

$$L_1 h = 0$$

is equivalent to

$$\left[h' + \frac{3}{2(s+s_0)} h\right]' = 0.$$

Therefore

$$h' + \frac{3}{2(s+s_0)}h = C_1.$$

Multiplying by the integrating factor

$$(s+s_0)^{3/2}$$

gives

$$\frac{d}{ds}\left[(s+s_0)^{3/2}h(s)\right] = C_1(s+s_0)^{3/2}.$$

Integration yields

$$(s+s_0)^{3/2}h(s) = \frac{2C_1}{5}(s+s_0)^{5/2} + C_2,$$

and hence

$$\boxed{h(s) = \frac{2C_1}{5}(s+s_0) + C_2(s+s_0)^{-3/2}.}$$

Thus the two homogeneous modes are

$$h_{\text{grow}}(s) = s + s_0$$

and

$$h_{\text{tr}}(s) = (s+s_0)^{-3/2}.$$

The second mode is precisely the translation mode, since

$$F_1'(s) = -\frac{c_\nu}{2}(s+s_0)^{-3/2}.$$

Therefore

$$\boxed{\ker L_1 = \text{span}\{F_1'\}}$$

in any profile space that excludes the growing solution $s + s_0$.
We now fix translation by imposing the phase condition

$$\boxed{h(0) = 0.}$$

Since

$$F_1'(0) = -\frac{c_\nu}{2}s_0^{-3/2} \neq 0,$$

the translation mode does not satisfy this phase condition. Hence

$$\boxed{\ker\left(L_1\big|_{h(0)=0}\right) = \{0\}.}$$

We next prove surjectivity and obtain the inverse explicitly.
Consider

$$L_1 h = f.$$

Introduce

$$g(s) = h'(s) + \frac{3}{2(s+s_0)} h(s).$$

Then

$$-\nu g'(s) = f(s),$$

so that, imposing the decaying condition

$$g(\infty) = 0,$$

we obtain

$$\boxed{g(s) = \frac{1}{\nu} \int_s^\infty f(\tau)\, d\tau.}$$

Therefore h solves

$$h' + \frac{3}{2(s+s_0)} h = \frac{1}{\nu} \int_s^\infty f(\tau)\, d\tau.$$

Multiplication by $(s+s_0)^{3/2}$ gives

$$\frac{d}{ds}\left[(s+s_0)^{3/2} h(s)\right] = \frac{(s+s_0)^{3/2}}{\nu} \int_s^\infty f(\tau)\, d\tau.$$

Using the phase condition

$$h(0) = 0,$$

we obtain

$$\boxed{(L_1^{-1} f)(s) = \frac{1}{\nu (s+s_0)^{3/2}} \int_0^s (\sigma+s_0)^{3/2} \int_\sigma^\infty f(\tau)\, d\tau\, d\sigma.}$$

Thus no solvability condition of the form

$$\int_0^\infty \psi(s) f(s)\, ds = 0$$

is required in these spaces.
To make the bounded inverse estimate explicit, fix

$$0 < \eta < \frac{3}{2}$$

and define

$$\boxed{Y_\eta = \left\{ f : \sup_{s\geq 0} (s+s_0)^{2+\eta} |f(s)| < \infty \right\}.}$$

Let

$$\|f\|_{Y_\eta} = \sup_{s\geq 0} (s+s_0)^{2+\eta} |f(s)|.$$

Set

$$M = \|f\|_{Y_\eta}.$$

Then

$$|f(\tau)| \leq M(\tau + s_0)^{-2-\eta},$$

and consequently

$$\begin{aligned} \left| \int_s^\infty f(\tau)\, d\tau \right| &\leq M \int_s^\infty (\tau + s_0)^{-2-\eta}\, d\tau \\ &= \frac{M}{1+\eta} (s+s_0)^{-1-\eta}. \end{aligned}$$

Hence

$$\boxed{|g(s)| \leq \frac{M}{\nu(1+\eta)} (s+s_0)^{-1-\eta}.}$$

Substituting this into the inverse formula gives

$$|h(s)| \leq \frac{M}{\nu(1+\eta)(s+s_0)^{3/2}} \int_0^s (\sigma + s_0)^{1/2-\eta}\, d\sigma.$$

For

$$0 < \eta < \frac{3}{2},$$

we have

$$\int_0^s (\sigma + s_0)^{1/2-\eta}\, d\sigma = \frac{(s+s_0)^{3/2-\eta} - s_0^{3/2-\eta}}{\frac{3}{2} - \eta}.$$

Therefore

$$|h(s)| \leq \frac{M}{\nu(1+\eta)(\frac{3}{2}-\eta)} (s+s_0)^{-\eta},$$

and hence

$$\boxed{\sup_{s\geq 0}(s+s_0)^{\eta}|h(s)| \leq C_{\nu,\eta}M.}$$

Since

$$h' = g - \frac{3}{2(s+s_0)}h,$$

we obtain

$$\boxed{|h'(s)| \leq C_{\nu,\eta}M(s+s_0)^{-1-\eta}.}$$

Furthermore,

$$g' = -\frac{1}{\nu}f.$$

Differentiating

$$g = h' + \frac{3}{2(s+s_0)}h$$

gives

$$-\frac{1}{\nu}f = h'' + \frac{3}{2(s+s_0)}h' - \frac{3}{2(s+s_0)^2}h.$$

Therefore

$$\boxed{h'' = -\frac{1}{\nu}f - \frac{3}{2(s+s_0)}h' + \frac{3}{2(s+s_0)^2}h.}$$

The preceding estimates imply

$$\boxed{|h''(s)| \leq C_{\nu,\eta}M(s+s_0)^{-2-\eta}.}$$

Define

$$\boxed{X_{\eta} = \left\{h : h(0) = 0, \quad \|h\|_{X_{\eta}} < \infty\right\},}$$

where

$$\boxed{\|h\|_{X_{\eta}} = \sup_{s\geq 0}(s+s_0)^{\eta}|h(s)| + \sup_{s\geq 0}(s+s_0)^{1+\eta}|h'(s)| + \sup_{s\geq 0}(s+s_0)^{2+\eta}|h''(s)|.}$$

Then

$$\boxed{\|L_1^{-1}f\|_{X_{\eta}} \leq C_{\nu,\eta}\|f\|_{Y_{\eta}}.}$$

Consequently,

$$\boxed{L_1 : X_{\eta} \longrightarrow Y_{\eta}}$$

is a bounded isomorphism.

This also resolves the apparent adjoint obstruction encountered in the logarithmic formulation.

Indeed, introduce

$$x = \log s$$

and

$$G(x) = s^{1/2} F(s)$$

at $\alpha = 1$.

For the normalized explicit profile,

$$Z_1(x) = \frac{G_1(x)}{H_*} = (1 + e^{-x})^{-1/2},$$

where

$$H_* = \sqrt{\frac{3\nu}{2}}.$$

Let

$$t(x) = Z_1(x)^2 = \frac{1}{1 + e^{-x}}.$$

Then

$$t' = t(1 - t)$$

and

$$Z_1' = \frac{1}{2} Z_1 (1 - Z_1^2) = \frac{1}{2} \sqrt{t}(1 - t).$$

The logarithmic linearization is

$$\boxed{\frac{1}{\nu} \mathscr{L}_1 h = -h'' + \left(2 - \frac{3}{2} t\right) h' + \left(-\frac{3}{4} + \frac{3}{4} t + \frac{3}{2} t^2\right) h.}$$

As

$$x \to -\infty, \qquad t \to 0,$$

the limiting equation is

$$-h'' + 2h' - \frac{3}{4} h = 0,$$

or

$$h'' - 2h' + \frac{3}{4} h = 0.$$

Its characteristic equation is

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0,$$

with roots

$$\boxed{\lambda_-^{(1)} = \frac{1}{2}, \qquad \lambda_-^{(2)} = \frac{3}{2}.}$$

Both modes decay as $x \to -\infty$.
As

$$x \to +\infty, \qquad t \to 1,$$

the limiting equation is

$$-h'' + \frac{1}{2}h' + \frac{3}{2}h = 0,$$

or

$$h'' - \frac{1}{2}h' - \frac{3}{2}h = 0.$$

Its roots are

$$\boxed{\lambda_+^{(1)} = -1, \qquad \lambda_+^{(2)} = \frac{3}{2}.}$$

Thus exactly one mode decays as $x \to +\infty$.
For a sufficiently small two-sided exponential weight

$$0 < \eta < \frac{1}{2},$$

the dimensions of the admissible asymptotic subspaces are therefore

$$d_- = 2$$

at the left endpoint and

$$d_+ = 1$$

at the right endpoint.
For a second-order scalar equation the corresponding Fredholm index is

$$\boxed{\operatorname{ind} \mathscr{L}_1 = d_- + d_+ - 2 = 2 + 1 - 2 = 1.}$$

The one-dimensional kernel is exactly the translation mode

$$\boxed{h_{\mathrm{tr}} = G_1'.}$$

Hence

$$\dim \ker \mathscr{L}_1 = 1$$

and, since the index is 1,

$$\boxed{\dim \operatorname{coker} \mathscr{L}_1 = 0.}$$

This explains why the formal adjoint mode previously obtained does not generate an actual solvability condition in the chosen weighted spaces.

In particular, the formal adjoint solution

$$\psi(x) = e^{-3x/2}$$

grows as

$$x \to -\infty.$$

For the small weights

$$0 < \eta < \frac{1}{2},$$

this function does not belong to the dual weighted space associated with the range. Therefore the functional

$$f \longmapsto \int_{\mathbb{R}} e^{-3x/2} f(x)\, dx$$

is not a bounded cokernel functional on that range space.

Thus the previously proposed condition

$$\int_{\mathbb{R}} e^{-3x/2} f(x)\, dx = 0$$

is not required.

After imposing one phase condition that removes G_1', the Fredholm index decreases by one:

$$\boxed{\operatorname{ind} \mathscr{L}_1^{\text{phase}} = 0.}$$

Since its kernel is trivial,

$$\boxed{\ker \mathscr{L}_1^{\text{phase}} = \{0\},}$$

and the cokernel is also trivial. Hence

$$\boxed{\mathscr{L}_1^{\text{phase}} : X_\eta^{\text{phase}} \longrightarrow Y_\eta}$$

is an isomorphism, provided the weighted realization is chosen consistently with the preceding endpoint asymptotics.

This is exactly consistent with the explicit half-line inverse derived above.

We can now return to the fractional problem.
For

$$1 < \alpha < 1 + \delta,$$

set

$$a_\alpha = \alpha - \frac{1}{2}$$

and

$$F(s) = s^{-a_\alpha} G(\log s).$$

The one-sided Mellin realization satisfies

$$(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda - 2\alpha},$$

where

$$\boxed{K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi} \sin \pi(\alpha - \lambda).}$$

Therefore

$$(-\partial_{ss})^\alpha F = s^{-a_\alpha - 2\alpha} \mathcal{M}_{a_\alpha} G,$$

with

$$\boxed{\mathcal{M}_{a_\alpha}(e^{\lambda x}) = K_\alpha^+(-a_\alpha + \lambda) e^{\lambda x}.}$$

The Fourier–Mellin symbol is

$$m_\alpha(\xi) = K_\alpha^+ \left(-\alpha + \frac{1}{2} + i\xi \right),$$

hence

$$\boxed{m_\alpha(\xi) = \frac{\Gamma(\frac{3}{2} - \alpha + i\xi)\Gamma(3\alpha - \frac{1}{2} - i\xi)}{\pi} \sin \pi \left(2\alpha - \frac{1}{2} - i\xi \right).}$$

At the anchor $\alpha = 1$,

$$m_1(\xi) = -\left(\frac{3}{2} - i\xi \right) \left(\frac{1}{2} - i\xi \right),$$

so

$$\boxed{m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4}.}$$

This agrees exactly with the direct logarithmic conjugation

$$F = s^{-1/2} G(\log s),$$

because

$$F'' = s^{-5/2} \left(G'' - 2G' + \frac{3}{4} G \right),$$

and therefore

$$\boxed{-\partial_{ss} F = s^{-5/2} \left(-G'' + 2G' - \frac{3}{4} G \right).}$$

Thus

$$\boxed{\mathcal{M}_{1/2} = -D^2 + 2D - \frac{3}{4}.}$$

Differentiating the Mellin symbol with respect to α gives

$$\boxed{\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi) \left[-\psi\left(\frac{3}{2} - \alpha + i\xi \right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi \right) + 2\pi \cot \pi \left(2\alpha - \frac{1}{2} - i\xi \right) \right].}$$

For fixed

$$0 < \delta < \frac{1}{4},$$

the Gamma-function arguments remain in pole-free right half-planes for

$$1 \le \alpha \le 1 + \delta.$$

Stirling estimates give

$$|m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha},$$

while the standard digamma estimate gives

$$|\partial_\alpha m_\alpha(\xi)| \le C_\delta \langle \xi \rangle^{2\alpha} \left(1 + \log \langle \xi \rangle \right).$$

Fix

$$0 < \sigma < 1.$$

If

$$0 < \delta \le \frac{\sigma}{4},$$

then

$$2\alpha \le 2 + 2\delta \le 2 + \frac{\sigma}{2},$$

and the logarithmic factor can be absorbed into an additional fractional power. Consequently,

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.}$$

By the mean-value theorem,

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \le C_{\delta,\sigma}|\alpha-1|\langle\xi\rangle^{2+\sigma}.}$$

Plancherel therefore gives

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \le C_{\delta,\sigma}|\alpha-1|\|h\|_{H^{2+\sigma}}.}$$

In particular,

$$\boxed{\alpha \longmapsto \mathcal{M}_\alpha \in \mathcal{L}\left(H^{2+\sigma}, L^2\right)}$$

is C^1 in a sufficiently small interval about $\alpha = 1$.
For $\gamma = 0$, the autonomous logarithmic profile equation is

$$\boxed{\nu\mathcal{M}_{a_\alpha}G = G^2G' - c_\alpha G^3,}$$

where

$$\boxed{c_\alpha = \frac{2\alpha-1}{2}.}$$

The desired heteroclinic satisfies

$$\boxed{G(-\infty) = 0, \qquad G(+\infty) = H_*(\alpha).}$$

At the constant right-hand equilibrium,

$$G = H_*(\alpha),$$

the profile equation gives

$$\nu K_\alpha^+(-a_\alpha)H_* = -c_\alpha H_*^3.$$

Hence

$$\boxed{H_*^2(\alpha) = -\frac{\nu K_\alpha^+(-a_\alpha)}{c_\alpha}.}$$

At $\alpha = 1$,

$$K_1^+\left(-\frac{1}{2}\right) = -\frac{3}{4},$$

and

$$c_1 = \frac{1}{2}.$$

Therefore

$$H_*^2(1) = \frac{3\nu}{2},$$

so

$$\boxed{H_*(1) = \sqrt{\frac{3\nu}{2}},}$$

exactly as required by the explicit profile.

Thus $H_*(\alpha)$ is determined by the equilibrium relation and is not an additional free amplitude parameter.

The left endpoint rate is determined by the Mellin zero

$$-a_\alpha + \lambda = \alpha - 1.$$

Since

$$a_\alpha = \alpha - \frac{1}{2},$$

we obtain

$$\boxed{q_-(\alpha) = \lambda = 2\alpha - \frac{3}{2}.}$$

At the anchor,

$$q_-(1) = \frac{1}{2},$$

which agrees with

$$Z_1(x) = (1 + e^{-x})^{-1/2} = e^{x/2} - \frac{1}{2}e^{3x/2} + \frac{3}{8}e^{5x/2} + O(e^{7x/2})$$

as

$$x \to -\infty.$$

At the right endpoint,

$$Z_1(x) = 1 - \frac{1}{2}e^{-x} + \frac{3}{8}e^{-2x} + O(e^{-3x}),$$

so the right decay rate at the anchor is

$$\boxed{r_+(1) = 1.}$$

We have therefore established the two principal analytic ingredients needed for continuation from $\alpha = 1$:

$$\boxed{\text{phase-fixed invertibility of the anchor linearization}}$$

and

$$\boxed{C^1 \text{ perturbative control of the Mellin multiplier near } \alpha = 1.}$$

There remains, however, one structural point that must be checked before invoking the Banach-space implicit-function theorem.

The Mellin identity is written for the one-sided power

$$s_+^\lambda,$$

whereas the explicit anchor

$$F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}$$

has

$$F_1(0) > 0.$$

For every $\alpha > 1$, the desired left asymptotic is instead

$$F_\alpha(s) \sim A_\alpha s^{\alpha-1}, \qquad s \downarrow 0,$$

and therefore

$$F_\alpha(0) = 0.$$

Thus the trace at $s = 0$ changes at the endpoint $\alpha = 1$.

If the notation

$$(-\partial_{ss})^\alpha$$

denotes the whole-line fractional Laplacian applied to the zero extension of F from $s > 0$, then this change in boundary trace can alter the operator domain at $\alpha = 1$. In particular, the zero extension of F_1 has a jump at $s = 0$, while the profiles for $\alpha > 1$ vanish there.

Therefore the final continuation theorem cannot yet be obtained merely by combining the two estimates above. One must first verify that the precise one-sided realization used in the manuscript admits a common Banach-space formulation across

$$\boxed{\alpha = 1.}$$

Equivalently, we must verify that the map

$$\boxed{(\alpha, G) \longmapsto \nu \mathcal{M}_{a_\alpha} G - G^2 G' + c_\alpha G^3}$$

is genuinely C^1 on a fixed weighted domain containing the explicit anchor G_1, with the manuscript's exact exterior-extension convention.

Once that domain issue is established, the phase-fixed inverse above and the Mellin-symbol perturbation estimate give precisely the local implicit-function framework required to prove

$$\boxed{\exists\, \delta > 0 \quad \text{such that} \quad 1 < \alpha < 1 + \delta}$$

admits a continuation of the explicit $\alpha = 1$ profile.

Thus the decisive next calculation is not another numerical profile computation. It is the verification of the common one-sided fractional operator domain at $\alpha = 1$, followed by the weighted C^1 estimate for the full nonlinear profile map.

25 Part 8

Consolidated small-supercritical continuation and generalized one-phase extension

This part supersedes the earlier adjoint-cokernel detour. The endpoint index count and the direct half-line inverse proved in Parts I, J, and 7 show that, for the small two-sided weights used here, the formal adjoint solution $e^{-3x/2}$ is not an element of the dual range space. Consequently no additional amplitude parameter is introduced merely to repair a nonexistent cokernel. The unphased operator has Fredholm index $+1$, its admissible kernel is the translation mode, and the phase-fixed operator has index zero.

The purpose of this part is twofold. We use identities that are genuinely uniform in α and therefore belong in an interval theorem. Second, it separates the statements that can now be proved from the estimates that still have to be supplied before a finite-time blow-up theorem can be asserted for all $1 < \alpha < 1 + \delta$.

72. Exact generalized one-phase reduction for an interval of exponents

Let

$$1 < \alpha < 1 + \delta < \frac{5}{4}$$

and let

$$s_N = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N}, \qquad u_N(x, y, z, t) = U_N(s_N).$$

Put

$$\kappa^2 = C_1^2 + C_2^2 + C_3^2.$$

For every scalar one-phase function $F(s_N)$, the torus Fourier modes lie on the one-dimensional lattice generated by (C_1, C_2, C_3). Hence the Fourier multiplier $|k|^{2\alpha}$ restricts exactly to $\kappa^{2\alpha}|n|^{2\alpha}$. Therefore

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha F(s_N) = \kappa^{2\alpha}(-\partial_{ss})^\alpha F(s_N).} \tag{246}$$

Likewise the fractional product commutator reduces exactly to its one-dimensional counterpart:

$$\boxed{\mathcal{R}_\alpha^{3D}(F(s_N), G(s_N)) = \kappa^{2\alpha}\mathcal{R}_\alpha^{1D}(F, G)(s_N).} \tag{247}$$

Thus there is no multidimensional symbol-approximation error inside the strict one-phase class. This statement is uniform for every compact interval of $\alpha > 0$ for which the multiplier is defined.

For the distinguished phase

$$C_1 = C_2 = 0, \qquad C_3 = C_4 = 1,$$

we have

$$\boxed{s_N = t - z - \theta_{*,N}}, \qquad \kappa = 1. \tag{248}$$

Along the characteristic with $\dot{z} = b_N$ and $\dot{t} = 1$,

$$\boxed{\dot{s}_N = 1 - b_N.} \tag{249}$$

This identity is purely kinematic and does not depend on the isolated value $\alpha = 9/8$.

73. Exact product equation and no-double-counting form

For

$$b = b_3 = u_x u_y,$$

the velocity equations imply

$$\boxed{b_t + u \cdot \nabla b + u_y P_x + u_x P_y + \nu\big[u_y(-\Delta)^\alpha u_x + u_x(-\Delta)^\alpha u_y\big] = u_y f_x + u_x f_y.} \tag{250}$$

With

$$\mathcal{R}_\alpha(f, g) = (-\Delta)^\alpha(fg) - f(-\Delta)^\alpha g - g(-\Delta)^\alpha f,$$

this becomes

$$\boxed{b_t + u \cdot \nabla b + u_y P_x + u_x P_y + \nu(-\Delta)^\alpha b = u_y f_x + u_x f_y + \nu\mathcal{R}_\alpha(u_x, u_y).} \tag{251}$$

Equations (250)–(251) are exact for every $\alpha > 1$ and are the correct starting point for the interval argument.

74. Interval similarity exponent and finite-time conversion

The one-phase terminal balance gives the parameter-dependent exponent

$$\boxed{r_\alpha = 2\alpha - 1.} \tag{252}$$

Accordingly the normalized terminal amplitude must be written as

$$\boxed{W_{N,\alpha}(s) = \frac{s^{2\alpha-1} b_N(s)}{p_{N,\alpha}}, \qquad b_N(s) = p_{N,\alpha} s^{-(2\alpha-1)} W_{N,\alpha}(s).} \tag{253}$$

No fixed power inherited from a single exponent should appear in the interval formulation.

The characteristic time law can now be proved once a two-sided terminal bound is available. Suppose that for some $s_0 > 0$ and constants $0 < c_- \le c_+ < \infty$ independent of the characteristic,

$$c_- s^{-r_\alpha} \le b(s) \le c_+ s^{-r_\alpha}, \qquad 0 < s < s_0,$$

with $r_\alpha = 2\alpha - 1 > 1$. Shrinking s_0 if necessary gives $b \ge 2$. Since $\dot{s} = 1 - b$,

$$\frac{1}{2} b \le b - 1 = -\dot{s} \le b.$$

Hence

$$\frac{1}{b(s)} \le -\frac{dt}{ds} \le \frac{2}{b(s)}.$$

Integrating from 0 to s gives

$$\frac{s^{r_\alpha+1}}{c_+(r_\alpha+1)} \le T_* - t(s) \le \frac{2s^{r_\alpha+1}}{c_-(r_\alpha+1)}.$$

Because $r_\alpha + 1 = 2\alpha$,

$$\boxed{T_* - t \asymp s^{2\alpha}.} \tag{254}$$

Combining with $b \asymp s^{-(2\alpha-1)}$ yields

$$\boxed{b(t) \asymp (T_* - t)^{-\frac{2\alpha-1}{2\alpha}}.} \tag{255}$$

This proves the time-conversion step uniformly on any compact α-interval on which the constants in the terminal two-sided estimate are uniform.

75. General Mellin profile equation and equilibrium

The one-dimensional profile equation relevant to the interval is

$$\boxed{\nu(-\partial_{ss})^\alpha F = F^2 F' + \frac{\gamma}{2s} F^3, \qquad s > 0.} \tag{256}$$

For the nonsingular continuation route developed in the preceding parts we set

$$\boxed{\gamma = 0.} \tag{257}$$

This avoids introducing the singular perturbation F^3/s at the endpoint $\alpha = 1$.

Let

$$a = \alpha - \frac{1}{2}, \qquad F(s) = s^{-a} G(\log s).$$

Then

$$\boxed{\nu \mathcal{M}_a G = G^2 G' - c_\alpha G^3, \qquad c_\alpha = \frac{2\alpha - 1}{2}.} \tag{258}$$

The Mellin multiplier is

$$\boxed{m_\alpha(\xi) = K_\alpha^+\left(-\alpha + \frac{1}{2} + i\xi\right),} \tag{259}$$

where

$$\boxed{K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi}\sin\big(\pi(\alpha-\lambda)\big).} \tag{260}$$

The right equilibrium is not an independent amplitude. It is fixed by

$$\boxed{H_*^2(\alpha) = -\frac{\nu K_\alpha^+(\frac{1}{2}-\alpha)}{c_\alpha}.} \tag{261}$$

For $1 < \alpha < 5/4$, the coefficient $K_\alpha^+(\frac{1}{2}-\alpha)$ is negative, so $H_*^2(\alpha) > 0$. At $\alpha = 1$ this gives

$$H_*^2(1) = \frac{3}{2}\nu,$$

in agreement with the explicit anchor.

76. Mellin-symbol perturbation theorem near one

Fix $0 < \sigma < 1$. For $\delta > 0$ sufficiently small, the Gamma arguments in (260) stay in pole-free vertical strips for $1 \le \alpha \le 1+\delta$. Differentiating the exact symbol gives

$$\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi)\left[-\psi\left(\frac{3}{2} - \alpha + i\xi\right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi\right) + 2\pi\cot\big(\pi(2\alpha - \tfrac{1}{2} - i\xi)\big)\right]. \tag{262}$$

Stirling's formula in vertical strips and the standard digamma bound imply

$$|m_\alpha(\xi)| \le C_\delta\langle\xi\rangle^{2\alpha},$$

and, after choosing $\delta \le \sigma/4$,

$$\boxed{|\partial_\alpha m_\alpha(\xi)| \le C_{\delta,\sigma}\langle\xi\rangle^{2+\sigma}.} \tag{263}$$

The mean-value theorem therefore gives

$$\boxed{|m_\alpha(\xi) - m_1(\xi)| \le C_{\delta,\sigma}|\alpha-1|\langle\xi\rangle^{2+\sigma}.} \tag{264}$$

Consequently

$$\boxed{\|(\mathcal{M}_\alpha - \mathcal{M}_1)h\|_{L^2} \le C_{\delta,\sigma}|\alpha-1|\|h\|_{H^{2+\sigma}}.} \tag{265}$$

This is a completed general-α estimate and replaces any isolated high-frequency certification at one particular exponent.

77. Exact phase-fixed inverse at the classical anchor

At $\alpha = 1$, $\gamma = 0$,

$$\boxed{F_1(s) = \sqrt{\frac{3\nu}{2(s+s_0)}}.} \tag{266}$$

The linearization is

$$L_1 h = -\nu h'' - F_1^2 h' - 2F_1 F_1' h = -\left(\nu h' + F_1^2 h\right)'. \tag{267}$$

Its homogeneous solutions are

$$(s+s_0)^{-3/2}, \qquad (s+s_0),$$

so the only decaying kernel mode is translation. Imposing $h(0) = 0$ removes that mode. For a decaying right-hand side f, the phase-fixed solution is

$$\boxed{(L_1^{-1} f)(s) = \frac{1}{\nu(s+s_0)^{3/2}} \int_0^s (\sigma+s_0)^{3/2} \int_\sigma^\infty f(\tau)\, d\tau\, d\sigma.} \tag{268}$$

For

$$Y_\eta = \left\{ f : \sup_{s\geq 0} (s+s_0)^{2+\eta} |f(s)| < \infty \right\}, \qquad 0 < \eta < \frac{3}{2},$$

and the corresponding phase-fixed space

$$X_\eta = \left\{ h : h(0) = 0,\ \sup r^\eta |h| + \sup r^{1+\eta} |h'| + \sup r^{2+\eta} |h''| < \infty \right\}, \quad r = s + s_0,$$

(268) gives

$$\boxed{\|h\|_{X_\eta} \leq C_{\nu,\eta} \|f\|_{Y_\eta}.} \tag{269}$$

Thus phase-fixed invertibility at the local anchor is proved in these half-line weighted spaces. No adjoint solvability condition is needed there.

78. Operator-domain audit: the remaining obstruction to a literal IFT through $\alpha = 1$

We fix the one-sided power by

$$s_+^\mu = \begin{cases} s^\mu, & s > 0, \\ 0, & s < 0, \end{cases}$$

and uses the whole-line Fourier multiplier $|\xi|^{2\alpha}$ to define its fractional action. Its finite-interval construction likewise prescribes exterior data and solves in $H^\alpha(\mathbb{R})$ with the correction equal to zero outside the interval. Therefore the continuation theorem must respect a whole-line exterior-data realization; it cannot silently replace it by a different half-line operator.

This creates a genuine endpoint-domain issue. The explicit anchor satisfies

$$F_1(0^+) = \sqrt{\frac{3\nu}{2s_0}} > 0.$$

If it is extended by zero to $s < 0$, it has a jump at the origin. At the integer endpoint $\alpha = 1$, the distributional second derivative of this zero extension contains boundary distributions. Hence the pointwise ODE

$$-\nu F_1'' = F_1^2 F_1' \qquad (s > 0)$$

does not by itself prove that the zero-extended F_1 is an element of the same whole-line operator domain used for $\alpha > 1$.

Accordingly the following implication is *not* yet justified:

phase-fixed inverse at $\alpha = 1+$ multiplier continuity $\implies$ Banach IFT in one-sided domain.

A correct interval theorem must first prove one of the following equivalent kinds of domain-compatibility statement: construct a renormalized exterior extension whose operator graph varies continuously at $\alpha = 1$; subtract an explicit boundary singular part and apply the IFT to the remainder; or formulate the nonlinear map in a Mellin graph space in which the endpoint anchor and all nearby profiles lie in one common domain.

This is a sharper conclusion than the earlier statement that only a spectral inverse was missing: the spectral inverse is now explicit, while the common operator domain is the remaining profile-level analytic obstruction.

79. General finite-N reconstruction statement that survives the interval

We have a useful general principle that does not depend on an isolated exponent. If a terminal fractional coefficient has a strict sign in the limiting one-phase model and the finite-N reconstruction admits uniform localization and separation estimates, then that sign survives for all sufficiently large N. Abstractly, if

$$D_\alpha \le -d_0 < 0$$

uniformly for $\alpha \in [1, 1+\delta]$ and

$$|D_{N,\alpha} - D_\alpha| \le \varepsilon_N, \qquad \varepsilon_N \to 0$$

uniformly in the same interval, then for all sufficiently large N,

$$\boxed{D_{N,\alpha} \le -\frac{d_0}{2} < 0.} \tag{270}$$

The proof is immediate from

$$D_{N,\alpha} \le D_\alpha + \varepsilon_N \le -d_0 + \varepsilon_N.$$

This lemma may be used wherever the detailed reconstruction supplies the uniform approximation. It does not itself prove the reconstruction estimate; that estimate remains a separate finite-N obligation.

80. Similarity-entry and Euler-derivative closure: correct interval formulation

A Region-I/Region-II architecture remains relevant, but its fixed-exponent constants must be replaced by quantities uniform in α. Define a similarity defect $\mathfrak{d}_{N,\alpha}$ measuring the distance of the normalized solution from the continued profile. The required entry theorem has the form

$$\boxed{\mathfrak{d}_{N,\alpha}(t_e) \le C\big(s_e^{\mu} + \varepsilon_N + p_{N,\alpha}^{-1}\big), \qquad \mu > 0,} \tag{271}$$

with C and μ independent of $\alpha \in (1, 1+\delta)$ and large N.

For derivative closure introduce the Euler operator

$$E = s\partial_s, \qquad q = -s\frac{b_s}{b}.$$

A direct calculation gives the exact identity

$$\boxed{\frac{E^4 b}{b} = q^4 - 6q^2(Eq) + 4q(E^2 q) + 3(Eq)^2 - E^3 q.} \tag{272}$$

Hence uniform bounds on

$$q,\ Eq,\ E^2 q,\ E^3 q$$

imply

$$\boxed{s^4 |b_{ssss}| \le Cb.} \tag{273}$$

This identity is exponent-independent and is therefore legitimately imported into the interval manuscript. What remains is to prove the required uniform bounds for the actual finite-N solution rather than for the formal profile alone.

81. Uniform conditional interval blow-up theorem

The preceding results allow the outstanding scalar theorem to be stated without any isolated exponent.

Theorem 2 (Uniform conditional one-phase scalar theorem)**.** *Let $0 < \delta < 1/4$ and $\alpha \in (1, 1+\delta)$. Assume, uniformly in α and sufficiently large N, that:*

1. *the exact reconstructed scalar equation and the distinguished one-phase characteristic law $\dot{s}_N = 1 - b_N$ hold;*
2. *a positive pre-entry seed is produced;*
3. *a positive phase-normalized profile exists in a common operator domain and depends continuously on α;*
4. *the similarity-entry estimate* (271) *holds;*
5. *the similarity-error and Euler-derivative bootstrap closes with a positive uniform decay exponent;*

6. *the favorable terminal coefficient has a uniform positive gap;*

7. *the regularized solutions converge strongly enough to pass the quadratic transport and fractional terms to the limit.*

Then there exist constants $c_-, c_+ > 0$, uniform after possibly shrinking δ, such that along the terminal characteristic

$$c_- s^{-(2\alpha-1)} \le b_3(s) \le c_+ s^{-(2\alpha-1)}.$$

Consequently the characteristic reaches $s = 0$ in finite time and

$$\boxed{b_3(t) \asymp (T_* - t)^{-\frac{2\alpha-1}{2\alpha}}.}$$

Proof. The entry and bootstrap assumptions place the normalized amplitude in the terminal invariant region and keep it there. The uniform coefficient gap and remainder bound give two-sided positive bounds on the normalized amplitude, which are equivalent through (253) to $b_3(s) \asymp s^{-(2\alpha-1)}$. Equation (249) then gives the finite-time conversion proved in (254)–(255). Strong convergence passes the scalar law to the limiting object in the topology assumed in item 7. □

82. What is proved, what is imported, and what remains

The following pieces are now analytic and genuinely interval-valued:

$$\boxed{\begin{array}{c}\text{exact product/commutator algebra,}\\ \text{exact strict one-phase fractional reduction,}\\ r_\alpha = 2\alpha - 1,\\ \dot{s} = 1 - b,\\ \text{Mellin symbol and } O(|\alpha-1|) \text{ perturbation bound,}\\ \text{explicit phase-fixed inverse at } \alpha = 1,\\ \text{finite-time conversion } b(s) \asymp s^{-(2\alpha-1)} \Rightarrow b(t) \asymp (T_* - t)^{-(2\alpha-1)/(2\alpha)},\\ \text{Euler-derivative identity (272).}\end{array}} \tag{274}$$

The following statements are *not* completed:

1. a common one-sided fractional operator domain through $\alpha = 1$ and therefore the actual Banach-space IFT producing a numerical or analytic $\delta_{\text{prof}} > 0$;

2. the uniform similarity-entry estimate for the finite-N reconstructed solution;

3. the uniform similarity-error/Euler bootstrap;

4. the uniform positive terminal coefficient gap in the interval normalization;

5. the strong compactness/limit theorem needed to pass all nonlinear and fractional terms;

6. for a full velocity-pressure theorem, exact reconstruction of all three velocity components, pressure compatibility in every momentum equation, and convergence to fixed initial data with admissible forcing.

Thus the isolated profile-certification machinery at one special exponent is not needed for the small-supercritical strategy, but it cannot simply be replaced by an unconditional interval theorem until the common-domain and finite-N uniform estimates above are proved.

83. Recommended theorem architecture

We have the following order first prove the exact generalized one-phase identities (246)–(251). Next establish the profile continuation in a common one-sided operator domain using the explicit anchor, the phase-fixed inverse, and the Mellin perturbation estimate (264)–(265). Then propagate the same $\alpha \in (1, 1+\delta)$ through the similarity-entry and Euler-bootstrap estimates. Only after obtaining a uniform coefficient gap and the strong limiting theorem should the conditional theorem in Section 81 be promoted to an unconditional interval blow-up theorem.

26 Part 9: Common Mellin graph domain through $\alpha = 1$

84. Purpose of the common-domain argument

The preceding analysis isolated one profile-level obstruction: the physical half-line representatives have different traces at the left endpoint,

$$F_1(0^+) > 0, \qquad F_\alpha(0^+) = 0 \quad (\alpha > 1),$$

so a literal zero-extension Sobolev domain in the variable s is not a convenient fixed Banach domain through $\alpha = 1$. This does not, however, force the logarithmically conjugated Mellin operator to change domain. The correct fixed-domain formulation is obtained after the conjugation

$$F(s) = s^{-a_\alpha} G(\log s), \qquad a_\alpha = \alpha - \frac{1}{2}.$$

The point of this section is to prove that the Mellin realization itself has a common graph domain near $\alpha = 1$ and depends C^1 on α there. This is the precise statement required by the Banach-space continuation argument.

We emphasize the realization being used. On the Mellin core generated by powers, the one-sided operator is characterized by

$$(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda - 2\alpha},$$

where

$$K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi} \sin \pi(\alpha - \lambda).$$

After logarithmic conjugation this becomes the Fourier multiplier with symbol

$$m_\alpha(\xi) = K_\alpha^+\left(-\alpha + \frac{1}{2} + i\xi\right).$$

The common-domain theorem below concerns the closure of this Mellin realization. It does not identify it with a different restricted, zero-extension, spectral, or censored fractional Laplacian on a bounded interval. That distinction is essential because those realizations encode boundary data differently.

85. Fixed graph space

Fix

$$0 < \sigma < 1$$

and choose

$$0 < \delta_0 < \min\left\{\frac{1}{4}, \frac{\sigma}{4}\right\}.$$

Set

$$X_\sigma := H^{2+\sigma}(\mathbb{R}), \qquad Y := L^2(\mathbb{R}).$$

For $\alpha \in [1, 1+\delta_0]$ define initially on Schwartz functions

$$\widehat{\mathcal{M}_\alpha h}(\xi) := m_\alpha(\xi)\widehat{h}(\xi).$$

The symbol estimates already proved give

$$|m_\alpha(\xi)| \le C\langle\xi\rangle^{2\alpha} \le C\langle\xi\rangle^{2+\sigma/2} \le C\langle\xi\rangle^{2+\sigma}.$$

Consequently

$$\boxed{\|\mathcal{M}_\alpha h\|_{L^2} \le C\|h\|_{H^{2+\sigma}}, \qquad 1 \le \alpha \le 1+\delta_0.} \tag{282}$$

Thus every $\mathcal{M}_\alpha$ extends uniquely to a bounded map

$$\boxed{\mathcal{M}_\alpha : X_\sigma \longrightarrow Y} \tag{283}$$

on the same domain X_σ.

This already removes the apparent domain jump at the level at which the IFT is actually formulated: the varying trace of F_α is absorbed into the α-dependent conjugation and endpoint template, while the perturbation variable is placed in one fixed logarithmic graph space.

86. Closed realization and equivalence of graph norms

For each fixed $\alpha \in [1, 1+\delta_0]$, Stirling's formula also gives the large-frequency lower bound

$$|m_\alpha(\xi)| \ge c\langle\xi\rangle^{2\alpha} \qquad (|\xi| \ge R),$$

uniformly after increasing R and decreasing $c > 0$ if necessary. Therefore

$$\|h\|_{H^{2\alpha}} \le C\big(\|h\|_{L^2} + \|\mathcal{M}_\alpha h\|_{L^2}\big), \tag{284}$$

while the reverse inequality follows from the upper symbol bound. Hence the maximal L^2 multiplier realization has domain $H^{2\alpha}$ and is closed. For continuation we deliberately use the stronger common core $H^{2+\sigma}$, since

$$2\alpha \le 2 + \frac{\sigma}{2} < 2 + \sigma.$$

In particular, no loss of closedness occurs by evaluating the nonlinear map on X_σ.

At the anchor,

$$m_1(\xi) = \xi^2 + 2i\xi - \frac{3}{4},$$

so

$$\boxed{\mathcal{M}_1 = -\partial_x^2 + 2\partial_x - \frac{3}{4}} \tag{285}$$

on $H^{2+\sigma}$, exactly agreeing with the direct logarithmic calculation.

87. C^1 dependence on the fixed domain

The exact derivative formula is

$$\partial_\alpha m_\alpha(\xi) = m_\alpha(\xi)\left[-\psi\left(\frac{3}{2} - \alpha + i\xi\right) + 3\psi\left(3\alpha - \frac{1}{2} - i\xi\right) + 2\pi\cot\big(\pi(2\alpha - \tfrac{1}{2} - i\xi)\big)\right].$$

The Gamma arguments remain in pole-free vertical strips for $1 \le \alpha \le 1 + \delta_0$, and the sine factor remains a positive distance from its real zero at $\alpha = 5/4$. The digamma and cotangent estimates then give

$$|\partial_\alpha m_\alpha(\xi)| \le C_{\sigma,\delta_0}\langle\xi\rangle^{2+\sigma}.$$

Dominated convergence in Fourier space yields

$$\boxed{\alpha \longmapsto \mathcal{M}_\alpha \in \mathcal{L}(X_\sigma, Y) \quad \text{is } C^1.} \tag{286}$$

Moreover,

$$\boxed{\|\mathcal{M}_\alpha - \mathcal{M}_1\|_{X_\sigma \to Y} \le C_{\sigma,\delta_0}|\alpha - 1|.} \tag{287}$$

Thus the common-domain issue is settled for the Mellin realization used by the profile equation.

88. Why the changing physical trace is not a contradiction

The physical profile is reconstructed by

$$F_\alpha(s) = s^{-a_\alpha} G_\alpha(\log s).$$

The left Mellin root is

$$q_-(\alpha) = 2\alpha - \frac{3}{2},$$

so if

$$G_\alpha(x) \sim A_\alpha e^{q_-(\alpha)x} \qquad (x \to -\infty),$$

then

$$F_\alpha(s) \sim A_\alpha s^{q_-(\alpha)-a_\alpha} = A_\alpha s^{\alpha-1}.$$

At $\alpha = 1$ this exponent is zero and the anchor has a nonzero finite trace; for every $\alpha > 1$ it vanishes at the endpoint. The discontinuity is thus a discontinuity of the *physical trace map* under an α-dependent conjugation, not a discontinuity of the logarithmic multiplier domain. The IFT need not place all F_α in a fixed zero-extension trace space; it places the logarithmic perturbations in a fixed Mellin graph space.

This observation also explains why directly zero-extending the classical anchor is the wrong test for the continuation map. A jump of that extension can create boundary distributions for the local second derivative, but those distributions are not part of the Mellin closure characterized by (282)–(287). The continuation theorem must therefore state explicitly that its one-sided fractional operator is the Mellin realization determined by K_α^+.

89. Affine heteroclinic space with fixed perturbation domain

The profile itself is not in $L^2(\mathbb{R})$ because it approaches the nonzero equilibrium $H_*(\alpha)$ as $x \to +\infty$. We therefore separate the endpoint data from the unknown perturbation.

Choose a smooth cutoff χ with

$$\chi(x) = 0 \quad (x \le -1), \qquad \chi(x) = 1 \quad (x \ge 1),$$

and define the endpoint template

$$\overline{G}_\alpha(x) = A_\alpha e^{q_-(\alpha)x}(1 - \chi(x)) + H_*(\alpha)\chi(x). \tag{288}$$

The amplitudes $A_\alpha > 0$ and $H_*(\alpha) > 0$ are fixed by the chosen phase normalization and the equilibrium relation. Near $\alpha = 1$ they are smooth functions of α; for H_* this follows directly from

$$H_*^2(\alpha) = -\frac{\nu K_\alpha^+(\frac{1}{2} - \alpha)}{c_\alpha}, \qquad c_\alpha = \frac{2\alpha - 1}{2}$$

for the $\gamma = 0$ continuation route.

Write

$$G = \overline{G}_\alpha + h.$$

For a small two-sided exponential weight $0 < \eta < 1/2$, let

$$X_{\eta,\sigma} = \{h : \omega_\eta h \in H^{2+\sigma}(\mathbb{R})\}, \qquad Y_\eta = \{f : \omega_\eta f \in L^2(\mathbb{R})\}. \tag{289}$$

The endpoint rates at $\alpha = 1$ are $1/2$ on the left and 1 on the right. Hence, after shrinking δ_0, one may choose one fixed η satisfying

$$0 < \eta < \frac{1}{2}$$

for every $\alpha \in [1, 1+\delta_0]$.

Conjugation by the exponential weight shifts the Fourier variable into a fixed complex strip. The Gamma arguments remain uniformly away from poles and the sine factor remains away from its zeros after decreasing η and δ_0 if necessary. Repeating the preceding symbol estimates on the shifted lines gives

$$\boxed{\|\mathcal{M}_\alpha - \mathcal{M}_1\|_{X_{\eta,\sigma}\to Y_\eta} \le C_{\eta,\sigma}|\alpha - 1|.} \tag{290}$$

For a smooth two-sided weight, commutators supported in the transition region are lower-order bounded operators; equivalently one may use separate one-sided exponential charts and a partition of unity. Thus the fixed weighted graph domain is compatible with the heteroclinic endpoint conditions.

90. Nonlinear map on the common domain

For $\gamma = 0$ define

$$\mathscr{F}(\alpha, h) := \nu\mathcal{M}_\alpha(\overline{G}_\alpha + h) - (\overline{G}_\alpha + h)^2(\overline{G}_\alpha + h)' + c_\alpha(\overline{G}_\alpha + h)^3. \tag{291}$$

The template is chosen only to encode the endpoints; its residual is an element of Y_η. In one dimension, $H^{2+\sigma}$ is a Banach algebra and embeds into C^1_b. Therefore multiplication and differentiation give

$$h \mapsto (\overline{G}_\alpha + h)^2(\overline{G}_\alpha + h)'$$

and

$$h \mapsto (\overline{G}_\alpha + h)^3$$

as C^1 maps from a neighborhood in $X_{\eta,\sigma}$ into Y_η. Together with (286) and (290),

$$\boxed{\mathscr{F} : [1, 1+\delta_0] \times X_{\eta,\sigma} \longrightarrow Y_\eta \quad \text{is } C^1.} \tag{292}$$

Add a phase functional $\Phi(h) = 0$ with

$$D\Phi(0)[G_1'] \neq 0.$$

The full map

$$\mathbb{F}(\alpha, h) = \big(\mathscr{F}(\alpha, h), \Phi(h)\big)$$

is therefore a C^1 map on fixed Banach spaces.

91. Phase-fixed derivative at the anchor

At $\alpha = 1$ the derivative in h is precisely the explicit linearized operator obtained earlier,

$$\mathscr{L}_1 h = \nu\left(-h'' + 2h' - \frac{3}{4}h\right) - 2G_1G_1'h - G_1^2h' + \frac{3}{2}G_1^2h. \tag{293}$$

The direct half-line factorization

$$L_1 h = -\nu\left[h' + \frac{3}{2(s+s_0)}h\right]'$$

and its explicit inverse show that, after the phase condition removes the translation mode, the anchor derivative is an isomorphism between the chosen weighted domain and range. The endpoint index count is consistent with this: the unphased operator has index $+1$, one-dimensional translation kernel, and trivial cokernel; phase fixing lowers the index to zero. No additional amplitude unfolding is required.

Denote the phase-fixed inverse norm by

$$C_0 := \|D_h\mathbb{F}(1,0)^{-1}\|. \tag{294}$$

The exact Green representation and the direct s-variable inverse give $C_0 < \infty$ for every fixed admissible weight.

92. Local profile continuation theorem

We can now state the consequence without the former common-domain caveat.

Theorem (small-supercritical Mellin-profile continuation). *Fix $\nu > 0$, take $\gamma = 0$, and use the one-sided Mellin realization specified by K_α^+. There exist $\eta \in (0, 1/2)$ and $\delta_{\mathrm{prof}} > 0$ such that for every*

$$1 \le \alpha < 1 + \delta_{\mathrm{prof}}$$

there is a unique phase-normalized profile G_α in a neighborhood of the explicit anchor G_1 satisfying

$$\nu\mathcal{M}_\alpha G_\alpha = G_\alpha^2 G_\alpha' - c_\alpha G_\alpha^3, \qquad c_\alpha = \frac{2\alpha - 1}{2},$$

with

$$G_\alpha(-\infty) = 0, \qquad G_\alpha(+\infty) = H_*(\alpha) > 0.$$

Moreover the phase-normalized branch is C^1 in α and

$$\|G_\alpha - G_1\|_{X_{\eta,\sigma}} \le C|\alpha - 1|.$$

After shrinking δ_{prof} if necessary, $G_\alpha(x) > 0$ for all $x \in \mathbb{R}$.

Proof. The common weighted domain and C^1 dependence of the fractional part are (286)–(292). The phase-fixed derivative at the anchor is an isomorphism by (293)–(294) and the explicit inverse established earlier. The Banach-space implicit-function theorem therefore gives a unique C^1 branch for $|\alpha - 1| < \delta_{\mathrm{prof}}$. The endpoint template gives the prescribed limits. Positivity follows by combining: (i) the positive left asymptotic coefficient, (ii) uniform closeness to the strictly positive anchor on a fixed compact interval, and (iii) closeness to the positive equilibrium on the right tail. These three regions cover $\mathbb{R}$ after decreasing δ_{prof}. □

93. Quantitative lower bound for the continuation interval

The IFT proof is constructive at the level of constants. Let C_0 be as in (294), and let C_{lin} satisfy

$$\|D_h\mathbb{F}(\alpha, 0) - D_h\mathbb{F}(1, 0)\| \le C_{\mathrm{lin}}|\alpha - 1|.$$

Let C_{nl} be a Lipschitz constant for the derivative with respect to h on a ball of radius ρ around the anchor. A sufficient choice is any $\delta_{\mathrm{prof}} > 0$ for which

$$C_0 C_{\mathrm{lin}} \delta_{\mathrm{prof}} \le \frac{1}{4}, \qquad C_0 C_{\mathrm{nl}} \rho \le \frac{1}{4}, \tag{295}$$

and the residual generated by the α-variation obeys

$$\|\mathbb{F}(\alpha, 0)\| \le \frac{\rho}{2C_0} \qquad (1 \le \alpha \le 1 + \delta_{\mathrm{prof}}). \tag{296}$$

Hence the analysis proves existence of a strictly positive interval $\delta_{\mathrm{prof}} > 0$. Producing a sharp decimal value requires explicit numerical upper bounds for $C_0, C_{\mathrm{lin}}, C_{\mathrm{nl}}$ and is not needed for the qualitative interval theorem.

94. Consequence for the remaining manuscript ledger

The profile-level common-domain item in Section 82 is now closed, provided the fractional profile operator is understood in the Mellin realization explicitly specified above. The outstanding work is therefore no longer profile existence near $\alpha = 1$. What remains for the full finite-time theorem is uniform control of the passage from the reconstructed finite-N dynamics to the profile regime:

1. the uniform similarity-entry estimate;
2. the similarity-error/Euler bootstrap;
3. a uniform positive terminal coefficient gap;
4. strong compactness sufficient to pass the nonlinear and fractional terms;
5. and, for a full three-component Navier–Stokes statement, the remaining velocity-pressure reconstruction and compatibility estimates.

Thus the small-supercritical strategy has advanced from a conditional profile-continuation program to an actual local profile theorem on a nonempty interval

$$\boxed{1 < \alpha < 1 + \delta_{\mathrm{prof}}.}$$

The theorem is deliberately not stated for a restricted zero-extension fractional Laplacian on a bounded interval; changing to such a realization would require a separate boundary-domain analysis and would define a different continuation problem.

95. Revised theorem architecture

We should use the following logical order. The exact generalized one-phase identities and characteristic reduction come first. The Mellin profile theorem above then supplies the positive connector uniformly for α in a nonempty interval above one. The terminal scaling law

$$r_\alpha = 2\alpha - 1$$

and time conversion can then be applied uniformly on that interval. The only remaining conditional hypotheses in the interval blow-up theorem are the finite-N entry, bootstrap, coefficient-gap, and limiting estimates listed in Section 94. No isolated-α profile certification is used in this architecture.

27 Part 10: Uniform finite-N entry on the small-supercritical interval

96. Interval notation and the uniformity problem

Fix once and for all a compact continuation interval

$$I_\delta = [1, 1+\delta], \qquad 0 < \delta \le \delta_{\mathrm{prof}},$$

where the positive phase-normalized profile branch from Part 9 exists. Put

$$r_\alpha = 2\alpha - 1, \qquad m_\alpha = r_\alpha + 1 = 2\alpha. \tag{297}$$

Then

$$1 \le r_\alpha \le 1 + 2\delta, \qquad 2 \le m_\alpha \le 2 + 2\delta. \tag{298}$$

All constants in Parts 10–13 are required to be independent of $\alpha \in I_\delta$ and of all sufficiently large N unless dependence is explicitly displayed.

Let $p_{N,\alpha} \to \infty$ denote the finite-N terminal amplitude scale and let X be the Region-I logarithmic coordinate. Assume that the exact finite-N characteristic reduction supplies, on Region I,

$$\frac{db_{N,\alpha}}{dX} \ge (1-\theta_0)\frac{b_{N,\alpha}^2}{h_1 + \gamma_{N,\alpha} b_{N,\alpha}}, \qquad 0 < \theta_0 < 1, \tag{299}$$

with

$$0 < h_0 \le H_{N,\alpha} \le h_1, \qquad 0 \le \gamma_{N,\alpha} \le \frac{C_\gamma}{p_{N,\alpha}}. \tag{300}$$

The point of the present section is that the bounded-excursion part of the entry argument is already uniform in α once (299)–(300) are uniform. No isolated value of α enters the proof.

97. Uniform bounded logarithmic excursion

Let

$$a_0 = 1 - \theta_0 > 0$$

and define

$$\Phi_{N,\alpha}(b) = \frac{h_1}{a_0}\left(\frac{1}{b_-} - \frac{1}{b}\right) + \frac{\gamma_{N,\alpha}}{a_0}\log\frac{b}{b_-}. \tag{301}$$

Then

$$\Phi'_{N,\alpha}(b) = \frac{h_1 + \gamma_{N,\alpha} b}{a_0 b^2}.$$

Combining this identity with (299) gives

$$\frac{d}{dX}\Phi_{N,\alpha}(b_{N,\alpha}(X)) \geq 1. \tag{302}$$

Let X_e be the first point at which

$$b_{N,\alpha}(X_e) = Mp_{N,\alpha}, \qquad M > 1. \tag{303}$$

Integration of (302) yields

$$\boxed{X_e - X_- \leq \frac{h_1}{a_0}\left(\frac{1}{b_-} - \frac{1}{Mp_{N,\alpha}}\right) + \frac{\gamma_{N,\alpha}}{a_0}\log\frac{Mp_{N,\alpha}}{b_-}.} \tag{304}$$

Using (300),

$$0 \leq \frac{\gamma_{N,\alpha}}{a_0}\log\frac{Mp_{N,\alpha}}{b_-} \leq \frac{C_\gamma}{a_0 p_{N,\alpha}}\log\frac{Mp_{N,\alpha}}{b_-}. \tag{305}$$

If

$$\inf_{\alpha\in I_\delta} p_{N,\alpha} \longrightarrow \infty, \tag{306}$$

then the right-hand side of (305) tends to zero uniformly in α. Consequently

$$\boxed{\sup_{\alpha\in I_\delta}(X_e - X_-) \leq L_* + o_N(1), \qquad L_* := \frac{h_1}{(1-\theta_0)b_-}.} \tag{307}$$

This closes the *bounded-excursion component* of the finite-N entry problem uniformly on the continuation interval.

98. Entry before Region-I exit

Assume Region I is known to remain valid for

$$0 \leq X - X_- \leq L_R$$

with a strict uniform margin

$$\boxed{L_R - L_* = \ell_0 > 0.} \tag{308}$$

By (307), choose N_0 so large that

$$\sup_{\alpha\in I_\delta}|o_N(1)| < \frac{\ell_0}{2} \qquad (N \geq N_0).$$

Then

$$X_e - X_- < L_* + \frac{\ell_0}{2} < L_R \tag{309}$$

for every $\alpha \in I_\delta$ and $N \geq N_0$. Hence

$$\boxed{\text{the amplitude } b_{N,\alpha} = Mp_{N,\alpha} \text{ is reached before Region-I exit, uniformly in } \alpha.} \tag{310}$$

This part of the former "finite-N entry hypothesis" is therefore a theorem under the explicit uniform Region-I inequalities (299)–(300), (306), and (308).

99. What remains in the entry bridge

The conclusion (310) is an amplitude-entry statement, not yet a profile-entry statement. Define

$$W_{N,\alpha}(s) = \frac{s^{r_\alpha} b_{N,\alpha}(s)}{p_{N,\alpha}} \tag{311}$$

and let B_α denote the continued positive similarity profile. A natural interval defect is

$$\mathfrak{d}_{N,\alpha}(t) = \|W_{N,\alpha}(t) - B_\alpha\|_{\mathcal{X}} + \mathfrak{r}_{N,\alpha}(t), \tag{312}$$

where $\mathfrak{r}_{N,\alpha}$ contains the reconstructed-variable norms. The genuinely remaining entry estimate is

$$\boxed{\mathfrak{d}_{N,\alpha}(t_e) \le C_{\mathrm{ent}} \left(s_e^{\mu_0} + \varepsilon_N + p_{N,\alpha}^{-1} \right), \qquad \mu_0 > 0,} \tag{313}$$

uniformly in α. We have the architecture leading to (313), but not an interval-uniform proof of (313). Thus Part 10 closes the logarithmic-excursion and Region-I-exit portions of entry, while isolating (313) as the only nonalgebraic profile-entry estimate.

28 Part 11: Uniform terminal bootstrap and coefficient gap

100. Uniform terminal equation

Write the normalized finite-N terminal equation in the form

$$\partial_\tau W_{N,\alpha} = \mathcal{A}_\alpha(W_{N,\alpha}) + \mathcal{E}_{N,\alpha}, \tag{314}$$

where $\tau = -\log s$, $\mathcal{A}_\alpha$ is the autonomous similarity vector field having equilibrium/profile B_α, and $\mathcal{E}_{N,\alpha}$ contains reconstruction, finite-N, transverse, and cutoff errors. Put

$$V_{N,\alpha} = W_{N,\alpha} - B_\alpha. \tag{315}$$

Taylor expansion about the profile gives

$$\partial_\tau V_{N,\alpha} = L_\alpha V_{N,\alpha} + \mathcal{N}_\alpha(V_{N,\alpha}) + \mathcal{E}_{N,\alpha}, \tag{316}$$

where

$$\|\mathcal{N}_\alpha(V)\|_{\mathcal{X}} \le C_Q \|V\|_{\mathcal{X}}^2 \tag{317}$$

in a fixed ball, after decreasing δ if necessary. The C^1 profile branch from Part 9 makes C_Q uniform on I_δ.

101. Perturbative uniform spectral gap lemma

Suppose the phase-fixed anchor semigroup satisfies

$$\|e^{\tau L_1}\|_{\mathcal{X}\to\mathcal{X}} \le M_0 e^{-\lambda_0 \tau}, \qquad \tau \ge 0, \tag{318}$$

for some $\lambda_0 > 0$. The C^1 operator dependence proved in Parts 7–9 gives

$$\|L_\alpha - L_1\|_{\mathcal{X}\to\mathcal{X}} \le C_L|\alpha - 1|. \tag{319}$$

Duhamel's formula yields

$$e^{\tau L_\alpha} = e^{\tau L_1} + \int_0^\tau e^{(\tau-\sigma)L_1}(L_\alpha - L_1)e^{\sigma L_\alpha}\,d\sigma.$$

A standard Gronwall estimate therefore gives, whenever

$$M_0 C_L \delta \le \frac{\lambda_0}{2}, \tag{320}$$

$$\boxed{\|e^{\tau L_\alpha}\|_{\mathcal{X}\to\mathcal{X}} \le M_0 e^{-\lambda_* \tau}, \qquad \lambda_* := \frac{\lambda_0}{2}, \quad \alpha \in I_\delta.} \tag{321}$$

Thus any verified phase-fixed spectral gap at the explicit anchor persists on a sufficiently small nonempty α-interval. Equation (321) is the uniform linear input needed for the nonlinear bootstrap.

102. Nonlinear bootstrap lemma

Assume

$$\|\mathcal{E}_{N,\alpha}(\tau)\|_{\mathcal{X}} \le C_E e^{-\mu\tau} + \epsilon_N, \qquad \epsilon_N \to 0, \tag{322}$$

uniformly in α, with $\mu > 0$. Variation of constants in (316) gives

$$\begin{aligned}\|V(\tau)\|_{\mathcal{X}} \le{}& M_0 e^{-\lambda_*(\tau-\tau_e)}\|V(\tau_e)\|_{\mathcal{X}} \\ &+ M_0 C_Q \int_{\tau_e}^\tau e^{-\lambda_*(\tau-\sigma)}\|V(\sigma)\|_{\mathcal{X}}^2\,d\sigma \\ &+ M_0 \int_{\tau_e}^\tau e^{-\lambda_*(\tau-\sigma)}(C_E e^{-\mu\sigma} + \epsilon_N)\,d\sigma.\end{aligned} \tag{323}$$

Let

$$\lambda_b = \frac{1}{2}\min\{\lambda_*, \mu\}. \tag{324}$$

For a bootstrap assumption

$$\|V(\tau)\|_{\mathcal{X}} \le 2Ae^{-\lambda_b(\tau-\tau_e)} + 2A\epsilon_N, \tag{325}$$

the quadratic integral in (323) is bounded by

$$C(M_0, C_Q, \lambda_b)A^2\left(e^{-\lambda_b(\tau-\tau_e)} + \epsilon_N\right). \tag{326}$$

Hence there exist $A_0 > 0$, $\epsilon_0 > 0$ and $\varepsilon_{\mathrm{ent}} > 0$, depending only on the uniform constants, such that if

$$\|V(\tau_e)\|_{\mathcal{X}} \le \varepsilon_{\mathrm{ent}}, \qquad \epsilon_N \le \epsilon_0, \tag{327}$$

then the factor 2 in (325) improves to 1. By continuity,

$$\boxed{\|V_{N,\alpha}(\tau)\|_{\mathcal{X}} \le A_0 e^{-\lambda_b(\tau-\tau_e)} + A_0\epsilon_N, \qquad \tau \ge \tau_e,} \tag{328}$$

uniformly in $\alpha \in I_\delta$.

Thus the nonlinear bootstrap is reduced to two checkable inputs: the uniform linear decay (321) and the finite-N forcing estimate (322). It is not necessary to re-prove the nonlinear stability separately for every exponent.

103. Euler-derivative closure in the interval

Let

$$E = s\partial_s, \qquad q_{N,\alpha} = -s\frac{\partial_s b_{N,\alpha}}{b_{N,\alpha}}.$$

The exact identity already established in Section 80 is

$$\frac{E^4 b}{b} = q^4 - 6q^2(Eq) + 4q(E^2q) + 3(Eq)^2 - E^3q. \tag{329}$$

Suppose

$$q_{N,\alpha} = A_{N,\alpha}\frac{1+\eta_{N,\alpha}}{1+\rho_{N,\alpha}}, \tag{330}$$

with

$$\inf|1+\rho_{N,\alpha}| \ge c_\rho > 0 \tag{331}$$

and

$$\sup_{\alpha,N}\sum_{j=0}^{3}\left(\|E^jA_{N,\alpha}\|_\infty + \|E^j\eta_{N,\alpha}\|_\infty + \|E^j\rho_{N,\alpha}\|_\infty\right) < \infty. \tag{332}$$

Repeated quotient differentiation then gives

$$\boxed{\sup_{\alpha,N}\|E^jq_{N,\alpha}\|_\infty < \infty, \qquad 0 \le j \le 3,} \tag{333}$$

and (329) implies

$$\boxed{s^4|\partial_s^4 b_{N,\alpha}| \le Cb_{N,\alpha}} \tag{334}$$

uniformly. Thus the Euler-derivative portion of the bootstrap is completely closed once the regular-singular factors in (330) satisfy the explicit uniform bounds (331)–(332).

104. Coefficient-gap continuation lemma

Let D_α be the limiting terminal coefficient produced by the continued profile and suppose

$$D_1 \ge 2d_* > 0. \tag{335}$$

Since the profile branch and all algebraic coefficient functionals are C^1 in α, there is $C_D < \infty$ such that

$$|D_\alpha - D_1| \le C_D|\alpha - 1|. \tag{336}$$

Choose

$$\delta_{\mathrm{gap}} \le \frac{d_*}{C_D}. \tag{337}$$

Then

$$D_\alpha \ge d_* \qquad (\alpha \in [1, 1+\delta_{\mathrm{gap}}]). \tag{338}$$

If in addition the finite-N reconstruction obeys

$$\sup_{\alpha \in I_\delta} |D_{N,\alpha} - D_\alpha| \le \varepsilon_N^{(D)}, \qquad \varepsilon_N^{(D)} \to 0, \tag{339}$$

then for sufficiently large N,

$$\boxed{D_{N,\alpha} \ge \frac{d_*}{2} > 0 \quad \text{uniformly in } \alpha \in I_\delta.} \tag{340}$$

Therefore the coefficient-gap problem is reduced to two concrete quantities: a strict positive anchor margin (335) and the uniform reconstruction error (339). Continuity in α itself creates no additional obstruction.

29 Part 12: Uniform invariant region and limiting passage

105. General-α invariant-region inequality

Let

$$W_{N,\alpha} = \frac{s^{r_\alpha} b_{N,\alpha}}{p_{N,\alpha}}, \qquad r_\alpha = 2\alpha - 1. \tag{341}$$

Assume the exact normalized terminal equation implies on the boundary $W_{N,\alpha} = W_*$ the lower differential inequality

$$\frac{d}{dt} W_{N,\alpha} \ge \Lambda_{N,\alpha}(s) W_* \left[D_{N,\alpha} W_* - K_\alpha^{\mathrm{diss}} - R_{N,\alpha}(s) \right], \tag{342}$$

where $\Lambda_{N,\alpha}(s) > 0$, the dissipative coefficient satisfies

$$\sup_{\alpha \in I_\delta} K_\alpha^{\mathrm{diss}} \le K_* < \infty, \tag{343}$$

and

$$\sup_{\alpha, N} |R_{N,\alpha}(s)| \le C_R s^{\mu_R} + \epsilon_N^{(R)}, \qquad \mu_R > 0. \tag{344}$$

Using (340), choose

$$\boxed{W_* > \frac{4K_*}{d_*}.} \tag{345}$$

Then

$$\frac{d_*}{2} W_* - K_* > K_*.$$

Choose $s_* > 0$ and then N_* so that

$$C_R s_*^{\mu_R} + \epsilon_N^{(R)} \le K_* \qquad (N \ge N_*). \tag{346}$$

Equations (342)–(346) imply

$$\frac{d}{dt} W_{N,\alpha} > 0 \quad \text{whenever } W_{N,\alpha} = W_*,\ 0 < s \le s_*. \tag{347}$$

Hence

$$\boxed{W_{N,\alpha} \ge W_*} \tag{348}$$

is a forward-invariant terminal region uniformly for $\alpha \in I_\delta$ and $N \ge N_*$. This is the general interval form of the invariant-region mechanism and contains no special exponent.

106. Two-sided terminal amplitude bounds

The lower invariant bound gives

$$b_{N,\alpha}(s) \ge p_{N,\alpha} W_* s^{-r_\alpha}. \tag{349}$$

For the matching upper bound, suppose the normalized equation also yields, for $W \ge W^*$,

$$\frac{d}{dt} W_{N,\alpha} < 0 \tag{350}$$

uniformly in the same terminal layer. Then the strip

$$W_* \le W_{N,\alpha} \le W^* \tag{351}$$

is invariant, and

$$\boxed{p_{N,\alpha} W_* s^{-r_\alpha} \le b_{N,\alpha}(s) \le p_{N,\alpha} W^* s^{-r_\alpha}.} \tag{352}$$

If the normalization is chosen so that $p_{N,\alpha}$ is absorbed into the profile amplitude, (352) becomes the uniform physical statement

$$c_- s^{-r_\alpha} \le b_{N,\alpha}(s) \le c_+ s^{-r_\alpha}. \tag{353}$$

The upper barrier (350) must be derived from the exact normalized finite-N equation; it cannot be inferred from the lower barrier alone.

107. Uniform finite-time conversion

Assume (353) and the exact characteristic law

$$\dot{s} = 1 - b_{N,\alpha}. \tag{354}$$

For sufficiently small s, the lower bound in (353) gives $b \ge 2$, hence

$$\frac{1}{2} b \le b - 1 = -\dot{s} \le b. \tag{355}$$

Therefore

$$\frac{1}{c_+}s^{r_\alpha} \le \frac{dt}{-ds} \le \frac{2}{c_-}s^{r_\alpha}. \tag{356}$$

Integrating from 0 to $s(t)$ gives

$$\frac{1}{c_+(r_\alpha+1)}s^{r_\alpha+1} \le T_* - t \le \frac{2}{c_-(r_\alpha+1)}s^{r_\alpha+1}. \tag{357}$$

Since $r_\alpha + 1 = 2\alpha$ and α lies in a compact interval, all prefactors are uniform. Thus

$$\boxed{T_* - t \asymp s^{2\alpha}, \qquad b_{N,\alpha}(t) \asymp (T_* - t)^{-\frac{2\alpha-1}{2\alpha}}.} \tag{358}$$

This proves the finite-time conversion uniformly on the entire continued profile interval once the invariant two-sided terminal bounds are available.

108. Strong compactness criterion sufficient for the scalar limit

Let $0 < T < T_*$ be fixed. A convenient sufficient compactness package is

$$b_{N,\alpha} \to b_\alpha \quad \text{strongly in } L^2(0,T;H^{\alpha+\varepsilon}), \tag{359}$$

$$\partial_t b_{N,\alpha} \quad \text{bounded in } L^2(0,T;H^{-\alpha-\varepsilon}), \tag{360}$$

for some $\varepsilon > 0$, together with analogous bounds for the reconstructed transport velocity. By the compact embedding on the torus and the Aubin–Lions mechanism, after extraction,

$$b_{N,\alpha} \to b_\alpha \quad \text{strongly in } L^2(0,T;H^{\alpha}) \tag{361}$$

and hence strongly in L^2.

The quadratic term then passes by

$$\|b_N^2 - b^2\|_{L^1} \le \|b_N - b\|_{L^2}(\|b_N\|_{L^2} + \|b\|_{L^2}) \to 0. \tag{362}$$

Moreover

$$(-\Delta)^{\alpha/2}b_N \to (-\Delta)^{\alpha/2}b \quad \text{strongly in } L^2, \tag{363}$$

so the fractional energy pairings pass to the limit. If

$$u_{N,\alpha} \to u_\alpha \quad \text{strongly in } L^2(0,T;H^1) \tag{364}$$

and $\nabla b_{N,\alpha}$ is uniformly bounded in the conjugate space required by the scalar equation, then

$$u_{N,\alpha} \cdot \nabla b_{N,\alpha} \to u_\alpha \cdot \nabla b_\alpha \tag{365}$$

in distributions. Finally, any cutoff or reconstruction residual $\mathcal{R}_{N,\alpha}$ satisfying

$$\sup_{\alpha \in I_\delta} \|\mathcal{R}_{N,\alpha}\|_{L^1(0,T;H^{-m})} \to 0 \tag{366}$$

vanishes in the limit.

Thus (359)–(366) constitute a concrete sufficient strong-limit theorem. The remaining task is to derive these bounds from the exact finite-N reconstruction; compactness itself is then standard and no special exponent is required.

30 Part 13: Revised interval theorem and exact remaining ledger

109. Uniform finite-N theorem under explicit verifiable estimates

Theorem 3 (Small-supercritical finite-N terminal theorem)**.** *Let $I_\delta = [1, 1+\delta]$ be contained in the profile interval from Part 9. Assume the exact finite-N reconstructed scalar equation and characteristic law hold and suppose the following estimates are uniform in $\alpha \in I_\delta$ for all sufficiently large N:*

1. *the Region-I inequalities* (299)–(300)*, the growth* (306)*, and the strict validity margin* (308)*;*
2. *the profile-entry estimate* (313)*;*
3. *the terminal forcing estimate* (322) *and the anchor spectral-gap hypothesis* (318)*, with δ chosen to satisfy* (320)*;*
4. *the regular-singular Euler bounds* (331)–(332)*;*
5. *the positive anchor coefficient margin* (335) *and reconstruction convergence* (339)*;*
6. *the terminal remainder estimate* (344) *and a matching upper barrier of the form* (350)*;*
7. *the compactness estimates* (359)–(366) *on every preterminal interval $[0, T]$, $T < T_*$.*

Then the finite-N trajectories enter the continued profile neighborhood before leaving Region I, the terminal bootstrap closes uniformly, the coefficient gap remains strictly positive, and the normalized terminal strip is invariant. Consequently

$$b_{N,\alpha}(s) \asymp s^{-(2\alpha-1)}$$

with constants uniform on a possibly smaller interval, and

$$\boxed{b_{N,\alpha}(t) \asymp (T_{*,N,\alpha} - t)^{-\frac{2\alpha-1}{2\alpha}}.} \tag{367}$$

If the compactness hypotheses pass the characteristic and terminal lower bound to a limiting scalar solution, then the same exponent law holds for the limit up to its terminal time.

Proof. Part 10 gives uniform entry to the amplitude scale before Region-I exit; (313) places the trajectory in the profile neighborhood. Part 11, equations (321)–(328), closes the terminal stability bootstrap, while (333)–(334) close the Euler derivative estimates. Equations (335)–(340) give a uniform positive coefficient gap. Part 12 then gives the invariant terminal strip and the two-sided power law. Equation (358) converts that power law to physical time. Finally (359)–(366) are sufficient to pass the scalar equation and its controlled residuals on each preterminal interval. □ □

110. Which former Section 94 items are now reduced or closed

The four items listed in Section 94 should now be separated more finely.

1. **Finite-N entry.** The bounded logarithmic excursion and entry before Region-I exit are proved uniformly by (304)–(310). The only remaining nonalgebraic entry input is the similarity-distance estimate (313).

2. **Bootstrap.** Once the anchor spectral decay (318) is verified in the chosen phase-fixed terminal norm and the finite-N forcing obeys (322), the interval bootstrap follows by perturbation and Duhamel, giving (328). Euler fourth-derivative closure is already a consequence of the explicit regular-singular bounds (331)–(332).

3. **Coefficient gap.** A strict positive gap at the anchor persists for all $1 < \alpha < 1 + \delta_{\mathrm{gap}}$ by continuity. The finite-N gap then follows from the single uniform reconstruction estimate (339). Thus no separate exponent-by-exponent gap proof is needed.

4. **Limiting estimates.** The sufficient topology is made explicit in (359)–(366). What remains is not an abstract compactness theorem but the derivation of those uniform bounds from the reconstruction formulas.

Accordingly, after the profile theorem of Part 9, the genuinely construction- specific estimates still to be established are

$$\boxed{\text{(313), (318), (322), (331)–(332), (335), (339), (344), (350), and (359)–(366).}} \tag{368}$$

Several of these are continuity or perturbation estimates rather than new nonlinear existence problems.

111. No isolated exponent enters Parts 10–13

Every exponent used above is expressed through

$$r_\alpha = 2\alpha - 1$$

and every interval restriction is obtained by shrinking the already nonempty profile interval. In particular, no numerical profile, Mellin root, coefficient, or remainder exponent tied to a single isolated value of α is used in Parts 10–13. This is essential: the purpose of the is to establish a theorem on an open interval above the classical endpoint, not to repackage a one-exponent certification.

112. Revised status

The manuscript now proves the positive Mellin-profile branch on a nonempty small-supercritical interval and proves the uniform pre-entry excursion mechanism on that same interval under the explicit Region-I inequalities. It also proves abstract uniform bootstrap, coefficient-continuation, invariant-region, and limiting lemmas whose hypotheses are stated in norms that can be checked directly against the finite-N reconstruction.

It would still be mathematically premature to delete all conditional language from the scalar interval theorem until the reconstruction-specific estimates listed in (368) are verified from the exact finite-N formulas. The gain is that the former four broad hypotheses have now been reduced to a short list of precise inequalities, with the purely functional-analytic parts proved once and for all uniformly in α.

31 Part 14: Exact finite-N dynamics to the continued profile

113. Purpose and logical form of the bridge

The profile theorem of Part 9 supplies a positive phase-normalized branch B_α for $\alpha \in I_\delta = [1, 1+\delta]$. Parts 10–13 isolate the remaining problem: the exact finite-N trajectory must be shown to enter a neighborhood of B_α, remain there uniformly, and converge strongly enough that the terminal inequalities survive $N \to \infty$. This part gives a single bridge theorem that separates what follows from exact identities from what must be estimated from the reconstruction coefficients.

Put

$$r_\alpha = 2\alpha - 1, \qquad \tau = -\log s,$$

and normalize the exact finite-N scalar by

$$W_{N,\alpha}(\tau) = \frac{s^{r_\alpha} b_{N,\alpha}(s)}{p_{N,\alpha}}. \tag{369}$$

Let B_α be the continued profile and set

$$V_{N,\alpha} = W_{N,\alpha} - B_\alpha. \tag{370}$$

The exact finite-N equation can always be decomposed, without approximation, as

$$\boxed{\partial_\tau V_{N,\alpha} = L_\alpha V_{N,\alpha} + \mathcal{N}_\alpha(V_{N,\alpha}) + \mathcal{E}_{N,\alpha},} \tag{371}$$

where $L_\alpha = D\mathcal{A}_\alpha(B_\alpha)$, the nonlinear remainder is

$$\mathcal{N}_\alpha(V) = \mathcal{A}_\alpha(B_\alpha + V) - \mathcal{A}_\alpha(B_\alpha) - L_\alpha V, \tag{372}$$

and $\mathcal{E}_{N,\alpha}$ is defined to be the exact difference between the finite-N normalized vector field and the autonomous profile vector field. Thus (371) is an identity. No smallness has yet been assumed.

114. Exact decomposition of the finite-N defect

For the one-phase reconstruction the defect may be grouped as

$$\boxed{\mathcal{E}_{N,\alpha} = \mathcal{E}^{\mathrm{coeff}}_{N,\alpha} + \mathcal{E}^{\mathrm{rec}}_{N,\alpha} + \mathcal{E}^{\mathrm{frac}}_{N,\alpha} + \mathcal{E}^{\mathrm{cut}}_{N,\alpha}.} \tag{373}$$

Here $\mathcal{E}^{\mathrm{coeff}}$ is produced by replacing the finite-N coefficient functions by their terminal limits, $\mathcal{E}^{\mathrm{rec}}$ is the velocity/reconstruction residual, $\mathcal{E}^{\mathrm{frac}}$ is the difference between the exact fractional term and its Mellin one-phase limit, and $\mathcal{E}^{\mathrm{cut}}$ contains any regularization or localization terms. The exact one-phase identity

$$(-\Delta_{\mathbb{T}^3})^{\alpha} F(s_N) = \kappa^{2\alpha}(-\partial_{ss})^{\alpha} F(s_N) \tag{374}$$

shows that, in the strict generalized affine one-phase class, there is no independent transverse fractional error. In that class $\mathcal{E}^{\mathrm{frac}}_{N,\alpha}$ consists only of the finite-N coefficient, regularization, or cutoff discrepancy actually present in the reconstruction. This is stronger than the earlier multidimensional remainder bookkeeping.

Define the uniform defect modulus

$$\varepsilon^{\mathrm{br}}_N(s) := \sup_{\alpha \in I_\delta} \|\mathcal{E}_{N,\alpha}(s)\|_{\mathcal{X}}. \tag{375}$$

The bridge requires an estimate, not an assumption about the terminal shape, of the form

$$\boxed{\varepsilon^{\mathrm{br}}_N(s) \le C_{\mathrm{br}} s^{\mu_{\mathrm{br}}} + \epsilon_N, \qquad \mu_{\mathrm{br}} > 0, \qquad \epsilon_N \to 0.} \tag{376}$$

Every term on the right of (373) is to be estimated directly from the exact finite-N coefficients. Equation (376) is therefore noncircular: it does not assume $W_{N,\alpha} \to B_\alpha$.

115. Entry from an overlap interval rather than from amplitude alone

Large amplitude by itself cannot imply profile closeness. The correct bridge uses an overlap interval

$$J_{N,\alpha} = [\tau_-, \tau_e], \qquad \tau_e - \tau_- \le L_{\mathrm{ov}}, \tag{377}$$

whose length is uniformly bounded by Part 10. On this interval solve (371) backward from τ_e or, equivalently, forward from τ_-. Let $S_\alpha(\tau)$ denote the phase-fixed evolution generated by L_α and assume the uniform dichotomy/stability estimate already isolated in (321):

$$\|S_\alpha(\tau)\|_{\mathcal{X}\to\mathcal{X}} \le M_0 e^{-\lambda_* \tau}, \qquad \tau \ge 0. \tag{378}$$

Variation of constants gives the exact identity

$$V(\tau_e) = S_\alpha(\tau_e - \tau_-)V(\tau_-) + \int_{\tau_-}^{\tau_e} S_\alpha(\tau_e - \sigma)\big[\mathcal{N}_\alpha(V(\sigma)) + \mathcal{E}_{N,\alpha}(\sigma)\big]\, d\sigma. \tag{379}$$

Suppose the overlap starts in a fixed bounded set,

$$\sup_{N,\alpha} \|V_{N,\alpha}(\tau_-)\|_{\mathcal{X}} \le M_{\mathrm{ov}}, \tag{380}$$

and that the exact preterminal dynamics keep $\|V\|_{\mathcal{X}} \le R_{\mathrm{ov}}$ on $J_{N,\alpha}$. Since the profile branch is C^1 on a compact interval,

$$\|\mathcal{N}_\alpha(V)\|_{\mathcal{X}} \le C_Q \|V\|^2_{\mathcal{X}} \tag{381}$$

with C_Q independent of α.

Combining (376)–(381) yields

$$\begin{aligned}\|V(\tau_e)\|_{\mathcal{X}} \le{}& M_0 e^{-\lambda_*(\tau_e-\tau_-)} M_{\mathrm{ov}} \\ &+ \frac{M_0 C_Q}{\lambda_*} \sup_{J_{N,\alpha}} \|V\|_{\mathcal{X}}^2 \\ &+ \frac{M_0}{\lambda_*}\left(C_{\mathrm{br}} s_e^{\mu_{\mathrm{br}}} + \epsilon_N\right)\end{aligned} \tag{382}$$

when the defect is monotone in the terminal direction; the obvious supremum version holds otherwise. Thus the desired entry estimate (313) follows once the overlap has enough stable time and the preterminal normalized error is small enough. More precisely, if

$$M_0 e^{-\lambda_* L_{\mathrm{stab}}} M_{\mathrm{ov}} \le \frac{\varepsilon_{\mathrm{ent}}}{4}, \tag{383}$$

$$\frac{M_0 C_Q}{\lambda_*} R_{\mathrm{ov}}^2 \le \frac{\varepsilon_{\mathrm{ent}}}{4}, \tag{384}$$

and

$$\frac{M_0}{\lambda_*}\left(C_{\mathrm{br}} s_e^{\mu_{\mathrm{br}}} + \epsilon_N\right) \le \frac{\varepsilon_{\mathrm{ent}}}{2}, \tag{385}$$

then

$$\boxed{\|W_{N,\alpha}(\tau_e) - B_\alpha\|_{\mathcal{X}} \le \varepsilon_{\mathrm{ent}}.} \tag{386}$$

This is the rigorous form of the finite-N-to-profile connection. It also shows why the amplitude condition $b = Mp_N$ alone was insufficient: one needs the stable overlap estimate (379).

116. A quantitative bridge theorem

Theorem 4 (Uniform finite-N profile-entry bridge)**.** *Assume the continued profile branch of Part 9, the uniform phase-fixed linear estimate (378), the quadratic estimate (381), and the exact defect estimate (376). Assume further that the exact Region-I dynamics produce an overlap interval satisfying (377), (380), and a bootstrap bound* $\sup_J \|V\|_{\mathcal{X}} \le R_{\mathrm{ov}}$. *If the overlap contains a stable subinterval of length* L_{stab} *for which (383)–(385) hold, then the finite-N trajectory enters the terminal profile ball (386), uniformly for* $\alpha \in I_\delta$ *and all sufficiently large* N.

Proof. Equation (379) is variation of constants applied to the exact identity (371). Use (378) on the homogeneous term, (381) on the nonlinear term, and (376) on the finite-N defect. Integration of the exponential kernel gives (382). The three contributions are bounded by (383), (384), and (385), respectively. Their sum is at most $\varepsilon_{\mathrm{ent}}$, proving (386). □ □

32 Part 15: Uniform propagation from entry to the terminal layer

117. Bootstrap begins exactly at the bridge time

At $\tau = \tau_e$, (386) supplies the initial smallness required in (327). Consequently the Part 11 bootstrap applies without any additional matching assumption and gives

$$\boxed{\|W_{N,\alpha}(\tau) - B_\alpha\|_{\mathcal{X}} \le A_0 e^{-\lambda_b(\tau-\tau_e)} + A_0\epsilon_N, \qquad \tau \ge \tau_e.} \tag{387}$$

Thus, once (376) and the overlap inequalities are verified from the exact coefficients, the former entry hypothesis (313) and the subsequent nonlinear profile bootstrap become one continuous argument.

118. Exact logarithmic-slope closure along the profile tube

The exact characteristic identities from the generalized affine manuscript are

$$\dot b = \Gamma_{N,\alpha} b^2 + F_{N,\alpha} - \mathcal{R}_{N,\alpha} - \mathcal{D}_{N,\alpha}, \tag{388}$$

$$-\dot s = H_{N,\alpha} + \gamma_{N,\alpha} b. \tag{389}$$

Define

$$A_{N,\alpha} = \frac{s\Gamma_{N,\alpha}}{\gamma_{N,\alpha}}, \qquad \eta_{N,\alpha} = \frac{F_{N,\alpha} - \mathcal{R}_{N,\alpha} - \mathcal{D}_{N,\alpha}}{\Gamma_{N,\alpha} b^2}, \qquad \rho_{N,\alpha} = \frac{H_{N,\alpha}}{\gamma_{N,\alpha} b}. \tag{390}$$

Then, with $E = s\partial_s$,

$$\boxed{q_{N,\alpha} := -s\frac{b_s}{b} = A_{N,\alpha}\frac{1+\eta_{N,\alpha}}{1+\rho_{N,\alpha}}.} \tag{391}$$

This is exact. Hence (331)–(332) are not independent dynamical hypotheses: they follow from uniform coefficient bounds

$$\inf_{N,\alpha,s} |1 + \rho_{N,\alpha}| \ge c_\rho > 0, \tag{392}$$

$$\sup_{N,\alpha} \sum_{j=0}^{3} \left(\|E^j A_{N,\alpha}\|_\infty + \|E^j \eta_{N,\alpha}\|_\infty + \|E^j \rho_{N,\alpha}\|_\infty \right) \le C_q. \tag{393}$$

Repeated quotient differentiation gives

$$\sup_{N,\alpha} \|E^j q_{N,\alpha}\|_\infty \le C_j, \qquad 0 \le j \le 3, \tag{394}$$

and therefore

$$\boxed{s^4 |\partial_s^4 b_{N,\alpha}| \le C b_{N,\alpha}.} \tag{395}$$

Thus the derivative closure is reduced entirely to coefficient estimates in (392)–(393), which can be checked before assuming the terminal power law.

119. Defect decay is preserved by the profile tube

Suppose the normalized reconstruction residual has the exact factorization

$$R^{\mathrm{norm}}_{N,\alpha} := \frac{\mathcal{R}_{N,\alpha}}{\Gamma_{N,\alpha} b^2_{N,\alpha}} = s^{\mu_R} a_{N,\alpha}(s), \qquad \mu_R > 0, \tag{396}$$

with

$$\sup_{N,\alpha} \|E^j a_{N,\alpha}\|_\infty \le C_j, \qquad 0 \le j \le 3. \tag{397}$$

Then

$$\boxed{|E^j R^{\mathrm{norm}}_{N,\alpha}| \le C'_j s^{\mu_R}, \qquad 0 \le j \le 3.} \tag{398}$$

The same conclusion holds for every defect term admitting such a positive terminal power. Since α ranges in a compact interval, the minimum of finitely many continuous positive remainder exponents remains positive after shrinking δ if necessary. Hence the forcing estimate (322) follows from the coefficient-level factorizations, with

$$\mu = \inf_{\alpha \in I_\delta} \mu_{\mathrm{rem}}(\alpha) > 0. \tag{399}$$

No isolated exponent is required.

33 Part 16: Coefficient gap and invariant strip from profile convergence

120. Uniform convergence of coefficient functionals

Let $\mathscr{D}_\alpha(W, \mathbf{c})$ denote the exact normalized coefficient functional, where $\mathbf{c}_{N,\alpha}$ denotes the finite-N reconstruction coefficients. Define

$$D_\alpha = \mathscr{D}_\alpha(B_\alpha, \mathbf{c}_\alpha), \qquad D_{N,\alpha} = \mathscr{D}_\alpha(W_{N,\alpha}, \mathbf{c}_{N,\alpha}). \tag{400}$$

On a fixed profile tube, local Lipschitz continuity gives

$$|D_{N,\alpha} - D_\alpha| \le C_D \left(\|W_{N,\alpha} - B_\alpha\|_{\mathcal{X}} + \|\mathbf{c}_{N,\alpha} - \mathbf{c}_\alpha\|_{\mathcal{C}} \right). \tag{401}$$

Using (387), if

$$\sup_{\alpha \in I_\delta} \|\mathbf{c}_{N,\alpha} - \mathbf{c}_\alpha\|_{\mathcal{C}} \le \epsilon^{\mathrm{c}}_N \to 0, \tag{402}$$

then

$$\boxed{\sup_{\alpha \in I_\delta} |D_{N,\alpha} - D_\alpha| \le C_D A_0 e^{-\lambda_b(\tau - \tau_e)} + C_D(A_0 \epsilon_N + \epsilon^{\mathrm{c}}_N).} \tag{403}$$

Therefore (339) follows in the deep terminal layer. If the anchor coefficient has a strict margin $D_1 \ge 2d_* > 0$, continuity of the profile branch gives, after reducing δ,

$$D_\alpha \ge d_*, \tag{404}$$

and then (403) gives

$$\boxed{D_{N,\alpha} \ge d_*/2} \tag{405}$$

for all sufficiently large N and sufficiently deep terminal time.

121. Two-sided invariant strip

The lower boundary argument of Part 12 uses (405) and the decaying normalized remainder. For the upper boundary one applies the same exact normalized equation with the inequalities reversed. Suppose on a fixed tube

$$\dot{W} = \Lambda_{N,\alpha}(s) W \left[D_{N,\alpha} W - K_\alpha^{\mathrm{diss}} + R_{N,\alpha}(s, W) \right], \qquad \Lambda_{N,\alpha} > 0, \tag{406}$$

with

$$|R_{N,\alpha}(s, W)| \le C_R s^{\mu_R} + \epsilon_N^{(R)} \tag{407}$$

for W in a compact interval. Choose W_- so that the bracket is positive at $W = W_-$ and W_+ so that the exact orientation of the normalized equation makes the vector field point downward at $W = W_+$. Then

$$\boxed{W_- \le W_{N,\alpha}(\tau) \le W_+} \tag{408}$$

is forward invariant. The second sign must be checked from the exact finite-N normalized equation; it is not a consequence of the lower sign. Once both signs hold, however,

$$\boxed{c_- s^{-r_\alpha} \le b_{N,\alpha}(s) \le c_+ s^{-r_\alpha}, \qquad r_\alpha = 2\alpha - 1,} \tag{409}$$

with constants uniform on a possibly smaller interval.

34 Part 17: Uniform passage to the finite-N limit

122. Compactness on every preterminal interval

Fix $T < T_*$. Suppose the exact reconstruction provides the uniform energy package

$$\sup_{N,\alpha} \|b_{N,\alpha}\|_{L^2(0,T;H^{\alpha+\varepsilon})} \le C_T, \tag{410}$$

$$\sup_{N,\alpha} \|\partial_t b_{N,\alpha}\|_{L^2(0,T;H^{-\alpha-\varepsilon})} \le C_T, \tag{411}$$

and corresponding bounds for the reconstructed transport velocity. Then Aubin–Lions gives, after extraction,

$$b_{N,\alpha} \to b_\alpha \quad \text{strongly in } L^2(0,T;H^\alpha), \tag{412}$$

which is enough for the quadratic and fractional terms as described in Part 12. If the residuals satisfy

$$\sup_{\alpha \in I_\delta} \|\mathcal{R}_{N,\alpha}\|_{L^1(0,T;H^{-m})} \to 0, \tag{413}$$

the limiting scalar equation follows distributionally on $[0, T]$.

123. Passing the terminal lower bound without losing it in the limit

Preterminal compactness alone does not automatically pass a lower bound at a moving terminal point. A sufficient uniform statement is the following. Assume there exist $s_0, c_- > 0$, independent of N and α, such that

$$b_{N,\alpha}(s) \geq c_- s^{-r_\alpha}, \qquad 0 < s \leq s_0, \tag{414}$$

and the characteristic maps $t \mapsto s_{N,\alpha}(t)$ converge locally uniformly on every $[0, T]$, $T < T_*$, to $s_\alpha(t)$. If, in addition, $b_{N,\alpha} \to b_\alpha$ locally uniformly along the distinguished characteristic away from $s = 0$, then for every fixed $s \in (0, s_0]$,

$$\boxed{b_\alpha(s) \geq c_- s^{-r_\alpha}.} \tag{415}$$

The constants do not deteriorate with N, so the lower terminal barrier is not lost in the limit.

Using the upper bound in (409) and the exact characteristic law $\dot{s} = 1 - b$, the calculation of Part 12 gives

$$\boxed{T_{*,\alpha} - t \asymp s_\alpha(t)^{2\alpha}, \qquad b_\alpha(t) \asymp (T_{*,\alpha} - t)^{-\frac{2\alpha-1}{2\alpha}}.} \tag{416}$$

Thus the limiting scalar diverges in finite time whenever the hypotheses of this part, including the uniform terminal barrier, are verified.

124. Uniformity in the exponent

All constants above may be chosen uniformly after replacing δ by

$$\boxed{\delta_* = \min\{\delta_{\text{prof}}, \delta_{\text{lin}}, \delta_{\text{gap}}, \delta_{\text{rem}}, \delta_{\text{comp}}\} > 0,} \tag{417}$$

provided each listed interval is nonempty. The exponent functions $r_\alpha = 2\alpha - 1$ and 2α vary continuously and remain bounded away from zero. Therefore no step in Parts 14–17 requires selecting an isolated value of α.

35 Part 18: Closed bridge theorem and remaining coefficient-level checks

125. Finite-N to profile to limit theorem

Theorem 5 (Uniform connection theorem)**.** *Let $1 < \alpha < 1 + \delta_*$, with $\delta_* > 0$ as in (417), and suppose the continued profile B_α exists. Assume the exact finite-N reconstruction satisfies the coefficient-level estimates (376), (378), (380), (383)–(385), (392)–(393), (396)–(397), (402), (407), and (410)–(414), together with the upper-boundary sign required for (408). Then, uniformly for sufficiently large N:*

1. *the exact finite-N trajectory enters the profile neighborhood;*

2. *the profile error remains small and decays according to (387);*

3. *the logarithmic derivatives close according to (394)–(395);*

4. *the terminal coefficient gap is positive according to (405);*

5. *the normalized terminal strip (408) is invariant;*

6. *the two-sided power law (409) holds;*

7. *a subsequence converges strongly on every preterminal interval; and*

8. *the limiting scalar satisfies the finite-time law (416).*

Proof. The entry statement is Theorem 116. Equation (387) is the Part 11 nonlinear bootstrap initialized by (386). Equations (391)–(395) give derivative closure from the exact characteristic coefficients. Equations (400)–(405) transfer the positive profile coefficient to finite N. Equations (406)–(409) give the invariant strip. Equations (410)–(413) yield strong preterminal compactness, while the uniform barrier (414) and characteristic convergence give (415). The finite-time conversion then gives (416). □ □

126. What has and has not been proved by the bridge argument

Parts 14–18 remove an important logical ambiguity. The similarity-entry estimate is not to be inferred from large amplitude. It follows from the exact defect equation (371), a stable overlap interval, and coefficient-level defect smallness. Likewise, the Euler bounds and coefficient gap are consequences of exact identities plus uniform coefficient convergence.

The argument does *not* manufacture the coefficient estimates. For the particular finite-N construction one must still verify directly from its explicit formulas:

$$\boxed{\begin{array}{c}\text{the defect estimate (376), the overlap bounds (380), (383)–(385),}\\ \text{the coefficient derivative bounds (392)–(393), the residual factorization (396)–(397), coefficient convergence (402),}\\ \text{the upper invariant-boundary sign in (406)–(408), and the uniform compactness/barrier hypotheses (410)–(414).}\end{array}} \tag{418}$$

These are now coefficient-level checks, not a missing profile theory. If the explicit reconstruction formulas establish (418), the scalar finite-N to profile to limiting blow-up chain is closed on the nonempty interval $1 < \alpha < 1 + \delta_*$. The separate reconstruction of a full three-component velocity-pressure solution remains an additional theorem and is not asserted by the scalar bridge alone.

36 Part 19: Term-by-term attack on the coefficient ledger (418)

127. Purpose: separate identities, theorems, and genuinely missing estimates

Equation (418) is a ledger rather than a single analytic inequality. It contains several logically different assertions. Some follow from exact one-phase identities already proved in the

manuscript; some follow from elementary quotient/Euler calculus once coefficient bounds are known; and some require new information about the particular finite-N reconstruction. The purpose of this part is to remove every item that can be removed without inserting a new hypothesis.

Throughout

$$I_\delta = [1, 1+\delta], \qquad r_\alpha = 2\alpha - 1, \qquad \tau = -\log s,$$

and $\delta > 0$ is reduced, when necessary, but is independent of N. No isolated value of α is used in this part.

128. Exact one-phase fractional reduction removes an entire defect block

For the generalized affine phase

$$s_N = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N}, \qquad \kappa^2 = C_1^2 + C_2^2 + C_3^2,$$

Fourier multiplier calculus gives the exact identity

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha F(s_N) = \kappa^{2\alpha}(-\partial_{ss})^\alpha F(s_N).} \tag{419}$$

Consequently there is no transverse fractional approximation error inside the strict one-phase class. If the same fractional realization is used in the finite-N equation and in the profile equation, then

$$\boxed{\mathcal{E}^{\mathrm{frac}}_{N,\alpha} = 0.} \tag{420}$$

If a cutoff or regularization is retained, the only fractional contribution to the bridge defect is the explicit cutoff/regularization commutator. Thus the fractional part of (376) is not an independent hypothesis.

Proof. For a one-phase Fourier mode e^{ins_N} the spatial frequency is $n(-C_1, -C_2, -C_3)$ and hence has modulus $|n|\kappa$. The torus multiplier $|k|^{2\alpha}$ therefore acts as $\kappa^{2\alpha}|n|^{2\alpha}$. Summing the Fourier series proves (419). Subtracting the identical one-dimensional fractional term in the profile vector field proves (420). □ □

129. Reconstruction residual: exact normalization and positive-power criterion

Write the exact scalar characteristic equation as

$$\dot{b} = \Gamma_{N,\alpha} b^2 + F_{N,\alpha} - \mathcal{R}_{N,\alpha} - \mathcal{D}_{N,\alpha}, \tag{421}$$

and introduce

$$\mathfrak{R}_{N,\alpha} := \frac{\mathcal{R}_{N,\alpha}}{\Gamma_{N,\alpha} b_{N,\alpha}^2}. \tag{422}$$

The reconstruction estimates derived earlier have the structural form

$$\frac{|\mathcal{R}_{N,\alpha}|}{|\Gamma_{N,\alpha}|b_{N,\alpha}^2} \le C\left(\frac{1}{|\Gamma_{N,\alpha}|b_{N,\alpha}} + \mathfrak{c}_{N,\alpha}(s)\right), \tag{423}$$

where $\mathfrak{c}_{N,\alpha}$ denotes the remaining explicit coefficient corrections. In the terminal profile tube

$$b_{N,\alpha} = p_{N,\alpha}s^{-r_\alpha}W_{N,\alpha}, \qquad W_{N,\alpha} \ge W_- > 0, \tag{424}$$

and if

$$|\Gamma_{N,\alpha}(s)| \ge g_- s^{-1}, \qquad g_- > 0, \tag{425}$$

then

$$\frac{1}{|\Gamma_{N,\alpha}|b_{N,\alpha}} \le \frac{1}{g_- W_- p_{N,\alpha}} s^{r_\alpha+1} = \frac{1}{g_- W_- p_{N,\alpha}} s^{2\alpha}. \tag{426}$$

Since $\alpha \in I_\delta$, $2\alpha \ge 2$, and therefore

$$\boxed{\frac{1}{|\Gamma_{N,\alpha}|b_{N,\alpha}} \le C p_{N,\alpha}^{-1} s^2.} \tag{427}$$

Thus the algebraic reconstruction term already has a uniform positive terminal power. Only the explicit coefficient correction $\mathfrak{c}_{N,\alpha}$ remains to be estimated from its reconstruction formula.

More generally, if

$$\mathfrak{c}_{N,\alpha}(s) = s^{\mu_c} c_{N,\alpha}(s), \qquad \mu_c > 0, \qquad \sup_{N,\alpha} \sum_{j=0}^{3} \|E^j c_{N,\alpha}\|_\infty < \infty, \tag{428}$$

then (423)–(428) imply

$$\boxed{|E^j \mathfrak{R}_{N,\alpha}| \le C_j \left(p_{N,\alpha}^{-1} s^2 + s^{\mu_c}\right), \qquad 0 \le j \le 3.} \tag{429}$$

This proves the residual factorization (396)–(397) whenever the remaining explicit reconstruction coefficient possesses the positive-power expansion (428). The important point is that the $1/(\Gamma b)$ term itself is now closed uniformly in α.

37 Part 20: Exact Lambert coefficient calculus and Euler closure

130. A general Euler-quotient lemma

The finite-N Lambert reconstruction enters the characteristic dynamics only through dimensionless coefficient combinations. Let

$$A_{N,\alpha} = \frac{s\Gamma_{N,\alpha}}{\gamma_{N,\alpha}}, \quad \eta_{N,\alpha} = \frac{F_{N,\alpha} - \mathcal{R}_{N,\alpha} - \mathcal{D}_{N,\alpha}}{\Gamma_{N,\alpha} b_{N,\alpha}^2}, \quad \rho_{N,\alpha} = \frac{H_{N,\alpha}}{\gamma_{N,\alpha} b_{N,\alpha}}. \tag{430}$$

Then the exact characteristic quotient gives

$$\boxed{q_{N,\alpha} := -s\frac{b_s}{b} = A_{N,\alpha}\frac{1+\eta_{N,\alpha}}{1+\rho_{N,\alpha}}.} \tag{431}$$

Theorem 6 (Uniform Euler quotient closure)**.** *Assume for some* $c_\rho > 0$

$$|1 + \rho_{N,\alpha}| \geq c_\rho \tag{432}$$

and

$$\sup_{N,\alpha} \sum_{j=0}^{3} \left(\|E^j A_{N,\alpha}\|_\infty + \|E^j \eta_{N,\alpha}\|_\infty + \|E^j \rho_{N,\alpha}\|_\infty \right) \leq C_*. \tag{433}$$

Then there are constants $C_j = C_j(C_*, c_\rho)$ *such that*

$$\boxed{\|E^j q_{N,\alpha}\|_\infty \leq C_j, \qquad 0 \leq j \leq 3.} \tag{434}$$

Proof. Apply E repeatedly to (431). Every derivative is a finite sum of products of $E^k A$, $E^k \eta$, and $E^k \rho$, divided by a power of $1+\rho$ no larger than four. Equations (432)–(433) bound every such term uniformly. □ □

The fourth derivative of b then follows algebraically:

$$\boxed{\frac{E^4 b}{b} = q^4 - 6q^2 Eq + 4qE^2 q + 3(Eq)^2 - E^3 q.} \tag{435}$$

Hence (432)–(433) prove the derivative closure required in (392)–(395). The only construction-specific obligation is therefore to prove (432)–(433) for the explicit Lambert coefficients; no further PDE estimate is hidden in this step.

131. Uniform increasing-depth Lambert logarithmic sensitivity

The fixed-depth representation (437) is useful only while the iterates remain inside the branch core. It is not the correct object to propagate when the depth $k = k(N)$ tends to infinity. The exact renormalized branch-distance map is instead

$$\Delta_{N,k+1} = \mathcal{F}(\Delta_{N,k}), \qquad \mathcal{F}(\Delta) = 1 + W_0\left(-e^{-1-\Delta}\right). \tag{436}$$

Write

$$q = \mathcal{F}(\Delta) \in (0, 1).$$

Since $W_0(-e^{-1-\Delta}) = q - 1$, the Lambert identity gives the exact inverse relation

$$\boxed{\Delta = -q - \log(1 - q).} \tag{437}$$

Differentiating (437) yields

$$\frac{d\Delta}{dq} = \frac{q}{1-q}, \qquad \mathcal{F}'(\Delta) = \frac{1-q}{q}. \tag{438}$$

Define the Euler logarithmic sensitivity

$$S_{N,k} := E \log \Delta_{N,k}, \qquad E = s\partial_s. \tag{439}$$

Then the exact chain rule gives

$$\boxed{S_{N,k+1} = A(\Delta_{N,k})S_{N,k}, \qquad A(\Delta) := \frac{\Delta\mathcal{F}'(\Delta)}{\mathcal{F}(\Delta)} = \frac{\Delta(1-q)}{q^2}.} \tag{440}$$

Thus the increasing-depth problem reduces to an exact scalar multiplier.

Theorem 7 (Global contraction of the Lambert logarithmic sensitivity)**.** *For every $\Delta > 0$, with $q = \mathcal{F}(\Delta) \in (0,1)$,*

$$0 < A(\Delta) < \frac{1}{2}. \tag{441}$$

Moreover A extends smoothly to $\Delta = 0$ with $A(0) = 1/2$. Consequently, for every finite N and every depth $k \geq 0$,

$$\boxed{|S_{N,k}| \leq 2^{-k}|S_{N,0}|.} \tag{442}$$

The estimate is uniform for arbitrary choices $k = k(N)$, including $k(N) \to \infty$.

Proof. Positivity is immediate from (440). To prove the upper bound, (441) is equivalent to

$$2(1-q)\big[-q - \log(1-q)\big] < q^2.$$

Set

$$H(q) = q^2 - 2(1-q)\big[-q - \log(1-q)\big].$$

Then $H(0) = 0$ and

$$H'(q) = -2q - 2\log(1-q) = 2\big[-q - \log(1-q)\big] > 0$$

for $0 < q < 1$. Hence $H(q) > 0$, proving $A < 1/2$. The branch expansion $-q - \log(1-q) = q^2/2 + q^3/3 + O(q^4)$ gives $A \to 1/2$ as $q \downarrow 0$. Iteration of (440) now proves (442). □ □

This theorem contains the branch-core exponent-halving law as its small-Δ limit, but it remains valid after the orbit leaves the branch core. Indeed,

$$\mathcal{F}(\Delta) = \sqrt{2\Delta} - \frac{2}{3}\Delta + O(\Delta^{3/2}), \qquad A(\Delta) = \frac{1}{2} + O(\sqrt{\Delta}), \tag{443}$$

whereas near the positive fixed point Δ_*,

$$A(\Delta_*) = \mathcal{F}'(\Delta_*) = \frac{1-\Delta_*}{\Delta_*} < 1. \tag{444}$$

Thus the square-root description is a branch-core asymptotic, while (440)–(442) are exact and increasing-depth uniform.

131A. Higher Euler derivatives: triangular induction

For the Euler bootstrap one needs three further derivatives of $S_{N,k}$. Introduce

$$B_1(\Delta) = \Delta A'(\Delta), \qquad B_2(\Delta) = \Delta \partial_\Delta B_1(\Delta), \qquad B_3(\Delta) = \Delta \partial_\Delta B_2(\Delta). \tag{445}$$

On any invariant interval $0 \le \Delta \le \Delta_+$ with $\Delta_+ < 1$ (and in particular on $[0, \Delta_*]$), the functions A, B_1, B_2, B_3 extend continuously to zero and are uniformly bounded. Applying E to (440) gives exactly

$$ES_{k+1} = A_k ES_k + B_{1,k} S_k^2, \tag{446}$$

$$E^2 S_{k+1} = A_k E^2 S_k + 3B_{1,k} S_k ES_k + B_{2,k} S_k^3, \tag{447}$$

and

$$\begin{aligned} E^3 S_{k+1} = {} & A_k E^3 S_k + 4B_{1,k} S_k E^2 S_k + 3B_{1,k} (ES_k)^2 \\ & + 6B_{2,k} S_k^2 ES_k + B_{3,k} S_k^4, \end{aligned} \tag{448}$$

where $A_k = A(\Delta_{N,k})$ and $B_{j,k} = B_j(\Delta_{N,k})$.

Theorem 8 (Uniform increasing-depth Euler derivative theorem)**.** *Assume the terminal input obeys*

$$\sup_N \sum_{m=0}^{3} |E^m S_{N,0}| \le M_0 \tag{449}$$

on the coupled terminal window, and assume the Lambert orbit remains in a fixed compact invariant interval $0 \le \Delta_{N,k} \le \Delta_+ < 1$. Then there exist constants $C_m = C_m(M_0, \Delta_+)$, independent of N and k, such that

$$\boxed{|E^m S_{N,k}| \le C_m 2^{-k}, \qquad 0 \le m \le 3.} \tag{450}$$

Consequently, for every sequence $k(N) \to \infty$,

$$\boxed{\max_{0 \le m \le 3} \|E^m (E \log \Delta_{N,k(N)})\|_\infty \longrightarrow 0.} \tag{451}$$

Proof. The case $m = 0$ is (442). Let $M_j = \sup_{0 \le \Delta \le \Delta_+} |B_j(\Delta)|$. Equation (446), the contraction $A_k \le 1/2$, and (442) give

$$|ES_{k+1}| \le \frac{1}{2} |ES_k| + M_1 M_0^2 \, 4^{-k}.$$

Discrete convolution with $2^{-(k-\ell)}$ yields $|ES_k| \le C_1 2^{-k}$. Substitution of this estimate into (447) gives

$$|E^2 S_{k+1}| \le \frac{1}{2} |E^2 S_k| + C 4^{-k},$$

and therefore $|E^2 S_k| \le C_2 2^{-k}$. Finally (448) and the three already proved estimates give

$$|E^3 S_{k+1}| \le \frac{1}{2} |E^3 S_k| + C 4^{-k},$$

so $|E^3 S_k| \le C_3 2^{-k}$. All constants depend only on the uniform input bound and the compact coefficient bounds, not on N or on the terminal depth. Taking $k = k(N) \to \infty$ proves (451). □ □

For the manuscript's terminal input

$$\Delta_{N,0}(s) = \lambda_N s(1 + \zeta_N(s)), \tag{452}$$

one has

$$S_{N,0} = 1 + E \log(1 + \zeta_N). \tag{453}$$

Hence (449) follows directly from a uniform denominator gap $|1+\zeta_N| \geq c_\zeta > 0$ and bounded first four Euler derivatives of ζ_N. The Lambert part of the old condition (439) is therefore replaced by the exact theorem (440)–(451); no small multiplicative remainder $\omega_{N,k(N)}$ is required once the orbit leaves the branch core.

131B. Consequence for the coefficient bootstrap

Every reconstruction coefficient that depends smoothly on a finite collection of $\Delta_{N,k}$ and their logarithmic Euler sensitivities inherits uniform Euler bounds from (450), provided its algebraic denominators remain separated from zero. Thus the Lambert contribution to (433) is now a theorem of the exact recurrence. The remaining content of (433) is only the non-Lambert coefficient/cutoff convergence and the stated denominator gaps.

38 Part 21: Coefficient convergence and the positive gap

132. Continuity in α transfers a strict anchor margin

Let D_α be the terminal coefficient functional evaluated on the continued profile and limiting reconstruction coefficients. Since Part 9 provides a C^1 profile branch and the coefficient formulas are smooth in α on a compact pole-free interval, D_α is continuous. Thus, if

$$D_1 \geq 2d_* > 0, \tag{441}$$

then there is $\delta_{\mathrm{gap}} > 0$ such that

$$\boxed{D_\alpha \geq \frac{3}{2} d_*, \qquad 1 \leq \alpha \leq 1 + \delta_{\mathrm{gap}}.} \tag{442}$$

This is an interval statement and uses no isolated exponent.

Suppose now that the explicit finite-N coefficient vector satisfies

$$\sup_{\alpha \in I_\delta} \|\mathbf{c}_{N,\alpha} - \mathbf{c}_\alpha\|_{\mathcal{C}} \leq \epsilon_N^c, \qquad \epsilon_N^c \to 0, \tag{443}$$

and that $W_{N,\alpha}$ is in the profile tube. Local Lipschitz continuity then gives

$$|D_{N,\alpha} - D_\alpha| \leq C_D \left(\|W_{N,\alpha} - B_\alpha\|_{\mathcal{X}} + \epsilon_N^c \right). \tag{444}$$

Choose the terminal tube and N so that the right side is at most $d_*/2$. Combining (442)–(444) proves

$$\boxed{D_{N,\alpha} \geq d_* > 0} \tag{445}$$

uniformly on a possibly smaller nonempty interval.

Thus the coefficient-gap problem has been reduced to one explicit convergence statement, (443). The gap itself is not an additional independent hypothesis.

133. What the finite-N Lambert expansion proves about (443)

A finite-depth expansion of the form (436)–(437) proves coefficient convergence for every coefficient that is a smooth function of finitely many quantities

$$\Delta_{N,j}, \quad E\Delta_{N,j}, \dots, E^m\Delta_{N,j}, \qquad j \le k, \quad m \le 3, \tag{446}$$

provided the corresponding remainders converge uniformly. Indeed, the mean value theorem on a compact coefficient set yields

$$\|\mathbf{c}_{N,\alpha} - \mathbf{c}_\alpha\|_{\mathcal{C}} \le C \max_{j\le k,\, m\le 3} \|E^m\Delta_{N,j} - E^m\Delta_j\|. \tag{447}$$

Therefore fixed-depth coefficient convergence is a theorem once the displayed Lambert asymptotics are uniform. For increasing depth $k(N)$, (447) again shows exactly which uniform cascade estimate is required; it cannot be replaced by pointwise-in-N convergence.

39 Part 22: Closing the two invariant boundaries

134. The lower and upper signs must come from the same scalar field

Write the exact normalized terminal equation as

$$\partial_\tau W = \Lambda_{N,\alpha}(s) W\big[D_{N,\alpha}W - K_{N,\alpha}(W) + R_{N,\alpha}(s,W)\big], \qquad \Lambda_{N,\alpha} > 0. \tag{448}$$

Assume on a compact interval $[W_-, W_+]$

$$D_{N,\alpha} \ge d_* > 0, \qquad |R_{N,\alpha}(s,W)| \le \varepsilon_R(s,N), \quad \varepsilon_R \to 0, \tag{449}$$

and suppose the limiting bracket

$$\mathscr{B}_\alpha(W) = D_\alpha W - K_\alpha(W) \tag{450}$$

has strict opposite boundary signs

$$\inf_{\alpha\in I_\delta} \mathscr{B}_\alpha(W_-) > 2\sigma, \qquad \sup_{\alpha\in I_\delta} \mathscr{B}_\alpha(W_+) < -2\sigma \tag{451}$$

for some $\sigma > 0$. Uniform coefficient convergence and (449) imply, for large N and sufficiently small s,

$$\mathscr{B}_{N,\alpha}(W_-) \ge \sigma, \qquad \mathscr{B}_{N,\alpha}(W_+) \le -\sigma. \tag{452}$$

Consequently the vector field points inward on both boundaries and

$$\boxed{W_- \le W_{N,\alpha}(\tau) \le W_+} \tag{453}$$

is forward invariant.

This proves the upper-boundary part of (406)–(408) once the strict limiting upper sign in (451) is verified from the continued profile vector field. It also shows why a positive lower coefficient gap alone cannot prove the upper barrier: the nonlinear dissipative function $K_\alpha(W)$ must dominate at W_+.

40 Part 23: The defect estimate (376) reduced to explicit finite-N moduli

135. A noncircular defect theorem

Let

$$\omega_N^{\mathrm{coeff}}(s) := \sup_{\alpha\in I_\delta} \|\mathbf{c}_{N,\alpha}(s) - \mathbf{c}_\alpha\|_{\mathcal{C}}, \tag{454}$$

$$\omega_N^{\mathrm{cut}}(s) := \sup_{\alpha\in I_\delta} \|\mathcal{E}_{N,\alpha}^{\mathrm{cut}}(s)\|_{\mathcal{X}}. \tag{455}$$

On a fixed profile tube the normalized vector field is locally Lipschitz in its coefficient vector. Using (420) and (429),

$$\boxed{\|\mathcal{E}_{N,\alpha}(s)\|_{\mathcal{X}} \le C\omega_N^{\mathrm{coeff}}(s) + C\left(p_{N,\alpha}^{-1}s^2 + s^{\mu_c}\right) + \omega_N^{\mathrm{cut}}(s).} \tag{456}$$

Therefore the bridge estimate (376) follows if the two genuinely construction-specific moduli obey

$$\omega_N^{\mathrm{coeff}}(s) \le Cs^{\mu_1} + \epsilon_N^{(1)}, \qquad \omega_N^{\mathrm{cut}}(s) \le Cs^{\mu_2} + \epsilon_N^{(2)}, \tag{457}$$

with $\mu_1, \mu_2 > 0$ and $\epsilon_N^{(i)} \to 0$. Indeed, with

$$\mu_{\mathrm{br}} = \min\{2, \mu_c, \mu_1, \mu_2\} > 0, \qquad \epsilon_N = Cp_{N,-}^{-1} + C\epsilon_N^{(1)} + \epsilon_N^{(2)}, \tag{458}$$

where $p_{N,-} = \inf_{\alpha\in I_\delta} p_{N,\alpha} \to \infty$, one gets

$$\boxed{\varepsilon_N^{\mathrm{br}}(s) \le C_{\mathrm{br}}s^{\mu_{\mathrm{br}}} + \epsilon_N.} \tag{459}$$

Thus (376) is reduced to the explicit coefficient and cutoff moduli (457). There is no remaining fractional one-phase defect and the algebraic reconstruction contribution is already controlled by (427).

41 Part 24: Uniform compactness and passage to the limit

136. A sufficient compactness theorem adapted to the interval

Fix $T < T_*$. Let $\delta > 0$ be small and choose $\varepsilon > \delta$. Since $\alpha \le 1 + \delta$, a uniform bound in the fixed space $H^{1+\delta+\varepsilon}$ dominates every $H^{\alpha+\varepsilon/2}$ norm. Assume

$$\sup_{N,\alpha} \|b_{N,\alpha}\|_{L^2(0,T;H^{1+\delta+\varepsilon})} \le C_T, \tag{460}$$

$$\sup_{N,\alpha} \|\partial_t b_{N,\alpha}\|_{L^2(0,T;H^{-1-\delta-\varepsilon})} \le C_T. \tag{461}$$

For every $0 < \theta < \varepsilon/2$, Aubin–Lions gives relative compactness in

$$L^2(0,T;H^{1+\delta+\varepsilon-\theta}_{\mathrm{loc}}). \tag{462}$$

In particular, after extraction,

$$b_{N,\alpha} \to b_\alpha \quad \text{strongly in } L^2(0,T;H^{\alpha}_{\mathrm{loc}}) \tag{463}$$

uniformly for α in a smaller compact interval, in the sense needed for all subsequences with $\alpha_N \to \alpha$.

The fractional term passes weakly because

$$(-\Delta)^{\alpha} : H^{\alpha} \to H^{-\alpha} \tag{464}$$

is bounded, while the strong convergence supplies the quadratic products. If the exact scalar residual tends to zero in $L^1(0,T;H^{-m})$, the limiting scalar equation follows distributionally.

The estimates (460)–(461) are therefore a sufficient fixed-space replacement for (410)–(413). They are still energy estimates that must be derived from the finite-N equations; compactness itself is no longer an additional mystery once they are available.

137. Persistence of the terminal barrier

Assume the invariant strip (453) is established with constants independent of N and α. Then

$$b_{N,\alpha}(s) \ge c_- s^{-r_\alpha} \tag{465}$$

on a common terminal interval. If the distinguished characteristic maps converge locally uniformly and the scalar convergence is locally uniform away from $s = 0$, then for each fixed $s > 0$ one may pass $N \to \infty$ in (465):

$$\boxed{b_\alpha(s) \ge c_- s^{-r_\alpha}.} \tag{466}$$

The terminal lower bound is therefore stable under the limit; no diagonal limit at $s = 0$ is needed.

42 Part 25: Revised status of equation (418) after the increasing-depth theorem

138. What is now proved

The term-by-term attack, together with the exact increasing-depth Lambert theorem of Part 20, proves the following implications without selecting an isolated exponent:

$$\begin{array}{rcl}
\text{strict one phase} & \Longrightarrow & \mathcal{E}^{\mathrm{frac}} = 0, \\
\text{profile tube} + \Gamma \gtrsim s^{-1} & \Longrightarrow & (\Gamma b)^{-1} = O(p_N^{-1} s^2), \\
\text{positive-power coefficient remainder} & \Longrightarrow & \text{residual factorization}, \\
\text{exact Lambert recurrence} & \Longrightarrow & \|E^m (E \log \Delta_{N,k})\| \lesssim 2^{-k},\ 0 \le m \le 3, \\
(432)\text{–}(433) & \Longrightarrow & \text{all Euler derivative bounds}, \\
\text{strict limiting gap + coefficient convergence} & \Longrightarrow & D_{N,\alpha} \ge d_* > 0, \\
\text{two strict limiting boundary signs} & \Longrightarrow & \text{uniform invariant strip}, \\
(460)\text{–}(461) & \Longrightarrow & \text{strong preterminal compactness}.
\end{array} \tag{467}$$

139. Reduced construction-level ledger

The former item (i) in (468), the uniform increasing-depth Lambert remainder hypothesis, is removed. It is replaced by the exact recurrence theorem (440)–(451), which is stronger than the fixed-depth remainder formulation and is uniform for arbitrary $k = k(N)$. The remaining construction-level obligations are therefore

(i) coefficient and cutoff moduli (457), including their first three Euler derivatives where required, excluding the Lambert sensitivity already controlled by (450);
(ii) the strict upper limiting sign in (451) of the invariant-strip analysis (with the numbering there understood independently of the new Lambert equation (451)); and
(iii) the fixed-space energy estimates (460)–(461), together with vanishing of the exact scalar residual in the limiting topology.

(468)

Thus the increasing-depth Lambert estimate is no longer an unresolved global hypothesis. The overlap inequalities (380), (383)–(385) reduce to the non-Lambert coefficient/cutoff moduli, the already proved bounded Region-I excursion, and the stable profile semigroup.

140. Conditional closure theorem with the Lambert hypothesis removed

Theorem 9 (Reduced finite-N closure theorem after exact Lambert contraction)**.** *Let $1 < \alpha < 1 + \delta_*$ and let the positive continued profile branch of Part 9 be given. Assume the terminal input satisfies the uniform hypotheses of the increasing-depth theorem (449), (452)–(453),*

and the required algebraic denominator gaps. Suppose the explicit finite-N reconstruction establishes the three remaining construction-level estimates in (468), and suppose the limiting normalized profile vector field has the strict lower and upper invariant-boundary signs. Then the bridge defect satisfies (459), the finite-N trajectory enters the profile tube, the Euler bootstrap closes, the coefficient gap remains uniformly positive, the terminal strip is invariant, and a subsequence converges on every preterminal interval to a scalar solution satisfying

$$b_\alpha(s) \asymp s^{-(2\alpha-1)}. \tag{469}$$

Along the distinguished characteristic $\dot{s} = 1 - b_\alpha$ this yields

$$\boxed{T_{*,\alpha} - t \asymp s^{2\alpha}, \qquad b_\alpha(t) \asymp (T_{*,\alpha} - t)^{-(2\alpha-1)/(2\alpha)}.} \tag{470}$$

Proof. Part 20 proves the increasing-depth Lambert sensitivity and its first three Euler derivatives directly from the exact principal-branch recurrence, so no fixed-k remainder hypothesis is used. Equations (419)–(420) remove the one-phase fractional defect, and (423)–(429) control the algebraic reconstruction residual. The non-Lambert coefficient/cutoff modulus in (468)(i), combined with the exact Lambert bounds (450), gives the bridge defect estimate (459). The bridge theorem of Part 14 gives profile entry. Equations (430)–(435), together with (450) and the remaining coefficient bounds, close the Euler bootstrap. Equations (441)–(447) of Part 21 transfer the positive coefficient gap. The invariant-strip argument supplies the lower and upper barriers. Finally the fixed-space energy and residual estimates in (468)(iii) give the strong preterminal limit and preserve the terminal barrier. Integrating $\dot{s} = 1 - b_\alpha$ under (469) gives (470). □ □

43 Part 26: Two-regime interpretation and exact replacement of the old (439)

141. Branch core versus increasing depth

The exact proof clarifies the two regimes that were previously mixed in the fixed-depth formula (437). If $\Delta_{N,k} \ll 1$, then

$$\Delta_{N,k+1} = \sqrt{2\Delta_{N,k}}\big(1 + O(\sqrt{\Delta_{N,k}})\big), \tag{471}$$

so the local power law halves the terminal exponent at each step. This is the branch-core regime. When the orbit is no longer asymptotically small, the monomial representation should be abandoned; the exact sensitivity equation

$$S_{N,k+1} = A(\Delta_{N,k})S_{N,k}, \qquad 0 < A < \frac{1}{2}, \tag{472}$$

continues to contract globally. Therefore there is no loss of derivative control at the transition from fixed branch depth to increasing depth.

142. Uniformity for coupled sequences

Let $N_j \to \infty$ and let $k_j = k(N_j)$ be arbitrary. If k_j remains bounded, the fixed-depth branch expansion applies on a sufficiently small terminal window. If $k_j \to \infty$, (450) gives

$$\sup_j 2^{k_j} |E^m S_{N_j,k_j}| \le C_m, \qquad 0 \le m \le 3, \tag{473}$$

and hence

$$E^m S_{N_j,k_j} \to 0. \tag{474}$$

Thus every coupled path is covered: bounded depths by the local expansion, and unbounded depths by the exact contraction theorem. This is precisely the uniformity that the old condition (439) was intended to provide.

143. Updated mathematical status

The increasing-depth Lambert derivative problem is closed at the level needed by the coefficient bootstrap. What remains is not a Lambert branch-point problem. It is the verification, from the explicit reconstruction, of the three non-Lambert items in the revised ledger (468): coefficient/cutoff convergence, the strict upper invariant-boundary sign, and the fixed-space energy/residual estimates required for the limiting passage.

44 Part 27: Coefficient/cutoff convergence modulus after exact Lambert contraction

144. Purpose and logical scope

The reduced ledger (468) still lists the non-Lambert coefficient/cutoff modulus (457). This part attacks that item directly. The exact Lambert sensitivity problem has already been removed by Part 20, so the coefficient vector is now split into an exact Lambert block and a genuinely non-Lambert reconstruction block. The point of the present argument is twofold:

1. to prove a quantitative coefficient convergence theorem for every coefficient obtained by a smooth algebraic composition of the exact Lambert variables, their Euler sensitivities, positive terminal powers, and finite-N regularization parameters; and

2. to identify the exact scale-separation condition under which a cutoff commutator satisfies the second estimate in (457).

The second point is important: a moving cutoff cannot have a globally small Euler norm across its own transition annulus. Smallness follows either from support separation or from a vanishing amplitude multiplying the cutoff.

145. The coefficient algebra

Let

$$S_{N,j} := E \log \Delta_{N,j}, \qquad E = s\partial_s,$$

and let $\mathbf{Z}_{N,\alpha}(s)$ denote the finite collection of variables entering the normalized reconstruction coefficients. We write

$$\mathbf{Z}_{N,\alpha} = \big(\Delta_{N,k(N)}, S_{N,k(N)}, ES_{N,k(N)}, E^2S_{N,k(N)}, E^3S_{N,k(N)}, \mathbf{r}_{N,\alpha}, \boldsymbol{\varepsilon}_N\big), \tag{475}$$

where $\mathbf{r}_{N,\alpha}$ denotes the remaining non-Lambert terminal variables and $\boldsymbol{\varepsilon}_N$ the finite-N regularization parameters. Assume that each component of $\mathbf{r}_{N,\alpha}$ has a limiting value $\mathbf{r}_\alpha$ and admits the factorization

$$E^\ell\big(\mathbf{r}_{N,\alpha} - \mathbf{r}_\alpha\big) = s^{\mu_r}\mathbf{a}_{N,\alpha,\ell}(s) + \mathbf{e}_{N,\alpha,\ell}, \qquad 0 \le \ell \le 3, \tag{476}$$

with

$$\mu_r > 0, \qquad \sup_{N,\alpha,\ell} \|\mathbf{a}_{N,\alpha,\ell}\|_\infty \le C_r, \qquad \sup_{\alpha,\ell} |\mathbf{e}_{N,\alpha,\ell}| \le \epsilon_N^r \to 0. \tag{477}$$

This is precisely the positive-terminal-power structure already isolated in (396)–(399), but now written at the level of the actual coefficient inputs.

Let the coefficient vector be

$$\mathbf{c}_{N,\alpha}(s) = \mathfrak{C}_\alpha(\mathbf{Z}_{N,\alpha}(s)), \qquad \mathbf{c}_\alpha = \mathfrak{C}_\alpha(\mathbf{Z}_\alpha), \tag{478}$$

where $\mathfrak{C}_\alpha$ is C^4 on a compact set K containing all finite-N and limiting coefficient states, uniformly for $\alpha \in I_\delta$. All algebraic denominators occurring in $\mathfrak{C}_\alpha$ are assumed to satisfy the already stated positive gap conditions. Thus

$$\sup_{\alpha\in I_\delta} \|D^m\mathfrak{C}_\alpha\|_{L^\infty(K)} \le M_m, \qquad 1 \le m \le 4. \tag{479}$$

146. Quantitative convergence of the exact Lambert state

Part 20 gives, uniformly for arbitrary $k = k(N)$,

$$\max_{0\le m\le 3} \|E^m S_{N,k(N)}\|_\infty \le C_L 2^{-k(N)}. \tag{480}$$

For the terminal Lambert state itself there are two possibilities. In the branch-core regime the fixed-depth expansion gives the appropriate limiting state directly. In the increasing-depth regime the exact map

$$\Delta_{j+1} = \mathcal{F}(\Delta_j), \qquad \mathcal{F}(\Delta) = 1 + W_0(-e^{-1-\Delta}) \tag{481}$$

has the positive attracting fixed point Δ_*. On any compact trapping interval $J \Subset (0,1)$ containing Δ_* after the entry iterate, there is a number $\rho_* < 1$ such that

$$\sup_{\Delta\in J} |\mathcal{F}'(\Delta)| \le \rho_*. \tag{482}$$

Consequently, if $j_c(N)$ is the first iterate in J, then

$$|\Delta_{N,k(N)} - \Delta_*| \le C_* \rho_*^{k(N)-j_c(N)}. \tag{483}$$

Combining (480) and (483), define the exact Lambert modulus

$$\epsilon_N^L := C_L 2^{-k(N)} + C_* \rho_*^{k(N)-j_c(N)}. \tag{484}$$

For every increasing-depth sequence for which $k(N) - j_c(N) \to \infty$, one has $\epsilon_N^L \to 0$.

Remark. No fixed-k branch expansion is used in (483). It is a consequence of the exact principal-branch map after the orbit has entered the compact attracting region. The principal branch W_0 is real on $[-e^{-1}, \infty)$, so the recurrence remains on the real branch throughout the positive Δ-dynamics.

147. Coefficient modulus theorem

Theorem 10 (Uniform coefficient convergence through three Euler derivatives)**.** *Assume (476)–(479), the exact Lambert estimates (480)–(484), and*

$$\sup_\alpha |\boldsymbol{\varepsilon}_N| \le \epsilon_N^{\mathrm{reg}} \to 0. \tag{485}$$

Then there is a constant C_c, independent of N, s, and $\alpha \in I_\delta$, such that for $0 \le \ell \le 3$,

$$\boxed{\sup_{\alpha \in I_\delta} \|E^\ell(\mathbf{c}_{N,\alpha}(s) - \mathbf{c}_\alpha)\|_{\mathcal{C}} \le C_c \left(s^{\mu_r} + \epsilon_N^L + \epsilon_N^r + \epsilon_N^{\mathrm{reg}}\right).} \tag{486}$$

In particular, with

$$\epsilon_N^{(1)} := C_c(\epsilon_N^L + \epsilon_N^r + \epsilon_N^{\mathrm{reg}}), \qquad \mu_1 = \mu_r, \tag{487}$$

the coefficient half of (457), including the first three Euler derivatives, is proved.

Proof. For $\ell = 0$, the mean-value theorem on the compact set K gives

$$\|\mathfrak{C}_\alpha(\mathbf{Z}_{N,\alpha}) - \mathfrak{C}_\alpha(\mathbf{Z}_\alpha)\|_{\mathcal{C}} \le M_1 \|\mathbf{Z}_{N,\alpha} - \mathbf{Z}_\alpha\|. \tag{488}$$

The Lambert components are controlled by (480)–(484), the non-Lambert components by (476)–(477), and the regularization parameters by (485). This proves (486) for $\ell = 0$.

For one Euler derivative, the chain rule gives

$$E\mathbf{c}_{N,\alpha} = D\mathfrak{C}_\alpha(\mathbf{Z}_{N,\alpha}) E\mathbf{Z}_{N,\alpha}. \tag{489}$$

Subtract the limiting expression and add and subtract $D\mathfrak{C}_\alpha(\mathbf{Z}_\alpha)E\mathbf{Z}_{N,\alpha}$. The first resulting term is bounded by $M_2|\mathbf{Z}_N - \mathbf{Z}_\alpha||E\mathbf{Z}_N|$, and the second by $M_1|E(\mathbf{Z}_N - \mathbf{Z}_\alpha)|$. The uniform Euler bounds from Part 20 and (476)–(477) therefore give (486) for $\ell = 1$.

For $\ell = 2, 3$, repeated Euler differentiation gives a finite Faà di Bruno sum. Every difference between the finite-N and limiting expressions contains at least one factor of one of the forms

$$E^j(\mathbf{Z}_{N,\alpha} - \mathbf{Z}_\alpha), \qquad 0 \le j \le \ell, \tag{490}$$

or

$$D^m\mathfrak{C}_\alpha(\mathbf{Z}_{N,\alpha}) - D^m\mathfrak{C}_\alpha(\mathbf{Z}_\alpha), \qquad 1 \le m \le \ell. \tag{491}$$

The first is bounded by the right side of (486); the second is bounded by $M_{m+1}|\mathbf{Z}_N - \mathbf{Z}_\alpha|$. Every remaining factor is uniformly bounded by (479), Part 20, and (477). Summing the finitely many terms proves (486) for $\ell = 2, 3$. □ □

148. Application to quotient-type reconstruction coefficients

The preceding theorem applies directly to coefficients of the form already used in the Euler closure,

$$q_{N,\alpha} = A_{N,\alpha}\frac{1+\eta_{N,\alpha}}{1+\rho_{N,\alpha}}, \qquad 1+\rho_{N,\alpha} \ge c_0 > 0. \tag{492}$$

Indeed, on the compact set defined by the denominator gap, the map

$$(A,\eta,\rho) \longmapsto A\frac{1+\eta}{1+\rho} \tag{493}$$

is C^∞ with uniformly bounded derivatives of every fixed order. Hence if A_N, η_N, ρ_N are themselves built from the state (475) and positive-power remainders (476), then

$$\max_{0\le\ell\le3} \|E^\ell(q_{N,\alpha} - q_\alpha)\|_\infty \le C_q\left(s^{\mu_r} + \epsilon_N^L + \epsilon_N^r + \epsilon_N^{\mathrm{reg}}\right). \tag{494}$$

Thus the quotient denominators do not create a new convergence obstruction; they only require the positive denominator gap already isolated in Parts 19 and 20.

149. Why a moving cutoff is not uniformly small on its transition annulus

Let a standard finite-N cutoff have the form

$$\chi_N(s) = \chi\left(\frac{s}{s_N}\right), \qquad s_N \downarrow 0, \tag{495}$$

where χ is smooth and nonconstant on a fixed compact transition interval. Then

$$E^j\chi_N(s) = \left(r\frac{d}{dr}\right)^j \chi(r)\Big|_{r=s/s_N}. \tag{496}$$

Therefore

$$\sup_{s\asymp s_N} |E^j\chi_N(s)| \tag{497}$$

is generally of order one, not $o(1)$. Hence the second estimate in (457) cannot be justified merely from $s_N \to 0$ if the norm takes a supremum across the cutoff transition. This observation removes a possible hidden nonuniformity from the limiting argument.

There are two rigorous ways to obtain the required cutoff modulus.

150. Support-separation theorem for the cutoff defect

Assume

$$\chi(r) = 0 \quad (r \le 1), \qquad \chi(r) = 1 \quad (r \ge 2), \tag{498}$$

and suppose the bridge/profile window satisfies

$$s \ge s_{\mathrm{br}}(N), \qquad \frac{s_{\mathrm{br}}(N)}{s_N} \longrightarrow \infty. \tag{499}$$

Then for all sufficiently large N,

$$\chi_N(s) \equiv 1 \quad \text{throughout the bridge window,} \tag{500}$$

and therefore every local cutoff derivative vanishes there:

$$\boxed{E^j(1-\chi_N) = 0, \qquad 0 \le j \le 3.} \tag{501}$$

For a local normalized vector field this gives

$$\boxed{\omega_N^{\mathrm{cut}}(s) = 0} \tag{502}$$

on the bridge window.

For the nonlocal fractional operator, support separation does not make the commutator identically zero. Let $\mathcal{E}_{N,\alpha}^{\mathrm{cut}}$ denote the exact nonlocal cutoff commutator. If its kernel representation satisfies, for some $\mu_{\mathrm{cut}} > 0$,

$$\sup_{\alpha \in I_\delta} \max_{0 \le j \le 3} \|E^j \mathcal{E}_{N,\alpha}^{\mathrm{cut}}(s)\|_{\mathcal{X}} \le C_{\mathrm{cut}} \left(\frac{s_N}{s}\right)^{\mu_{\mathrm{cut}}}, \qquad s \ge s_{\mathrm{br}}(N), \tag{503}$$

then

$$\boxed{\omega_N^{\mathrm{cut}}(s) \le C_{\mathrm{cut}} \left(\frac{s_N}{s_{\mathrm{br}}(N)}\right)^{\mu_{\mathrm{cut}}} =: \epsilon_N^{\mathrm{cut}} \to 0.} \tag{504}$$

Thus in the support-separated formulation one may take μ_2 arbitrary in (457) for the local cutoff terms and absorb the nonlocal tail entirely into $\epsilon_N^{(2)} = \epsilon_N^{\mathrm{cut}}$.

Status of (503). Equation (503) is not a consequence of smoothness of χ alone. It is the precise nonlocal kernel estimate that must be checked for the fractional realization used. In the strict one-phase class the three-dimensional operator has already been reduced exactly to the one-dimensional fractional operator, so (503) is a one-dimensional cutoff commutator estimate rather than a new transverse problem.

151. Vanishing-amplitude alternative

If support separation is not used, suppose instead that every cutoff term has an explicit vanishing prefactor:

$$\mathcal{E}_{N,\alpha}^{\mathrm{cut}}(s) = a_N s^{\mu_2} \mathcal{K}_{N,\alpha}(s), \qquad \max_{0 \le j \le 3} \|E^j \mathcal{K}_{N,\alpha}\|_{\mathcal{X}} \le C_K, \tag{505}$$

with $\sup_N |a_N| < \infty$, or more generally $|a_N| \le C + o(1)$. Leibniz' rule then gives

$$\boxed{\max_{0\le j\le 3} \|E^j \mathcal{E}^{\mathrm{cut}}_{N,\alpha}(s)\|_{\mathcal{X}} \le C'_K s^{\mu_2}.} \tag{506}$$

If an additional finite-N error is present, it is added to the right side as $\epsilon_N^{(2)}$. This is exactly the cutoff half of (457).

152. Revised bridge modulus

Combining the coefficient theorem (486) with the defect decomposition (456) gives

$$\|\mathcal{E}_{N,\alpha}(s)\|_{\mathcal{X}} \le C\left[s^{\mu_r} + s^{\mu_c} + p_{N,\alpha}^{-1} s^2\right] + C\left(\epsilon_N^L + \epsilon_N^r + \epsilon_N^{\mathrm{reg}}\right) + \omega_N^{\mathrm{cut}}(s). \tag{507}$$

Under either the support-separated cutoff estimate (504) or the vanishing-amplitude estimate (506), there are $\mu_{\mathrm{br}} > 0$ and $\epsilon_N^{\mathrm{br}} \to 0$ such that

$$\boxed{\|\mathcal{E}_{N,\alpha}(s)\|_{\mathcal{X}} \le C_{\mathrm{br}} s^{\mu_{\mathrm{br}}} + \epsilon_N^{\mathrm{br}},} \tag{508}$$

where one may choose

$$\mu_{\mathrm{br}} = \min\{2, \mu_c, \mu_r, \mu_2\} \tag{509}$$

in the vanishing-amplitude formulation, while in the support-separated formulation the cutoff tail is absorbed into ϵ_N^{br}. This is the desired bridge forcing estimate (376).

153. What has and has not been proved about ledger item (i)

The coefficient part of the old item (468)(i) is no longer an independent analytic hypothesis once the explicit reconstruction coefficients are checked to belong to the smooth coefficient algebra (478) with the positive-power remainders (476). The proof is the chain-rule theorem (486), and the Lambert part is already supplied by Part 20.

The cutoff part has been reduced to a sharper dichotomy. A moving cutoff is *not* uniformly small across its transition annulus. Therefore a valid proof must establish either

$$\boxed{\frac{s_{\mathrm{br}}(N)}{s_N} \to \infty \quad \text{and the one-dimensional nonlocal tail estimate (503),}} \tag{510}$$

or the vanishing-amplitude factorization (505). No weaker appeal to $s_N \to 0$ is sufficient.

Consequently the revised construction-level ledger is

$$\boxed{\begin{array}{c}\text{(i) verify the explicit non-Lambert coefficient inputs satisfy (476)–(479),}\\ \text{and verify either the cutoff separation estimate (503) or (505);}\\ \text{(ii) prove the strict upper limiting sign of the invariant strip; and}\\ \text{(iii) prove the fixed-space energy estimates and vanishing scalar residual}\\ \text{needed for the strong limiting passage.}\end{array}} \tag{511}$$

The first line is now a finite list of substitutions into explicit reconstruction formulas rather than an abstract coefficient-convergence hypothesis.

154. Updated closure theorem after the coefficient attack

Theorem 11 (Finite-N bridge after coefficient convergence)**.** *Let $1 < \alpha < 1+\delta_*$ and assume the positive profile branch of Part 9, the exact Lambert contraction theorem of Part 20, and the denominator gaps of Parts 19–20. Suppose the explicit non-Lambert reconstruction variables obey (476)–(479) and (485). Suppose, in addition, that the cutoff is treated by either (503)–(504) or (505)–(506). Then the bridge defect satisfies (508) uniformly for $\alpha \in I_\delta$, and the profile-entry/bootstrap argument of Parts 14–16 applies without a separate coefficient modulus hypothesis.*

Proof. The exact Lambert state is controlled by (480)–(484). The non-Lambert coefficient inputs are controlled by (476)–(477), and the smooth coefficient algebra transfers these estimates through three Euler derivatives by (486). The reconstruction and positive-power terms are already controlled by (423)–(429) and (396)–(399). The cutoff contribution is controlled by (504) or (506). Substitution into (456) gives (507), hence (508). The Duhamel bridge estimate of Part 14 and the terminal bootstrap of Part 15 then apply. □ □

155. Updated mathematical status

The coefficient/cutoff item has been substantially reduced but should not be deleted unconditionally until the actual finite-N formulas are substituted into (476)–(479) and the actual cutoff realization is checked against (503) or (505). The important advance is that no new abstract convergence mechanism is missing: coefficient convergence follows from exact Lambert contraction plus smooth finite-dimensional composition, and cutoff convergence is equivalent to an explicit scale-separation/nonlocal-tail estimate or an explicit vanishing-amplitude factorization.

45 Part 28: Audit of the alleged cutoff and removal of the spurious cutoff modulus

156. Why the cutoff question must be re-audited from the reconstruction

Part 27 introduced, as a possible localization device, a moving function

$$\chi_N^{\mathrm{cut}}(s) = \chi^{\mathrm{cut}}(s/\sigma_N), \qquad \sigma_N \downarrow 0, \tag{512}$$

and reduced its nonlocal contribution to the model estimate (503). That reduction was logically correct *if such a moving cutoff is actually part of the finite-N reconstruction.* The source reconstruction must now be checked before (512) is treated as a genuine object.

The audit gives a different conclusion. The symbol s_N occurring in the actual one-phase reconstruction is the *phase variable*

$$\boxed{s_N = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N},} \tag{513}$$

not a positive cutoff scale. Likewise, the symbol $\chi_N(s)$ occurring in the square-root reconstruction is a *geometric coefficient* satisfying

$$\boxed{A_N(s)^2 = -2\chi_N(s)B_N(s), \qquad \chi_N(s) \sim -\chi_{0,N}s^{1/2},} \tag{514}$$

on the selected terminal branch. It is not a compactly supported transition function of the form (512).

Consequently the notation used in Part 27 conflated two different objects: the actual phase/geometric coefficient of the reconstruction and a hypothetical localizing cutoff. This section corrects that point. We henceforth reserve

$$s_N^{\mathrm{ph}} = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N} \tag{515}$$

for the finite-N phase, write χ_N^{geom} for the square-root geometric coefficient, and write χ_N^{cut} only if an independent localization cutoff is explicitly inserted.

157. No cutoff commutator is generated by the strict one-phase reduction

The strict one-phase architecture has the exact Fourier reduction

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha F(s_N^{\mathrm{ph}}) = \kappa^{2\alpha}(-\partial_{ss})^\alpha F(s_N^{\mathrm{ph}}), \qquad \kappa^2 = C_1^2 + C_2^2 + C_3^2.} \tag{516}$$

This identity applies to the actual one-phase profiles themselves. It does not require multiplication by a moving cutoff in order to eliminate transverse variables. Therefore, if the finite-N reconstruction is used exactly in its one-phase form, there is no additional term

$$[(-\partial_{ss})^\alpha, \chi_N^{\mathrm{cut}}]F_N \tag{517}$$

to estimate, because no χ_N^{cut} has been introduced.

Thus the quantity called $\mathcal{E}_{N,\alpha}^{\mathrm{cut}}$ in Part 27 is not an intrinsic remainder of the displayed finite-N one-phase reconstruction. The estimate (503) is a valid auxiliary lemma for a separately localized construction, but it is not a required hypothesis for the uncut strict one-phase construction.

Theorem 12 (Cutoff-free one-phase defect reduction)**.** *Assume that the finite-N scalar and reconstructed velocity coefficients are functions of the single phase s_N^{ph} and that no independent spatial localization multiplier is inserted. Then*

$$\boxed{\mathcal{E}_{N,\alpha}^{\mathrm{cut}} \equiv 0, \qquad \omega_N^{\mathrm{cut}} \equiv 0.} \tag{518}$$

In particular, (503)–(506) are unnecessary for the strict uncut one-phase bridge.

Proof. Every finite-N field entering the reduction is evaluated directly as a function of s_N^{ph}. Applying $(-\Delta)^\alpha$ therefore gives the exact one-dimensional multiplier (516). A cutoff commutator can arise only from a product $\chi_N^{\mathrm{cut}}F_N$. Since no such product occurs in the uncut reconstruction, the corresponding commutator and all its Euler derivatives vanish identically. Hence (518) follows. □

158. The actual geometric coefficient is part of the coefficient algebra

The nontrivial object is χ_N^{geom} in (514). It must not be discarded. Its contribution belongs to the ordinary reconstruction coefficient vector. Define

$$\mathbf{Z}_{N,\alpha}^{\mathrm{act}} = \big(\Delta_{N,k(N)}, S_{N,k(N)}, ES_{N,k(N)}, E^2S_{N,k(N)}, E^3S_{N,k(N)}, \chi_N^{\mathrm{geom}}, \mathbf{r}_{N,\alpha}, \boldsymbol{\varepsilon}_N\big). \tag{519}$$

The square-root velocity coefficient is the smooth algebraic map

$$A_N = \mathfrak{A}(B_N, \chi_N^{\mathrm{geom}}) := \sqrt{-2\chi_N^{\mathrm{geom}} B_N} \tag{520}$$

on every profile tube on which

$$-2\chi_N^{\mathrm{geom}} B_N \geq c_A > 0. \tag{521}$$

Hence, on such a tube, the coefficient-algebra theorem of Part 27 applies to A_N and to every rational reconstruction coefficient built from A_N whose denominators retain the previously established positive gaps.

Suppose the actual geometric coefficient has the terminal expansion

$$\chi_N^{\mathrm{geom}}(s) = -\chi_{0,\alpha} s^{\beta_\chi}\big(1 + r_{\chi,N,\alpha}(s)\big), \qquad \beta_\chi > 0, \tag{522}$$

with

$$\max_{0\leq j\leq 3} \|E^j r_{\chi,N,\alpha}\|_\infty \leq C_\chi s^{\mu_\chi} + \epsilon_N^\chi, \qquad \epsilon_N^\chi \to 0. \tag{523}$$

Then

$$E\log(-\chi_N^{\mathrm{geom}}) = \beta_\chi + E\log(1 + r_{\chi,N,\alpha}), \tag{524}$$

and therefore

$$\boxed{\max_{0\leq j\leq 2} \big\|E^j\big(E\log(-\chi_N^{\mathrm{geom}}) - \beta_\chi\big)\big\|_\infty \leq C\big(s^{\mu_\chi} + \epsilon_N^\chi\big).} \tag{525}$$

For the square-root behavior displayed in the reconstruction, $\beta_\chi = 1/2$. Thus the geometric χ_N produces a controlled terminal power, not a moving-cutoff error.

159. Direct propagation through the square-root reconstruction

Taking logarithms of (520) gives

$$E\log A_N = \frac{1}{2}E\log B_N + \frac{1}{2}E\log(-\chi_N^{\mathrm{geom}}). \tag{526}$$

Writing

$$q_{B,N} := -E\log B_N, \tag{527}$$

we obtain

$$\boxed{E\log A_N = -\frac{1}{2}q_{B,N} + \frac{\beta_\chi}{2} + O(s^{\mu_\chi} + \epsilon_N^\chi).} \tag{528}$$

Repeated Euler differentiation and (525) yield, for $1 \le j \le 3$,

$$\boxed{\|E^j \log A_N - E^j \log A_\alpha\|_\infty \le C \left(\sum_{m=0}^{j-1} \|E^m (q_{B,N} - q_{B,\alpha})\|_\infty + s^{\mu_\chi} + \epsilon_N^\chi \right).} \tag{529}$$

Since $u_{x,N} = B_N / A_N$, the same estimate controls the logarithmic derivatives of $u_{x,N}$. Consequently the actual square-root geometry fits the same finite-dimensional smooth coefficient algebra as the Lambert and quotient blocks; it does not create a separate nonlocal cutoff modulus.

160. Revised bridge defect with no artificial cutoff term

After the correction above, equation (456) is replaced, for the strict uncut one-phase construction, by

$$\boxed{\|\mathcal{E}_{N,\alpha}(s)\|_{\mathcal{X}} \le C \omega_N^{\mathrm{coeff,act}}(s) + C \left(p_{N,\alpha}^{-1} s^2 + s^{\mu_c} \right),} \tag{530}$$

where $\omega_N^{\mathrm{coeff,act}}$ includes the actual geometric coefficient χ_N^{geom} through (519)–(529). In particular there is no ω_N^{cut} term.

If the explicit non-Lambert inputs, including χ_N^{geom}, satisfy positive-power estimates of the form (476)–(477) and (523), then

$$\omega_N^{\mathrm{coeff,act}}(s) \le C s^{\mu_{\mathrm{act}}} + \epsilon_N^{\mathrm{act}}, \tag{531}$$

where

$$\mu_{\mathrm{act}} = \min\{\mu_r, \mu_\chi, \mu_L\} > 0, \qquad \epsilon_N^{\mathrm{act}} \to 0. \tag{532}$$

Substitution into (530) gives

$$\boxed{\|\mathcal{E}_{N,\alpha}(s)\|_{\mathcal{X}} \le C_{\mathrm{br}} s^{\mu_{\mathrm{br}}} + \epsilon_N^{\mathrm{br}}, \qquad \mu_{\mathrm{br}} = \min\{2, \mu_c, \mu_{\mathrm{act}}\} > 0, \quad \epsilon_N^{\mathrm{br}} \to 0.} \tag{533}$$

This is exactly the bridge forcing estimate required by Parts 14–16, with no cutoff-tail hypothesis.

161. What is proved and what still requires substitution

The cutoff question is now resolved at the structural level: the displayed strict one-phase reconstruction contains no independent moving localization cutoff of the type postulated in (495). The s_N of the reconstruction is a phase coordinate, and χ_N is the square-root geometric coefficient. Therefore the nonlocal estimate (503) is not an outstanding condition for the uncut one-phase construction.

What remains inside the former coefficient/cutoff ledger item is only the coefficient substitution problem: the exact finite-N formulas for χ_N^{geom} and the other non-Lambert variables must satisfy the positive-power estimates (523) and (476)–(477), uniformly in $\alpha \in I_\delta$. This is a coefficient estimate, not a cutoff estimate.

Accordingly the revised ledger is

$$\boxed{\begin{array}{c}\text{(i) verify the remaining actual non-Lambert coefficient expansions,}\\ \text{including the geometric } \chi_N^{\mathrm{geom}}\text{, in (523) and (476)–(477);}\\ \text{(ii) prove the strict upper limiting sign of the invariant strip;}\\ \text{(iii) prove the fixed-space energy estimates and vanishing scalar residual}\\ \text{needed for the strong } N\to\infty \text{ passage.}\end{array}} \tag{534}$$

The artificial moving-cutoff condition (503) is removed from this ledger.

162. Corrected finite-N bridge theorem

Theorem 13 (Cutoff-free finite-N bridge)**.** *Let $1<\alpha<1+\delta_*$ and assume the positive profile branch of Part 9, the exact Lambert contraction theorem of Part 20, and the denominator gaps of Parts 19–20. Assume that the actual finite-N reconstruction is kept in the strict one-phase class and contains no additional localization multiplier. Assume that its non-Lambert coefficients, including χ_N^{geom}, satisfy (476)–(477), (521), and (523). Then the bridge defect satisfies (533) uniformly for $\alpha\in I_\delta$. Consequently the profile-entry and terminal bootstrap arguments of Parts 14–16 apply without either an abstract coefficient modulus or a cutoff-commutator hypothesis.*

Proof. The exact one-phase reduction (516) eliminates any transverse fractional remainder. The cutoff-free defect theorem (518) eliminates the artificial cutoff block. Exact Lambert sensitivity is controlled by Part 20. The actual geometric coefficient is propagated through the square-root reconstruction by (522)–(529), while the remaining non-Lambert variables are propagated by the smooth coefficient algebra of Part 27. The algebraic reconstruction remainder has already been reduced to $p_{N,\alpha}^{-1}s^2+s^{\mu_c}$. Combining these bounds gives (530)–(533), and the Duhamel bridge and terminal bootstrap then apply. □

163. Status after the cutoff audit

The preceding audit changes the interpretation of Part 27 in an essential way. Equations (495)–(506) should be read only as an auxiliary localization lemma for a modified construction in which an independent cutoff is deliberately inserted. They are not part of the strict finite-N one-phase reconstruction unless such an insertion is explicitly made.

For the actual uncut one-phase architecture, the remaining proof program is therefore narrower: establish the positive-power convergence of the genuine non-Lambert reconstruction coefficients, prove the strict upper invariant-boundary sign, and derive the fixed-space energy/residual estimates needed to pass $N\to\infty$. No separate cutoff scale or fractional cutoff commutator is required.

46 Part 29: Exact periodic geometric coefficient and the characteristic compatibility test

164. Exact source formula for the geometric coefficient

The preceding Part 28 deliberately stopped at the abstract expansion (522)–(523). The square-root reconstruction source gives more information. Its convenient periodic realization of the $\mathcal{J}_3$ geometry is

$$\boxed{\chi^{\mathrm{geom}}(x) = \sin(x - x_*), \qquad s_{\mathrm{geo}}(x) = \big(\chi^{\mathrm{geom}}(x)\big)^2 = \sin^2(x - x_*).} \tag{535}$$

On the one-sided real branch

$$\mathcal{J}_3 = (x_* - \delta, x_*), \tag{536}$$

we have $\chi^{\mathrm{geom}} < 0$, and therefore

$$\boxed{\chi^{\mathrm{geom}}(x) = -\sqrt{s_{\mathrm{geo}}(x)}} \tag{537}$$

exactly. Thus, if the similarity variable in (522) were s_{geo} itself, (523) would hold with

$$\chi_{0,\alpha} = 1, \qquad \beta_\chi = \frac{1}{2}, \qquad r_{\chi,N,\alpha} \equiv 0, \qquad \epsilon_N^\chi = 0. \tag{538}$$

In that purely geometric parametrization there is no coefficient remainder to estimate.

This is not yet the substitution required by the finite-N dynamics, however. The terminal characteristic variable used throughout Parts 10–28 is

$$\boxed{s_N^{\mathrm{ph}} = t - z - \theta_{*,N}, \qquad \dot{s}_N^{\mathrm{ph}} = 1 - b_N,} \tag{539}$$

whereas (535) defines a different quantity $s_{\mathrm{geo}} = \sin^2(x - x_*)$. The source reconstruction only imposes the terminal tube comparison

$$c_1 s_N^{\mathrm{ph}} \le \big(\chi^{\mathrm{geom}}(x)\big)^2 \le c_2 s_N^{\mathrm{ph}}, \tag{540}$$

not the identity $s_{\mathrm{geo}} = s_N^{\mathrm{ph}}$. Therefore the exact formula (537) cannot be substituted into (523) with $s = s_N^{\mathrm{ph}}$ unless the motion of x is proved to identify the two scales to sufficiently high order.

165. The correct finite-N remainder variable

Let $x_N(t)$ be the x component of the actual characteristic and set

$$s = s_N^{\mathrm{ph}}(t), \qquad X_N(s) = x_N(t(s)). \tag{541}$$

On the negative branch define

$$\boxed{R_{\chi,N}(s) := \frac{-\sin(X_N(s) - x_*)}{\sqrt{s}} - 1.} \tag{542}$$

Then the desired formula (522) with $\beta_\chi = 1/2$ is equivalent to

$$\boxed{\chi_N^{\mathrm{geom}}(s) = -s^{1/2}\big(1 + R_{\chi,N}(s)\big).} \tag{543}$$

Consequently the actual version of (523) is

$$\boxed{\max_{0\le j\le 3} |E^j R_{\chi,N}(s)| \le C_\chi s^{\mu_\chi} + \epsilon_N^\chi, \qquad E = s\partial_s.} \tag{544}$$

The tube estimate (540) gives only

$$\sqrt{c_1} \le 1 + R_{\chi,N}(s) \le \sqrt{c_2}, \tag{545}$$

and hence boundedness of $R_{\chi,N}$; it does not imply $R_{\chi,N} \to 0$, still less the three Euler derivative estimates in (544). Thus (523) cannot be declared proved from the tube comparison alone.

166. Exact characteristic equation for the geometric factor

The extended b_3 transport operator gives the characteristic equations

$$\dot{x} = u_z b_1 = A_N C_N^2, \qquad \dot{z} = b_N, \tag{546}$$

where

$$A_N^2 = -2\chi_N^{\mathrm{geom}} b_N, \qquad C_N = u_{z,N}. \tag{547}$$

Combining (539) and (546) yields the exact s–equation

$$\boxed{\frac{dX_N}{ds} = \frac{A_N C_N^2}{1 - b_N}.} \tag{548}$$

Therefore, with

$$\chi_N(s) = \sin(X_N(s) - x_*), \tag{549}$$

we have

$$\boxed{\frac{d\chi_N}{ds} = \cos(X_N - x_*)\frac{A_N C_N^2}{1 - b_N}.} \tag{550}$$

Equations (547) and (550) are the exact dynamical equations that must be used to test the proposed square-root relation between χ_N and the phase variable s.

167. Compatibility test against the proposed square-root law

Suppose, as required by (523), that along the characteristic

$$-\chi_N(s) = s^{1/2}(1 + o(1)), \tag{551}$$

and suppose the terminal scalar and reconstructed third component obey

$$b_N(s) \asymp s^{-r_\alpha}, \qquad r_\alpha = 2\alpha - 1 > 0, \qquad 0 < c_- \le C_N \le c_+ < \infty. \tag{552}$$

Then from (547),

$$A_N = \sqrt{-2\chi_N b_N} \asymp s^{1/4-r_\alpha/2}. \tag{553}$$

Since $b_N \to \infty$,

$$|1 - b_N| \asymp b_N \asymp s^{-r_\alpha}. \tag{554}$$

Hence (550), using $\cos(X_N - x_*) \to 1$, gives

$$\boxed{\left|\frac{d\chi_N}{ds}\right| \asymp s^{1/4+r_\alpha/2}.} \tag{555}$$

The right side is integrable at $s = 0$. If the characteristic reaches the geometric zero, $\chi_N(0) = 0$, integration gives

$$\boxed{|\chi_N(s)| \asymp s^{5/4+r_\alpha/2}.} \tag{556}$$

This exponent is strictly larger than $1/2$ for every $r_\alpha > 0$. In particular,

$$\frac{|\chi_N(s)|}{s^{1/2}} \asymp s^{3/4+r_\alpha/2} \longrightarrow 0, \tag{557}$$

which contradicts (551) and the two-sided tube condition (540).

The same conclusion follows directly by differentiating the asserted square-root law: (551) would require

$$\left|\frac{d\chi_N}{ds}\right| \sim \frac{1}{2}s^{-1/2}, \tag{558}$$

whereas the exact characteristic equation gives the vanishing quantity (555).

Theorem 14 (Characteristic incompatibility of the unmodified square-root tube)**.** *Assume the active characteristic equations (546), the square-root velocity law (547), the terminal lower and upper power bounds in (552), and a positive bounded third component C_N. Then an actual characteristic approaching a zero of $\chi(x) = \sin(x - x_*)$ cannot simultaneously satisfy*

$$c_1 s \le \chi_N(s)^2 \le c_2 s$$

for fixed positive c_1, c_2 all the way to $s = 0$. In particular, the positive-power estimate (523) with leading exponent $\beta_\chi = 1/2$ is not a consequence of the displayed finite-N reconstruction; under these hypotheses it is incompatible with the characteristic x–equation.

Proof. The assumptions imply (553)–(555). Integrating (555) from the endpoint $\chi_N(0) = 0$ yields (556), and hence $\chi_N(s)^2/s \to 0$, contradicting a positive lower bound c_1 in (540). □

168. What would be required to recover a square-root geometric scale

The preceding theorem does not say that the periodic $\mathcal{J}_3$ geometry is algebraically inconsistent. Equation (537) is an exact identity in the geometric variable $s_{\text{geo}} = \chi^2$. It says instead that the manuscript has not justified identifying that geometric variable with the phase variable $s_N^{\text{ph}} = t - z - \theta_{*,N}$ along the actual transport characteristic.

Indeed, from (550), a necessary balance for $|\chi_N| \asymp s^{1/2}$ is

$$\frac{A_N C_N^2}{|1-b_N|} \asymp s^{-1/2}. \tag{559}$$

Using $A_N \asymp s^{1/4-r_\alpha/2}$ and $|1-b_N| \asymp s^{-r_\alpha}$, this would require

$$\boxed{C_N^2 \asymp s^{-3/4-r_\alpha/2}, \qquad C_N \asymp s^{-3/8-r_\alpha/4}.} \tag{560}$$

Thus the square-root identification would demand a singular $u_z = C_N$, in conflict with the bounded positive terminal reconstruction $C_N \to c_0 > 0$ used in the present core estimates. Alternatively one must modify the characteristic geometry or keep s_{geo} and s_N^{ph} as genuinely distinct variables.

169. Consequence for estimate (523)

There are now two mathematically distinct statements.

First, in the intrinsic periodic geometric coordinate,

$$\boxed{\chi^{\mathrm{geom}} = -s_{\mathrm{geo}}^{1/2}} \tag{561}$$

exactly on the selected branch, so the analogue of (523) is trivial with zero remainder.

Second, in the phase coordinate required by the finite-N terminal dynamics,

$$s = s_N^{\mathrm{ph}}, \tag{562}$$

the estimate (523) is not presently true under the simultaneously asserted bounded-C_N characteristic reconstruction. Therefore it would be incorrect to remove the entire coefficient item from the proof ledger on the basis of (535) alone.

The appropriate replacement for (523) is a compatibility estimate for the map

$$s_N^{\mathrm{ph}} \longmapsto s_{\mathrm{geo,N}} := \sin^2(X_N(s_N^{\mathrm{ph}}) - x_*), \tag{563}$$

derived from the exact ODE (548). Under bounded C_N and the terminal power law, (556) gives instead

$$\boxed{s_{\mathrm{geo,N}}(s) \asymp s^{5/2+r_\alpha}.} \tag{564}$$

This is the scale produced by the active characteristic equations, not $s_{\mathrm{geo,N}} \asymp s$.

170. Revised coefficient ledger after exact substitution

The requested exact substitution therefore sharpens the ledger, but in the opposite direction from an unconditional deletion. The artificial cutoff item remains deleted by Part 28. The geometric coefficient itself is explicit, $\chi(x) = \sin(x - x_*)$, so there is no unknown finite-N formula for that function. The unresolved point is the dynamical identification of its argument $x_N(t)$ with the phase scale used by the similarity profile.

Accordingly the construction-level ledger is now

$$\boxed{\begin{array}{c}\text{(i) resolve the phase–geometry compatibility } s_{\mathrm{geo,N}} \text{ versus } s_N^{\mathrm{ph}} \text{ in (563),}\\ \text{or modify the terminal reconstruction so that the desired relation is dynamically preserved;}\\ \text{(ii) prove the strict upper invariant-boundary sign;}\\ \text{(iii) prove the fixed-space energy estimates and vanishing scalar residual}\\ \text{needed for the strong } N\to\infty \text{ passage.}\end{array}} \tag{565}$$

The old "coefficient/cutoff" wording should therefore be retired. There is no cutoff obstruction and no unknown formula for χ; there is a *phase–geometry compatibility obstruction* exposed by the exact characteristic substitution.

171. Updated theorem status

The positive Mellin profile branch, the exact Lambert sensitivity estimates, and the cutoff-free one-phase fractional reduction are unaffected by the calculation above. What changes is the finite-N bridge from that profile to the reconstructed velocity geometry. The profile equation may remain valid as a one-dimensional similarity equation while the proposed identification $\chi^2 \asymp s_N^{\mathrm{ph}}$ fails along the full characteristic flow.

Consequently a finite-time blow-up conclusion for the reconstructed scalar cannot use (523) as a proved input until (563) is reconciled with (548). Likewise, a full velocity theorem cannot simultaneously use bounded $C_N \to c_0 > 0$, the active x–characteristic $\dot{x} = A_N C_N^2$, and the square-root tube $\chi^2 \asymp s_N^{\mathrm{ph}}$ without an additional mechanism, because the exponents in (555)–(557) are incompatible.

This audit is useful because it replaces an abstract coefficient hypothesis by one explicit ODE compatibility problem. The next decisive calculation is therefore to determine whether the reconstruction can be reformulated with s_{geo} as the terminal similarity variable, or whether the transport field can be modified consistently so that (559) holds without destroying incompressibility, the b_3 product identity, or the momentum equations.

47 Part 30: Intrinsic geometric similarity variable and exact characteristic reparametrization

172. Definition of the intrinsic geometric variable

Part 29 showed that the periodic geometric coefficient is not an unknown finite-N object. On the selected branch it is exactly

$$\chi_N^{\mathrm{geom}} = \sin(x_N - x_*), \qquad \sigma_N := s_{\mathrm{geo},N} = \sin^2(x_N - x_*), \tag{566}$$

and hence

$$\boxed{\chi_N^{\mathrm{geom}} = -\sigma_N^{1/2}} \tag{567}$$

without a remainder. In this Part we therefore use σ_N itself as the terminal geometric variable and derive its exact characteristic law rather than identifying it with the affine phase distance

$$s_N^{\mathrm{ph}} = t - z - \theta_{*,N}. \tag{568}$$

Write

$$\chi_N = \sin(x_N - x_*), \qquad c_N^{\mathrm{geo}} := \cos(x_N - x_*). \tag{569}$$

On the one-sided branch approaching x_*,

$$c_N^{\mathrm{geo}} = \sqrt{1 - \sigma_N}. \tag{570}$$

The active characteristic equation from Part 29 is

$$\dot{x}_N = A_N C_N^2, \qquad A_N^2 = -2\chi_N b_N. \tag{571}$$

Using (567),

$$\boxed{A_N = \sqrt{2}\,\sigma_N^{1/4} b_N^{1/2}} \tag{572}$$

on the positive square-root branch.

173. Exact characteristic equation for s_{geo}

Differentiate (566) along the actual characteristic. Since

$$\frac{d}{dx}\sin^2(x - x_*) = 2\sin(x - x_*)\cos(x - x_*),$$

we obtain

$$\dot{\sigma}_N = 2\chi_N c_N^{\mathrm{geo}} \dot{x}_N. \tag{573}$$

Substituting (567), (570), (571), and (572) gives the exact identity

$$\boxed{\dot{\sigma}_N = -2\sqrt{2}\,(1 - \sigma_N)^{1/2} \sigma_N^{3/4} b_N^{1/2} C_N^2.} \tag{574}$$

Thus σ_N is a legitimate monotone terminal coordinate whenever $b_N > 0$ and $C_N \neq 0$. In particular, if

$$0 < c_- \le C_N \le c_+ < \infty, \tag{575}$$

then $\dot{\sigma}_N < 0$ for $0 < \sigma_N < 1$.

The affine phase still obeys

$$\dot{s}_N^{\mathrm{ph}} = 1 - b_N. \tag{576}$$

Consequently the exact change-of-coordinate equation is

$$\boxed{\frac{d\sigma_N}{ds_N^{\mathrm{ph}}} = \frac{-2\sqrt{2}\,(1 - \sigma_N)^{1/2} \sigma_N^{3/4} b_N^{1/2} C_N^2}{1 - b_N}.} \tag{577}$$

Equation (577), rather than $\sigma_N \asymp s_N^{\mathrm{ph}}$, is the correct finite-$N$ compatibility law.

174. Exact b_N equation in the geometric variable

Let the scalar characteristic equation be written in the form already used in the terminal analysis,

$$\dot{b}_N = \Gamma_N b_N^2 + 2C_N G_N - \mathcal{R}_N, \tag{578}$$

where

$$\Gamma_N = \partial_\xi \log Q_N \tag{579}$$

is the Lambert logarithmic sensitivity. Dividing (578) by (574) yields the exact geometric-coordinate equation

$$\boxed{\frac{db_N}{d\sigma_N} = -\frac{\Gamma_N b_N^2 + 2C_N G_N - \mathcal{R}_N}{2\sqrt{2}\,(1-\sigma_N)^{1/2}\sigma_N^{3/4} b_N^{1/2} C_N^2}.} \tag{580}$$

Equivalently,

$$\boxed{\frac{db_N}{d\sigma_N} = -\frac{\Gamma_N}{2\sqrt{2}\,C_N^2}(1-\sigma_N)^{-1/2}\sigma_N^{-3/4} b_N^{3/2}\left[1+\mathfrak{e}_N\right],} \tag{581}$$

where

$$\mathfrak{e}_N := \frac{2C_N G_N - \mathcal{R}_N}{\Gamma_N b_N^2}. \tag{582}$$

Thus the previously established condition $\mathfrak{e}_N = o(1)$ remains exactly the condition which makes the Lambert term dominant; changing coordinates does not discard the reconstruction remainder.

175. Induced geometric power law from the already established phase law

Suppose on a terminal interval that

$$b_N(s) \asymp s^{-r_\alpha}, \qquad r_\alpha = 2\alpha - 1 > 0, \qquad C_N \asymp 1, \tag{583}$$

where here and only here $s = s_N^{\mathrm{ph}}$. Equation (577), or equivalently the calculation of Part 29, gives

$$\boxed{\sigma_N \asymp s^{\lambda_\alpha}, \qquad \lambda_\alpha := \frac{5}{2} + r_\alpha = 2\alpha + \frac{3}{2}.} \tag{584}$$

Hence

$$s \asymp \sigma_N^{1/\lambda_\alpha}, \tag{585}$$

and therefore

$$\boxed{b_N(\sigma_N) \asymp \sigma_N^{-\rho_\alpha}, \qquad \rho_\alpha := \frac{r_\alpha}{\lambda_\alpha} = \frac{2\alpha-1}{2\alpha+3/2} = \frac{4\alpha-2}{4\alpha+3}.} \tag{586}$$

For example, at the continuation anchor $\alpha = 1$,

$$\rho_1 = \frac{2}{7}. \tag{587}$$

This exponent is not r_α. Thus merely renaming s_{geo} as the old similarity variable would change the profile scaling.

176. Transformation of the Lambert coefficient

The finite-depth Lambert analysis gives, in the affine phase coordinate,

$$\Gamma_N(s) = \frac{\gamma_{N,\alpha}}{s}\,[1 + o(1)] \tag{588}$$

with the appropriate positive finite-depth coefficient $\gamma_{N,\alpha}$. Combining (585) and (588) gives

$$\boxed{\Gamma_N(\sigma) \asymp \sigma^{-1/\lambda_\alpha},} \tag{589}$$

not σ^{-1}. More precisely, if

$$s = a_{N,\alpha}\sigma^{1/\lambda_\alpha}\big(1 + o(1)\big), \tag{590}$$

then

$$\Gamma_N(\sigma) = \frac{\gamma_{N,\alpha}}{a_{N,\alpha}}\sigma^{-1/\lambda_\alpha}\big(1 + o(1)\big). \tag{591}$$

Substitution in (581) is consistent with the exponent (586). Indeed, writing $b \asymp \sigma^{-\rho}$, the left side scales as $\sigma^{-\rho-1}$, while the leading right side scales as

$$\sigma^{-1/\lambda_\alpha - 3/4 - 3\rho/2}. \tag{592}$$

Equality of powers gives

$$\rho + 1 = \frac{1}{\lambda_\alpha} + \frac{3}{4} + \frac{3\rho}{2}, \tag{593}$$

or

$$\rho = \frac{1}{2} - \frac{2}{\lambda_\alpha}. \tag{594}$$

Using $\lambda_\alpha = 5/2 + r_\alpha$ and the particular phase exponent assumed in (583) shows that the geometric-coordinate equation imposes an additional coefficient/exponent consistency condition. Therefore the old phase-profile power law cannot simply be transplanted into σ without rederiving the full scalar balance. Equation (580) is the exact starting point for that rederivation.

177. The decisive fractional-operator obstruction

There is a more fundamental issue. The successful Mellin profile equation in the preceding Parts uses the affine one-phase variable

$$s_N^{\mathrm{ph}} = C_4 t - C_3 z - C_2 y - C_1 x - \theta_{*,N}, \tag{595}$$

for which the manuscript proved the exact reduction

$$(-\Delta_{\mathbb{T}^3})^\alpha F(s_N^{\mathrm{ph}}) = \kappa^{2\alpha}(-\partial_{ss})^\alpha F(s_N^{\mathrm{ph}}). \tag{596}$$

The intrinsic geometric coordinate

$$\sigma = \sin^2(x - x_*) \tag{597}$$

is not affine. Therefore a field $F(\sigma)$ is not a function of the affine one-phase variable used in (596), and the identity (596) cannot be invoked with s replaced by σ.

For the ordinary Laplacian this failure is already explicit. If $U(x) = F(\sigma(x))$, then

$$U_x = \sigma_x F_\sigma, \qquad U_{xx} = \sigma_x^2 F_{\sigma\sigma} + \sigma_{xx} F_\sigma. \tag{598}$$

Since

$$\sigma_x = \sin(2(x - x_*)), \qquad \sigma_x^2 = 4\sigma(1-\sigma), \qquad \sigma_{xx} = 2(1-2\sigma), \tag{599}$$

we obtain

$$\boxed{-\partial_{xx} F(\sigma) = -4\sigma(1-\sigma)F_{\sigma\sigma} - 2(1-2\sigma)F_\sigma.} \tag{600}$$

Thus even at $\alpha = 1$ the geometric-coordinate operator is a degenerate variable-coefficient operator, not $-\partial_{\sigma\sigma}$. For noninteger α the transformed fractional operator is nonlocal and cannot be obtained by the formal substitution $(-\partial_{ss})^\alpha \mapsto (-\partial_{\sigma\sigma})^\alpha$.

Theorem 15 (Geometric reparametrization theorem)**.** *The intrinsic variable*

$$\sigma = s_{\mathrm{geo}} = \chi^2 = \sin^2(x - x_*)$$

resolves the geometric coefficient identity exactly and yields the exact monotone characteristic equations (574), (577), and (580). It therefore removes the false requirement $s_{\mathrm{geo}} \asymp s_N^{\mathrm{ph}}$. However, it does not *preserve the affine one-phase fractional reduction or the previously proved Mellin profile equation. Consequently the existing profile theorem cannot be transferred to s_{geo} by a change of notation alone.*

Proof. The exact identity $\chi = -\sigma^{1/2}$ is (567). Differentiation along the actual characteristic and the square-root reconstruction give (574), and combining it with (576) and (578) gives (577) and (580). Hence no comparison between σ and s_N^{ph} is required to parametrize the characteristic. On the other hand, (598)–(600) show already for $\alpha = 1$ that the spatial operator in σ differs from the affine one-dimensional operator. Since the noninteger fractional Laplacian is nonlocal, the affine identity (596) likewise does not survive the nonlinear map $x \mapsto \sin^2(x - x_*)$. The Mellin profile equation proved earlier is therefore not the profile equation associated with (580). □

178. What is proved and what is not proved by choosing s_{geo}

The proposed repair succeeds at one level and fails at another. It succeeds in the precise sense

$$\boxed{\chi_N^{\mathrm{geom}} = -s_{\mathrm{geo},N}^{1/2}} \tag{601}$$

with zero finite-N coefficient remainder. Thus the geometric part of the old estimate (523) is completely closed in its natural coordinate.

It also gives a regular exact characteristic reparametrization through (574) and (580). Hence the contradiction in Part 29 disappears: that contradiction came from imposing $s_{\mathrm{geo}} \asymp s_N^{\mathrm{ph}}$, which is no longer imposed.

What does not follow is that the already proved Mellin heteroclinic is a profile for the new geometric-coordinate equation. To obtain a theorem by this route one would have to analyze

the new operator induced by $x \mapsto \sin^2(x - x_*)$ and prove a corresponding profile theorem for (580), including the transformed fractional term. This is a new analytic problem, not a remaining coefficient estimate.

179. Revised proof ledger after the geometric-variable test

The exact test permits a sharper ledger. The old coefficient/cutoff item is now deleted as a finite-N coefficient issue:

$$\boxed{\text{the cutoff is absent and } \chi = -s_{\mathrm{geo}}^{1/2} \text{ exactly.}} \tag{602}$$

The phase–geometry contradiction of Part 29 is also removed if the two variables are kept distinct. But there are now two alternative proof routes:

$$\boxed{\begin{array}{ll} \text{Route A:} & \text{retain the affine phase } s_N^{\mathrm{ph}} \text{ and modify/justify the} \\ & \text{velocity geometry without asserting } s_{\mathrm{geo}} \asymp s_N^{\mathrm{ph}}; \\ \text{Route B:} & \text{use } s_{\mathrm{geo}} \text{ intrinsically and prove a new profile theorem} \\ & \text{for the variable-coefficient/nonlocal operator induced by (597).} \end{array}} \tag{603}$$

Therefore it is not yet mathematically justified to say that only the upper invariant-boundary sign and the energy/residual limit remain. That shorter ledger would be valid only after one of Routes A or B is closed.

The rigorous remaining ledger is

$$\boxed{\begin{array}{c} \text{(i) close Route A or Route B in (603);} \\ \text{(ii) prove the strict upper invariant-boundary sign;} \\ \text{(iii) prove the fixed-space energy estimates and the vanishing residual} \\ \text{needed for the strong } N \to \infty \text{ passage.} \end{array}} \tag{604}$$

This formulation preserves the successful Mellin continuation theorem while preventing the intrinsic geometric identity from being used as though it were an affine-phase identity.

180. Updated theorem status after the s_{geo} reparametrization

The calculation above proves that s_{geo} is an exact and monotone local terminal coordinate for the reconstructed characteristic and that the geometric coefficient has no positive-power remainder in this coordinate. It also proves that the old incompatibility is not an algebraic failure of the $\mathcal{J}_3$ geometry; it is a mismatch between two different coordinates.

At the same time, the calculation prevents an unjustified conclusion. The positive Mellin profile branch established earlier belongs to the affine one-phase fractional operator. The nonlinear geometric coordinate produces a different spatial operator. Hence a finite-time singularity theorem based on the existing profile still requires a rigorous bridge between the affine profile dynamics and the intrinsic geometric reconstruction, or a new profile analysis directly in s_{geo}.

This is the exact outcome of the proposed reparametrization: it removes the false identification $s_{\mathrm{geo}} \asymp s_N^{\mathrm{ph}}$, closes the geometric coefficient remainder itself, and isolates the remaining issue as an operator/profile compatibility problem rather than a cutoff or unknown coefficient problem.

48 Part 31: Fractional operator induced by the intrinsic geometric coordinate

181. The correct square-root coordinate behind s_{geo}

Set

$$y = x - x_*, \qquad \sigma = \sin^2 y,$$

and restrict to the one-sided branch $y \in (-\delta, 0)$. Introduce

$$\boxed{r = -\sin y = \sqrt{\sigma}.} \tag{605}$$

Then $r \in (0, \sin\delta)$ and

$$\frac{dr}{dy} = -\cos y = -\sqrt{1 - r^2}. \tag{606}$$

Consequently r is a smooth local coordinate with

$$r = -y + O(y^3), \qquad y = -r + O(r^3), \qquad r \downarrow 0. \tag{607}$$

Thus the nonlinear map $y \mapsto \sigma$ is quadratic, but its square root is locally a smooth diffeomorphism. This is the structural point that permits a comparison with the affine Mellin operator.

182. Exact check at $\alpha = 1$

For $U(y) = f(r(y))$ one has

$$U_y = -\sqrt{1 - r^2}\, f_r$$

and hence

$$\boxed{-U_{yy} = -(1 - r^2) f_{rr} + r f_r.} \tag{608}$$

Equivalently, if $F(\sigma) = f(\sqrt{\sigma})$, (608) is exactly the same operator as (600). Its terminal decomposition is

$$\boxed{-U_{yy} = -f_{rr} + \mathcal{R}_1 f, \qquad \mathcal{R}_1 f = r^2 f_{rr} + r f_r.} \tag{609}$$

For a Mellin monomial $f(r) = r^\lambda$,

$$-f_{rr} = -\lambda(\lambda - 1) r^{\lambda - 2},$$

whereas

$$\mathcal{R}_1 f = \lambda^2 r^\lambda. \tag{610}$$

Therefore the geometric correction is two powers less singular than the principal affine operator. In particular,

$$\frac{\mathcal{R}_1(r^\lambda)}{-\partial_{rr} r^\lambda} = O(r^2) \tag{611}$$

whenever the denominator is nonzero.

183. Exact induced fractional operator

Let $\Phi(r) = -\arcsin r$, so that $y = \Phi(r)$ on the chosen branch, and define the pullback

$$(Tf)(y) = f(r(y)). \tag{612}$$

The exact geometric fractional operator is

$$\boxed{\mathcal{L}_\alpha^{\mathrm{geo}} := T^{-1}(-\partial_{yy})^\alpha T.} \tag{613}$$

For the periodic problem the same definition is made with the periodic one-dimensional fractional Laplacian. Formula (613), rather than $(-\partial_{\sigma\sigma})^\alpha$, is the operator induced by the intrinsic geometry.

The principal symbol follows from conjugation by the diffeomorphism $y = \Phi(r)$:

$$\boxed{\ell_\alpha^{\mathrm{geo}}(r,\xi) = (1-r^2)^\alpha |\xi|^{2\alpha} + \ell_{\alpha,<}(r,\xi),} \tag{614}$$

where $\ell_{\alpha,<}$ has order at most $2\alpha - 1$ locally. Since

$$(1-r^2)^\alpha = 1 - \alpha r^2 + O(r^4), \tag{615}$$

the principal symbol converges to the affine symbol $|\xi|^{2\alpha}$ at the terminal point.

184. Blow-up rescaling and operator convergence

For $\varepsilon > 0$ let

$$(S_\varepsilon f)(r) = f(r/\varepsilon)$$

and define the rescaled geometric operator

$$\mathcal{L}_{\alpha,\varepsilon}^{\mathrm{geo}} := \varepsilon^{2\alpha} S_\varepsilon^{-1} \mathcal{L}_\alpha^{\mathrm{geo}} S_\varepsilon. \tag{616}$$

On every fixed compact R-interval, the rescaled coordinate map satisfies

$$\frac{\Phi(\varepsilon R)}{\varepsilon} = -R + O(\varepsilon^2 R^3), \qquad \Phi'(\varepsilon R) = -1 + O(\varepsilon^2 R^2). \tag{617}$$

Hence the conjugated kernels, or equivalently the local symbols, converge to those of the translation-invariant fractional Laplacian. In particular, for $0 < \sigma_0 < 1$ and compactly supported smooth f,

$$\boxed{\|\mathcal{L}_{\alpha,\varepsilon}^{\mathrm{geo}} f - (-\partial_{RR})^\alpha f\|_{H^{-\sigma_0}} \longrightarrow 0 \qquad (\varepsilon \downarrow 0),} \tag{618}$$

uniformly for α in a sufficiently small compact interval $[1, 1+\delta]$. The same conclusion holds locally for the periodic operator: the difference between the periodic kernel and its Euclidean singular kernel is smooth near the diagonal and is annihilated at leading order by the factor $\varepsilon^{2\alpha}$.

Proof. The map in (617) converges in C^∞ on compact R-sets to the reflection $R \mapsto -R$. The fractional Laplacian is invariant under reflection. After rescaling, the singular part of its kernel has the form $|R - R'|^{-1-2\alpha}$ multiplied by Jacobian and metric factors converging uniformly to one. Away from the diagonal the kernel is smooth and its contribution is lower order after multiplication by $\varepsilon^{2\alpha}$. Dominated convergence after the standard second-difference cancellation gives (618) for smooth compactly supported f. Uniformity in $\alpha \in [1, 1+\delta]$ follows because the kernel constants and the cancelled integrands vary continuously on this compact interval. The periodic kernel is the Euclidean singular kernel plus a smooth local remainder, so the same argument applies. □

185. Mellin homogeneity in $r = \sqrt{\sigma}$

The preceding theorem identifies the correct terminal homogeneity. If

$$f(r) = r^\lambda,$$

then the affine leading operator satisfies

$$\boxed{(-\partial_{rr})^\alpha r_+^\lambda = K_\alpha^+(\lambda) r^{\lambda - 2\alpha}.} \tag{619}$$

Since $r = \sigma^{1/2}$, this may also be written

$$(-\partial_{rr})^\alpha \sigma^{\lambda/2} = K_\alpha^+(\lambda) \sigma^{(\lambda-2\alpha)/2}. \tag{620}$$

Thus the apparent change from a 2α Mellin shift in the affine variable to an α shift in σ is exactly the quadratic change of variable; it is not a new scaling law. The existing Mellin symbol is recovered *without alteration* when the intrinsic geometry is expressed in its square-root coordinate $r = \sqrt{s_{\mathrm{geo}}}$.

186. Logarithmic conjugation and comparison with the existing profile operator

Set

$$r = e^X, \qquad f(r) = r^{-a_\alpha} G(X), \qquad a_\alpha = \alpha - \frac{1}{2}. \tag{621}$$

For the affine leading operator,

$$(-\partial_{rr})^\alpha f = r^{-a_\alpha - 2\alpha} \mathcal{M}_{a_\alpha} G, \tag{622}$$

with exactly the Mellin multiplier already used in Parts 8–9,

$$\mathcal{M}_{a_\alpha}(e^{\lambda X}) = K_\alpha^+(-a_\alpha + \lambda) e^{\lambda X}. \tag{623}$$

Combining (618) and (622) gives the terminal operator decomposition

$$\boxed{\mathcal{L}_\alpha^{\mathrm{geo}}\big(r^{-a_\alpha} G(\log r)\big) = r^{-a_\alpha - 2\alpha} \left[\mathcal{M}_{a_\alpha} G + \mathcal{Q}_{\alpha,r} G\right],} \tag{624}$$

where the rescaled remainder $\mathcal{Q}_{\alpha,r}$ tends to zero on compact logarithmic windows translated toward $X = -\infty$. Thus the existing Mellin operator is the terminal tangent operator of the exact geometric fractional operator.

187. What this proves for Route B

Theorem 16 (Terminal Mellin tangent theorem for the intrinsic geometry)**.** *Let*

$$\sigma = \sin^2(x - x_*), \qquad r = \sqrt{\sigma} = -\sin(x - x_*)$$

on the selected one-sided branch. Then the exact fractional operator induced by the geometric coordinate is the conjugated operator (613). Under terminal blow-up rescaling $r = \varepsilon R$, its rescaled realization converges to $(-\partial_{RR})^\alpha$ uniformly for $\alpha \in [1, 1+\delta]$ on compact R-sets in the sense of (618). After the logarithmic Mellin conjugation (621), the tangent operator is precisely the existing Mellin operator $\mathcal{M}_{a_\alpha}$.

This theorem closes the *leading-operator* part of Route B. It proves that the nonlinear geometric coordinate does not generate a different terminal principal profile operator: after passing to its natural square-root coordinate, the old Mellin operator is recovered as the tangent operator.

It is important, however, to distinguish tangent-operator convergence from a full heteroclinic persistence theorem. To transfer the already constructed Mellin profile to the exact geometric equation one still needs a weighted estimate of the form

$$\boxed{\|\mathcal{Q}_{\alpha,r} G\|_{Y_\eta} \le C e^{\mu X_0} \|G\|_{X_{\eta,\sigma}}} \tag{625}$$

for logarithmic windows $X \le X_0 \ll 0$, together with the corresponding C^1 estimate for the linearization. Equation (618) proves convergence on fixed rescaled compact sets; (625) is the global weighted estimate needed by the Banach-space profile theorem.

188. Revised Route-B ledger

The operator/profile compatibility issue is therefore reduced to one precise analytic estimate. The following statements are now proved:

$$\boxed{\begin{array}{c} \chi = -s_{\mathrm{geo}}^{1/2} \quad \text{exactly}, \\ r = \sqrt{s_{\mathrm{geo}}} \text{ is a smooth terminal coordinate}, \\ \mathcal{L}_\alpha^{\mathrm{geo}} = T^{-1}(-\partial_{yy})^\alpha T, \\ \varepsilon^{2\alpha} S_\varepsilon^{-1} \mathcal{L}_\alpha^{\mathrm{geo}} S_\varepsilon \longrightarrow (-\partial_{RR})^\alpha, \\ \text{and the logarithmic tangent operator is } \mathcal{M}_{a_\alpha}. \end{array}} \tag{626}$$

Accordingly Route B is no longer an unidentified "new profile operator" problem. Its remaining profile–geometry bridge is the weighted global remainder estimate (625). Only after (625) and its linearized analogue are proved may the exact geometric profile be obtained from the existing Mellin heteroclinic by perturbation.

The construction-level ledger is therefore sharpened to

$$\boxed{\begin{array}{c} \text{(i) prove the weighted geometric-operator remainder estimate (625) and its } C^1 \text{ analogue;} \\ \text{(ii) prove the strict upper invariant-boundary sign;} \\ \text{(iii) prove the fixed-space energy estimates and vanishing residual for } N \to \infty. \end{array}} \tag{627}$$

189. Status of the profile–geometry bridge

The calculation in this Part resolves the principal scaling question raised at the end of Part 30. The coordinate σ itself is not an affine Mellin coordinate, but $r = \sqrt{\sigma}$ is asymptotically affine at the terminal point. The exact geometric fractional operator is therefore asymptotically conjugate to the same affine fractional operator that generated the successful Mellin profile equation. This is a genuine compatibility result and removes the principal-symbol obstruction identified in Part 30.

What remains is quantitative rather than structural: the local tangent convergence must be upgraded to the weighted global estimate (625) on the profile space. Until that estimate is established, Route B is strongly supported at the terminal operator level but is not yet a complete exact profile theorem for the nonlinear periodic geometry.

49 Part 32: Weighted geometric-operator remainder on the terminal chart

190. Precise formulation of the estimate left open in (625)

The estimate written schematically in (625) must first be interpreted on the actual terminal chart. The intrinsic square-root coordinate satisfies

$$0 < r = -\sin(x - x_*) < 1, \qquad X = \log r < 0, \tag{628}$$

so the exact geometric chart does not by itself cover the whole Mellin line $X \in \mathbb{R}$. Consequently (625) cannot literally be a global estimate on all of $\mathbb{R}$ for the exact map $r = -\sin(x - x_*)$. What is required by the finite-N terminal matching is the uniform estimate on a left terminal window

$$\Omega_{X_0} = \{X \le X_0\}, \qquad X_0 < 0. \tag{629}$$

We now prove that estimate, including its C^1 analogue in α and its linearized version in the profile variable.

Choose $0 < r_0 < 1/4$ and $X_0 \le \log r_0$. Let $\zeta \in C_c^\infty((-2r_0, 2r_0))$ be equal to one on $(-r_0, r_0)$. The local geometric operator is obtained by applying the exact periodic fractional Laplacian to

$$U(y) = \zeta(r(y)) f(r(y)) \tag{630}$$

and then restricting to $0 < r < r_0/2$. The complementary part of the periodic kernel is smooth on this restricted chart and will be denoted by $\mathcal{S}_\alpha$. This decomposition does not alter the exact operator; it only separates its singular near-diagonal part from its smooth far-field part.

For

$$f(r) = r^{-a_\alpha} G(\log r), \qquad a_\alpha = \alpha - \frac{1}{2}, \tag{631}$$

define the exact logarithmic remainder by

$$\boxed{\mathcal{Q}_\alpha^{\mathrm{geo}} G(X) := e^{(a_\alpha + 2\alpha)X} \mathcal{L}_\alpha^{\mathrm{geo}} f(e^X) - \mathcal{M}_{a_\alpha} G(X).} \tag{632}$$

This is the precise version of the schematic quantity $\mathcal{Q}_{\alpha,r}$ in (624)–(625).

191. Geometry of the terminal diffeomorphism

On the chosen branch

$$y = \Phi(r) = -\arcsin r.$$

Taylor expansion gives, uniformly for $0 \le r \le r_0$,

$$\Phi(r) = -r - \frac{r^3}{6} + O(r^5), \qquad \Phi'(r) = -(1-r^2)^{-1/2} = -1 - \frac{r^2}{2} + O(r^4), \tag{633}$$

$$\Phi''(r) = -r(1-r^2)^{-3/2} = O(r), \qquad \Phi^{(j)}(r) = O(1), \quad j \ge 3. \tag{634}$$

For $r, r' \in (0, r_0)$, the mean-value theorem and (633) imply

$$\Phi(r) - \Phi(r') = -(r-r')\big[1 + \Theta(r,r')\big], \qquad |\Theta(r,r')| \le C(r^2 + r'^2). \tag{635}$$

Likewise the Jacobian satisfies

$$|\Phi'(r')| = 1 + O(r'^2). \tag{636}$$

Thus every metric or Jacobian correction in the singular kernel begins at quadratic order in the terminal variables. The only apparently linear geometric coefficient is $\Phi'' = O(r)$; because it multiplies an operator one order lower, it also produces a quadratic error after Mellin scaling.

192. Local symbol decomposition with a quadratic terminal factor

Let $p_\alpha^{\mathrm{geo}}(r,\xi)$ be the full local symbol of the singular part of $\mathcal{L}_\alpha^{\mathrm{geo}}$ in the r coordinate. Standard one-dimensional change-of-variables calculus for the classical elliptic symbol $|\xi|^{2\alpha}$, together with (633)–(636), gives

$$p_\alpha^{\mathrm{geo}}(r,\xi) = |\xi|^{2\alpha} + r^2 p_{\alpha,2}(r,\xi) + r p_{\alpha,1}(r,\xi), \tag{637}$$

where $p_{\alpha,2}$ has order 2α and $p_{\alpha,1}$ has order at most $2\alpha - 1$. Uniformly for

$$\alpha \in I_\delta = [1, 1+\delta], \qquad 0 < \delta \le \frac{\sigma}{4}, \tag{638}$$

one has, for the finite number of symbol seminorms needed below,

$$|\partial_r^j \partial_\xi^\ell p_{\alpha,2}(r,\xi)| \le C_{j\ell} \langle \xi \rangle^{2\alpha - \ell}, \tag{639}$$

$$|\partial_r^j \partial_\xi^\ell p_{\alpha,1}(r,\xi)| \le C_{j\ell} \langle \xi \rangle^{2\alpha - 1 - \ell}. \tag{640}$$

Differentiation with respect to α adds at most one logarithm:

$$|\partial_\alpha \partial_r^j \partial_\xi^\ell p_{\alpha,k}| \le C_{j\ell} \langle \xi \rangle^{2\alpha + 1 - k - \ell} (1 + \log\langle \xi \rangle), \qquad k = 1, 2. \tag{641}$$

Because $2\alpha \le 2 + \sigma/2$ under (638),

$$\langle \xi \rangle^{2\alpha} (1 + \log\langle \xi \rangle) \le C_\sigma \langle \xi \rangle^{2+\sigma}. \tag{642}$$

Hence all constants in (639)–(641) may be chosen uniformly on I_δ as operators from $H^{2+\sigma}$ to L^2.

The significance of (637) is seen directly after logarithmic conjugation. A r-derivative of order m applied to (631) has the form

$$\partial_r^m f(r) = r^{-a_\alpha - m} P_{m,a_\alpha}(D_X) G(X), \qquad X = \log r, \tag{643}$$

with $P_{m,a}$ a polynomial of degree m. Therefore

$$r^{a_\alpha+2\alpha}\big[r^2 \operatorname{Op}(p_{\alpha,2}) f\big] = r^2 \mathfrak{P}_{\alpha,2} G, \tag{644}$$

while the lower-order term satisfies

$$r^{a_\alpha+2\alpha}\big[r \operatorname{Op}(p_{\alpha,1}) f\big] = r^2 \mathfrak{P}_{\alpha,1} G. \tag{645}$$

The operators $\mathfrak{P}_{\alpha,j}$ are uniformly bounded $H^{2+\sigma} \to L^2$ on logarithmic unit charts. Equations (644)–(645) are the fractional analogue of the exact $\alpha = 1$ identity (609): both the principal metric correction and the subprincipal curvature correction are two terminal powers smaller than the affine operator.

193. Smooth far-field part

The periodic kernel outside the local diagonal is smooth. For $0 < r < r_0/2$, Taylor expansion of that kernel in the observation variable gives

$$|\partial_r^j \mathcal{S}_\alpha f(r)| \le C_j \|f\|_{\mathcal{Z}}, \qquad 0 \le j \le 2, \tag{646}$$

where $\mathcal{Z}$ is any fixed norm controlling the chosen smooth extension in (630). After multiplication by the Mellin factor in (632),

$$r^{a_\alpha+2\alpha}|\mathcal{S}_\alpha f(r)| \le C r^{a_\alpha+2\alpha}\|f\|_{\mathcal{Z}}. \tag{647}$$

Since

$$a_\alpha + 2\alpha = 3\alpha - \frac{1}{2} \ge \frac{5}{2}, \tag{648}$$

the far-field contribution is smaller than r^2 throughout I_δ. This is the point at which nonlocality enters the proof: it does not disappear, but after the singular local kernel has been extracted its remaining contribution has strictly higher terminal order than the quadratic geometric error.

194. Weighted left-tail estimate

Let

$$\|h\|_{X_{\eta,\sigma}(X_0)} := \|\omega_\eta h\|_{H^{2+\sigma}((-\infty,X_0))}, \qquad \|g\|_{Y_\eta(X_0)} := \|\omega_\eta g\|_{L^2((-\infty,X_0))}. \tag{649}$$

On $X \le X_0 < 0$,

$$e^{2X} \le e^{2X_0}. \tag{650}$$

A partition into logarithmic unit intervals, application of (639)–(645) on each interval, and summation with the exponential weight therefore give

$$\|\omega_\eta \mathcal{Q}_\alpha^{\mathrm{loc}} G\|_{L^2(X\le X_0)} \le C e^{2X_0} \|G\|_{X_{\eta,\sigma}(X_0+2)}. \tag{651}$$

The two-unit enlargement only records the finite overlap of the local partition. From (647)–(648),

$$\|\omega_\eta \mathcal{Q}_\alpha^{\mathrm{far}} G\|_{L^2(X\le X_0)} \le C e^{(3\alpha-1/2)X_0} \|G\|_{X_{\eta,\sigma}} \le C e^{2X_0} \|G\|_{X_{\eta,\sigma}}, \tag{652}$$

after the fixed extension operator is included in the norm. Combining (651)–(652) proves the desired terminal estimate.

Theorem 17 (Uniform weighted geometric remainder)**.** *Fix $0 < \eta < 1/2$ and $\sigma > 0$. There exist $\delta_* > 0$, $X_* < 0$, and $C < \infty$ such that for*

$$1 \le \alpha \le 1 + \delta_*, \qquad \delta_* \le \frac{\sigma}{4}, \qquad X_0 \le X_*,$$

the exact geometric remainder (632), restricted to the terminal chart, satisfies

$$\boxed{\|\mathcal{Q}_\alpha^{\mathrm{geo}} G\|_{Y_\eta(X_0)} \le C e^{2X_0} \|G\|_{X_{\eta,\sigma}}.} \tag{653}$$

Thus the exponent in the schematic estimate (625) may be taken to be

$$\boxed{\mu = 2.} \tag{654}$$

The constant is uniform in α on the stated interval.

Proof. The singular local part is estimated by (637)–(645), which yields the quadratic factor $r^2 = e^{2X}$. The weighted summation gives (651). The nonlocal far-field part is smooth and, by (648), carries the stronger factor $r^{3\alpha-1/2}$, giving (652). Since $3\alpha - 1/2 \ge 5/2 > 2$, both pieces are bounded by the right side of (653). Uniformity in α follows from (638)–(642) and compactness of I_{δ_*}. □

195. The C^1 estimate in the fractional order

Differentiate (632) with respect to α. There are three contributions: variation of the geometric fractional symbol, variation of the Mellin symbol, and variation of the conjugating power $a_\alpha = \alpha - 1/2$. The first two are controlled by (641)–(642) and by the Mellin estimate already proved in Part 9. Differentiating the powers of r produces factors $\log r = X$. For $X \le X_0 < 0$ and every $0 < \varepsilon < 2$,

$$|X| e^{2X} \le C_\varepsilon e^{(2-\varepsilon)X_0} \quad \text{after weighted summation on } (-\infty, X_0]. \tag{655}$$

Consequently one loses an arbitrarily small amount of the quadratic exponent, but no derivative beyond the already reserved σ derivatives.

Theorem 18 (C^1 weighted remainder estimate)**.** *Under the hypotheses of the preceding theorem, for every fixed* $0 < \varepsilon < \min\{1, 2-2\eta\}$ *there is* $C_\varepsilon < \infty$ *such that*

$$\boxed{\|\partial_\alpha \mathcal{Q}_\alpha^{\mathrm{geo}} G\|_{Y_\eta(X_0)} \le C_\varepsilon e^{(2-\varepsilon)X_0} \|G\|_{X_{\eta,\sigma}}.} \tag{656}$$

In particular

$$\alpha \longmapsto \mathcal{Q}_\alpha^{\mathrm{geo}} \in \mathcal{L}(X_{\eta,\sigma}, Y_\eta(X_0)) \tag{657}$$

is C^1*, uniformly on sufficiently negative terminal windows, and its operator norm tends to zero as* $X_0 \to -\infty$*.*

Proof. The α derivative of an order-2α symbol introduces at most $\log\langle\xi\rangle$, absorbed by (642). The derivative of $r^{a_\alpha+2\alpha}$ and of r^{-a_α} introduces at most one factor $|X|$. The elementary bound $\sup_{X\le X_0} |X| e^{\varepsilon X} \le C_\varepsilon(1+|X_0|)e^{\varepsilon X_0}$, combined with e^{2X}, gives (655) after decreasing the exponent from 2 to $2-\varepsilon$. The far-field exponent in (648) remains larger than $2-\varepsilon$. This proves (656). □

196. Linearization in the profile variable

The fractional geometric operator and the Mellin operator are linear in the profile. Therefore the remainder (632) is linear in G and

$$\boxed{D_G \mathcal{Q}_\alpha^{\mathrm{geo}}[G]h = \mathcal{Q}_\alpha^{\mathrm{geo}} h.} \tag{658}$$

There is consequently no new nonlinear commutator in the profile linearization. Equations (653) and (656) immediately imply

$$\boxed{\|D_G \mathcal{Q}_\alpha^{\mathrm{geo}}[G]h\|_{Y_\eta(X_0)} \le C e^{2X_0} \|h\|_{X_{\eta,\sigma}},} \tag{659}$$

and

$$\boxed{\|\partial_\alpha D_G \mathcal{Q}_\alpha^{\mathrm{geo}}[G]h\|_{Y_\eta(X_0)} \le C_\varepsilon e^{(2-\varepsilon)X_0} \|h\|_{X_{\eta,\sigma}}.} \tag{660}$$

These are exactly the linearized estimates required when the geometric operator is inserted into the phase-fixed profile map.

197. What has and has not been closed

The estimate requested in (625) is now proved in the mathematically correct form dictated by the intrinsic chart:

$$\boxed{\|\mathcal{Q}_\alpha^{\mathrm{geo}}\|_{X_{\eta,\sigma}\to Y_\eta(X_0)} \le C e^{2X_0}, \qquad X_0 \to -\infty,} \tag{661}$$

with the C^1 estimate (656) and the profile-linearized estimates (659)–(660). Hence there is no longer an unresolved *terminal weighted operator remainder* in Route B.

There is, however, one distinction that must be retained. Since $r = -\sin(x - x_*)$ takes values only in a bounded interval, the exact geometric chart corresponds to $X = \log r < 0$ and does not represent the full Mellin heteroclinic line $X \in \mathbb{R}$. Therefore (661) proves the

operator bridge needed in the terminal overlap region, but by itself it does not identify the right-hand equilibrium $X \to +\infty$ of the old Mellin profile with an exact geometric endpoint. A global Route-B heteroclinic theorem would require one additional global chart or matching construction. The finite-N terminal argument, by contrast, uses precisely the left overlap window, and for that purpose (661) is the required estimate.

198. Revised ledger after the weighted remainder theorem

The first item in (627) can now be removed as a terminal-overlap hypothesis. The revised construction-level ledger is

$$\boxed{\begin{array}{c}\text{(i) prove the strict upper invariant-boundary sign;}\\ \text{(ii) prove the fixed-space energy estimates and the vanishing residual for } N \to \infty.\end{array}} \tag{662}$$

for the terminal finite-N matching problem. If the manuscript is to claim a *global exact geometric-profile heteroclinic* rather than terminal matching to the already constructed Mellin profile, the bounded-chart issue identified in Section 197 remains a separate global profile question and must not be hidden inside (662).

Theorem 19 (Terminal Route-B operator bridge)**.** *For $\alpha \in [1, 1+\delta_*]$, the exact fractional operator induced by $s_{\mathrm{geo}} = \sin^2(x - x_*)$, expressed in the natural coordinate $r = \sqrt{s_{\mathrm{geo}}}$, differs from the affine Mellin profile operator by an operator of weighted size $O(e^{2X_0})$ on every terminal logarithmic window $X \le X_0$. Its α derivative is $O(e^{(2-\varepsilon)X_0})$, and the same estimates hold for the profile linearization. Hence the geometric fractional operator is a uniformly small C^1 perturbation of the Mellin tangent operator in the finite-N terminal overlap region.*

199. Consequence for the next proof step

The weighted operator estimate no longer blocks the terminal matching. The next construction-specific question is therefore the upper invariant boundary: one must insert the now-controlled geometric fractional remainder into the exact finite-N scalar differential inequality and prove that at a sufficiently large normalized amplitude W^* the vector field points strictly inward, uniformly in N and in $\alpha \in [1, 1+\delta_*]$. That sign calculation is independent of the operator-domain issue settled in this Part.

50 Part 33: Exact audit of the strict upper invariant-boundary sign

200. Substitution of the geometric fractional remainder

Part 32 proves, on every terminal logarithmic window $X \le X_0 < 0$,

$$\|\mathcal{Q}_\alpha^{\mathrm{geo}}\|_{X_{\eta,\sigma} \to Y_\eta(X_0)} \le C_{\mathrm{geo}} e^{2X_0}, \tag{663}$$

uniformly for $\alpha \in [1, 1+\delta_*]$. In the normalized finite-N scalar field (448), this contributes to the remainder rather than changing the leading bracket. Thus the exact boundary field has the form

$$\partial_\tau W = \Lambda_{N,\alpha}(s) W \Big[D_{N,\alpha} W - K_{N,\alpha}(W) + R^{\mathrm{old}}_{N,\alpha}(s,W) + R^{\mathrm{geo}}_{N,\alpha}(s,W) \Big], \qquad \Lambda_{N,\alpha} > 0, \quad (664)$$

where, on every fixed compact W-interval on which the normalization is regular,

$$|R^{\mathrm{geo}}_{N,\alpha}(s,W)| \le C_{\mathrm{geo}} e^{2X_0}, \quad (665)$$

and the previously established finite-N moduli give

$$|R^{\mathrm{old}}_{N,\alpha}(s,W)| \le C_R s^{\mu_R} + \epsilon_N^{(R)}. \quad (666)$$

Consequently

$$|R^{\mathrm{tot}}_{N,\alpha}(s,W)| \le C_R s^{\mu_R} + \epsilon_N^{(R)} + C_{\mathrm{geo}} e^{2X_0}. \quad (667)$$

The geometric operator correction is therefore uniformly small after first choosing the terminal window $X_0 \ll 0$, then the terminal scale $s_* > 0$, and finally N sufficiently large.

201. The exact sign criterion at an upper boundary

Let

$$\mathscr{B}_\alpha(W) := D_\alpha W - K_\alpha(W) \quad (668)$$

be the limiting normalized bracket. A uniform upper boundary W^* is obtained if and only if one can find $W^* > 0$ and $\sigma_* > 0$ such that

$$\boxed{\sup_{\alpha \in I_\delta} \mathscr{B}_\alpha(W^*) \le -4\sigma_*.} \quad (669)$$

Indeed, coefficient convergence gives, after increasing N,

$$\sup_{\alpha \in I_\delta} |(D_{N,\alpha} W^* - K_{N,\alpha}(W^*)) - (D_\alpha W^* - K_\alpha(W^*))| \le \sigma_*. \quad (670)$$

Choose X_0, s_* and N_* so that

$$C_{\mathrm{geo}} e^{2X_0} + C_R s_*^{\mu_R} + \epsilon_N^{(R)} \le \sigma_*, \qquad N \ge N_*. \quad (671)$$

Then (664), (669)–(671) imply

$$\partial_\tau W\big|_{W=W^*} \le -2\sigma_* \Lambda_{N,\alpha}(s) W^* < 0, \quad (672)$$

uniformly for $\alpha \in I_\delta$, $N \ge N_*$ and $0 < s \le s_*$. Hence $W \le W^*$ is forward invariant at the upper boundary.

Theorem 20 (Uniform upper-boundary transfer)**.** *Assume the limiting bracket has the strict upper sign (669), the finite-N coefficient convergence (670), and the old normalized residual estimate (666). Then the weighted geometric fractional estimate of Part 32 preserves the sign: there exist $X_0 < 0$, $s_* > 0$ and N_* for which (672) holds uniformly. Thus the geometric fractional remainder is not an obstruction to the upper invariant boundary.*

202. Why a sufficiently large amplitude does not follow from the present positive gap

We must distinguish the transfer theorem above from existence of a value W^* satisfying (669). The information already proved is

$$D_\alpha \geq d_* > 0. \tag{673}$$

This controls the lower boundary, but by itself it drives $D_\alpha W$ *upward* as W increases. Therefore a large-W upper barrier requires quantitative growth of the dissipative nonlinear term. A sufficient condition is

$$\boxed{\liminf_{W\to\infty} \inf_{\alpha\in I_\delta} \frac{K_\alpha(W)}{W} > D^*, \qquad D^* := \sup_{\alpha\in I_\delta} D_\alpha.} \tag{674}$$

Under (674), choose $\vartheta > 0$ and W^* so large that

$$K_\alpha(W^*) \geq (D^* + \vartheta)W^* \quad \text{for every } \alpha \in I_\delta. \tag{675}$$

Then

$$\mathscr{B}_\alpha(W^*) \leq -\vartheta W^*, \tag{676}$$

so (669) follows with, for example, $4\sigma_* \leq \vartheta W^*$.

Conversely, the earlier simplified model (342), in which K_α^{diss} is merely bounded independently of W, cannot yield an upper barrier by taking W large. If

$$K_\alpha(W) \leq K_* < \infty, \tag{677}$$

then (673) gives

$$\mathscr{B}_\alpha(W) \geq d_* W - K_*, \tag{678}$$

which is positive for every $W > K_*/d_*$. The vector field then points in the same direction as at the lower boundary, not inward from above.

Theorem 21 (Large-amplitude obstruction under bounded dissipation)**.** *Under $D_\alpha \geq d_* > 0$ and the bounded-dissipation hypothesis (677), no sufficiently large constant W^* can satisfy the required upper sign (669). The $O(e^{2X_0})$ geometric remainder cannot repair this, because it can be made arbitrarily small and does not change the limiting sign (678).*

203. Audit of the construction actually present

The current source introduces the abstract nonlinear quantity $K_{N,\alpha}(W)$ in (448) and states in (450)–(453) that an upper boundary would follow from a strict limiting sign. It does not, however, subsequently supply an explicit construction-level formula for $K_\alpha(W)$ or a proved growth estimate of the form (674). Earlier Part 12 uses only the bounded coefficient K_α^{diss} in (343). Those statements are sufficient for the lower barrier but are insufficient for the upper one.

Therefore the requested substitution settles the effect of the new geometric fractional remainder but does *not* prove existence of a uniform W^* from the presently recorded hypotheses. The exact missing statement is now isolated as

$$\boxed{\text{prove (669) directly from the explicit normalized finite-}N\text{ dissipative term, or prove the uniform dominance condition (674).}} \tag{679}$$

This is a construction-level sign question, not an operator-remainder question.

204. Revised ledger after the upper-sign audit

Part 32 and the present Part prove that all already-controlled remainders, including the intrinsic geometric fractional correction, transfer any strict limiting upper sign to the finite-N equation. What is not yet supplied is the strict limiting upper sign itself. Accordingly the ledger becomes

(i) derive the explicit limiting function $K_\alpha(W)$ from the actual normalized finite-N dissipative term and verify (669), equivalently a sufficient condition such as (674); (ii) prove the fixed-space energy estimates and vanishing scalar residual needed for the strong $N \to \infty$ passage.

(680)

If the explicit $K_\alpha(W)$ is found to be bounded in W, then (678) shows that the proposed large-amplitude upper barrier is impossible in the present normalization and a different invariant variable or barrier mechanism is required. If instead $K_\alpha(W)$ grows faster than $D_\alpha W$ uniformly, then (674)–(676), together with (663), close the upper boundary immediately.

Theorem 22 (Status of the strict upper invariant-boundary problem)**.** *The weighted geometric fractional correction has now been fully inserted into the finite-N boundary field and is $O(e^{2X_0})$; hence it preserves every strict limiting upper sign. A uniform upper invariant boundary follows from (669), and in particular from the uniform dissipative dominance condition (674). This paper does not contain an explicit formula or growth theorem for $K_\alpha(W)$ sufficient to establish (669). Thus the upper-sign problem is reduced to the single explicit construction-level calculation (679), but it is not yet closed.*

51 Part 34: Explicit dissipative amplitude and the fixed-space limiting audit

205. The principal fractional term fixes the amplitude dependence exactly

The abstract notation $K_{N,\alpha}(W)$ in (448) can now be audited from the actual scalar equation rather than assigned an independent nonlinear growth law. The corrected product equation contains exactly one principal scalar fractional term,

$$-\nu(-\Delta)^\alpha b, \tag{681}$$

with the fractional product commutator placed in the separately controlled remainder. In the strict one-phase terminal coordinate the principal fractional operator is linear. Write

$$r_\alpha = 2\alpha - 1, \qquad b = p_{N,\alpha} W s^{-r_\alpha}. \tag{682}$$

At a constant-amplitude boundary $W = \text{const}$, Mellin homogeneity gives

$$(-\partial_{ss})^\alpha b = p_{N,\alpha} W K_\alpha^+(-r_\alpha) s^{-r_\alpha - 2\alpha}. \tag{683}$$

If the one-phase wave number is κ, then

$$(-\Delta)^\alpha b = \kappa^{2\alpha} p_{N,\alpha} W K_\alpha^+(-r_\alpha) s^{-r_\alpha - 2\alpha} + \mathcal{E}^{\text{geo}}_{N,\alpha}, \tag{684}$$

where Part 32 places the geometric correction in the small terminal remainder. Consequently the principal dissipative contribution is

$$\mathcal{D}^{\mathrm{prin}}_{N,\alpha} = -\nu\kappa^{2\alpha} p_{N,\alpha} W K^{+}_{\alpha}(-r_\alpha) s^{-r_\alpha - 2\alpha}. \tag{685}$$

The crucial point is that (685) is *linear in* W. After extracting the positive prefactor $\Lambda_{N,\alpha}(s)W$ used in (448), its contribution to the bracket is independent of W. Thus the function denoted abstractly by $K_{N,\alpha}(W)$ has, for the principal fractional term, the form

$$\boxed{K^{\mathrm{prin}}_{N,\alpha}(W) = k^{\mathrm{prin}}_{N,\alpha}(s),} \tag{686}$$

not a superlinear function of W. In the limiting autonomous normalization,

$$\boxed{K^{\mathrm{prin}}_{\alpha}(W) = k^{\mathrm{prin}}_{\alpha} \quad \text{is constant in } W.} \tag{687}$$

The exact numerical value of $k^{\mathrm{prin}}_{\alpha}$ depends on the positive normalizing prefactor used in (448), but its W-dependence does not: linearity of $(-\Delta)^{\alpha}$ forces (687).

206. The strict large-amplitude upper sign cannot come from principal dissipation

Suppose, as established earlier, that

$$D_\alpha \geq d_* > 0 \tag{688}$$

uniformly on I_δ. Absorb every already-controlled commutator, geometric, reconstruction, and coefficient defect into $R^{\mathrm{tot}}_{N,\alpha}$. Then the limiting constant-amplitude bracket produced by the principal scalar dissipation is

$$\mathscr{B}^{\mathrm{prin}}_{\alpha}(W) = D_\alpha W - k^{\mathrm{prin}}_{\alpha}. \tag{689}$$

If

$$k^* := \sup_{\alpha \in I_\delta} |k^{\mathrm{prin}}_{\alpha}| < \infty, \tag{690}$$

then

$$\mathscr{B}^{\mathrm{prin}}_{\alpha}(W) \geq d_* W - k^*. \tag{691}$$

Hence

$$\boxed{W > \frac{k^*}{d_*} \quad \Longrightarrow \quad \mathscr{B}^{\mathrm{prin}}_{\alpha}(W) > 0} \tag{692}$$

uniformly in α. This is the opposite of the required upper-boundary condition (669).

Theorem 23 (Explicit W-dependence of the principal dissipative term)**.** *For the corrected scalar equation in which the principal hyperdissipation is exactly $-\nu(-\Delta)^{\alpha} b$ and all fractional product commutators are kept in the remainder, substitution of $b = p_{N,\alpha} W s^{-r_\alpha}$ shows that the principal normalized dissipative quantity is constant in W. Therefore it cannot satisfy the superlinear dominance condition (674). Under the positive gap (688), no sufficiently large constant upper barrier can be generated by the principal fractional dissipation alone.*

This theorem resolves the ambiguity in Part 33. The abstract notation $K_\alpha(W)$ must not be treated as though the linear fractional operator could manufacture a superlinear amplitude law. Any additional W-dependence must come from a term outside the principal fractional block, and such a term must be derived explicitly rather than hidden in the remainder.

207. What would be required to recover an upper invariant boundary

There are only three logically distinct possibilities. First, an explicit reconstruction or commutator term retained in the leading field may produce a negative contribution $-\widetilde{K}_\alpha(W)$ satisfying

$$\liminf_{W\to\infty} \inf_{\alpha\in I_\delta} \frac{\widetilde{K}_\alpha(W)}{W} > D^*. \tag{693}$$

Second, the normalization may be changed so that the invariant variable has a bounded natural profile range and the upper boundary is placed at a finite profile value rather than at large amplitude. Third, one may prove a strict upper sign at a finite W^* directly from the full limiting profile vector field even though no large-W dominance occurs. Without one of these three mechanisms, the two-sided strip (453) is not proved.

208. Energy identity available at the velocity level

We next audit the second item of (680). If the finite-N reconstruction is an exact divergence-free solution of the hyperdissipative velocity equation on $\mathbb{T}^3$, then testing the velocity equation by u_N gives

$$\frac{1}{2}\frac{d}{dt}\|u_N\|_{L^2}^2 + \nu\|(-\Delta)^{\alpha/2}u_N\|_{L^2}^2 = \langle f_N, u_N\rangle. \tag{694}$$

The transport and pressure terms vanish by periodicity and incompressibility. If

$$\sup_N \|u_N(0)\|_{L^2} \le C_0, \qquad \sup_N \|f_N\|_{L^2(0,T;H^{-\alpha})} \le C_f, \tag{695}$$

then duality and Young's inequality give

$$|\langle f_N, u_N\rangle| \le \frac{\nu}{2}\|u_N\|_{H^\alpha}^2 + C_\nu\|f_N\|_{H^{-\alpha}}^2, \tag{696}$$

and therefore

$$\boxed{\sup_N \|u_N\|_{L^\infty(0,T;L^2)}^2 + \nu \sup_N \|u_N\|_{L^2(0,T;H^\alpha)}^2 \le C_T.} \tag{697}$$

Thus a standard fixed-space velocity energy bound follows once (695) is verified for the actual reconstruction.

209. Why the velocity energy estimate does not imply the scalar estimate (460)

The scalar is the product

$$b_N = u_{x,N}u_{y,N}. \tag{698}$$

In three dimensions, the Sobolev product rule gives schematically

$$H^\alpha \cdot H^\alpha \longrightarrow H^{2\alpha-3/2-\varepsilon} \tag{699}$$

for arbitrarily small $\varepsilon > 0$ in the relevant non-endpoint range. For α near 1, the right-hand exponent is near $1/2$, whereas (460) asks for the much stronger fixed-space control

$$b_N \in L^2(0,T;H^{1+\delta+\varepsilon_0}). \tag{700}$$

Consequently (697) alone cannot prove (460). In particular, the scalar fixed-space estimate is not a formal consequence of the basic Navier–Stokes energy identity.

A sufficient stronger package would be, for some $s_u > 3/2$ and $s_u \geq 1+\delta+\varepsilon_0$,

$$\sup_N \|u_N\|_{L^\infty(0,T;H^{s_u})} + \sup_N \|u_N\|_{L^2(0,T;H^{s_u+\alpha})} \leq C_T. \tag{701}$$

Since $H^{s_u}(\mathbb{T}^3)$ is then an algebra,

$$\|b_N\|_{H^{s_u}} \leq C\|u_{x,N}\|_{H^{s_u}}\|u_{y,N}\|_{H^{s_u}}, \tag{702}$$

which supplies the spatial part of (460). But (701) is a higher-order estimate and must be proved from the exact finite-N equations and their forcing; it is not contained in (694).

210. Time derivative and the scalar residual

From the exact scalar equation write

$$\partial_t b_N = -u_N \cdot \nabla b_N - \nu(-\Delta)^\alpha b_N + \mathcal{S}_N + \mathcal{R}_N, \tag{703}$$

where $\mathcal{S}_N$ denotes the explicitly retained amplification, forcing, pressure/reconstruction terms and $\mathcal{R}_N$ the exact approximation residual. If (700) holds and

$$\sup_N \|\mathcal{S}_N\|_{L^2(0,T;H^{-1-\delta-\varepsilon_0})} \leq C_T, \tag{704}$$

together with a corresponding transport bound, then

$$\boxed{\sup_N \|\partial_t b_N\|_{L^2(0,T;H^{-1-\delta-\varepsilon_0})} \leq C_T,} \tag{705}$$

which is (461). If in addition

$$\boxed{\|\mathcal{R}_N\|_{L^1(0,T;H^{-m})} \longrightarrow 0,} \tag{706}$$

then Parts 17 and 24 give the strong preterminal compactness and the limit equation.

The important logical point is that (706) is not implied by boundedness of $\mathcal{R}_N$ or by pointwise convergence of reconstruction coefficients. It requires an actual norm estimate for the exact residual after every finite-N velocity, pressure, commutator, and forcing term has been substituted.

211. Status of the two items in ledger (680)

The present audit produces a definitive answer for item (i), but it is an obstruction rather than the desired large-amplitude closure:

$$\boxed{K_\alpha^{\mathrm{prin}}(W) = k_\alpha^{\mathrm{prin}} \quad \text{and therefore principal hyperdissipation cannot force the large-}W\text{ upper sign.}} \tag{707}$$

Thus item (i) is replaced by the sharper finite-amplitude question

$$\boxed{\text{derive the full limiting bracket at finite } W \text{ and locate a strict upper zero/sign, or change the invariant normalization.}} \tag{708}$$

For item (ii), the basic velocity energy estimate (697) is available under the uniform data/forcing hypothesis (695), but it does not imply the stronger scalar compactness package (460)–(461). The remaining estimates are exactly (701), or another reconstruction-specific substitute strong enough to imply (700), together with (704)–(706).

Accordingly the rigorous ledger is now

$$\boxed{\begin{array}{c}\text{(i) resolve the finite-amplitude upper boundary using the full explicit normalized bracket, since principal dissipation is only linear in amplitude;}\\ \text{(ii) derive reconstruction-specific higher-order fixed-space bounds sufficient for (460)–(461); and}\\ \text{(iii) prove the exact residual estimate (706).}\end{array}} \tag{709}$$

Theorem 24 (Outcome of the explicit dissipation and energy audit)**.** *The corrected scalar equation determines the principal W-dependence of the fractional dissipative block: it is linear before division by the positive factor W and hence constant in the normalized bracket. Therefore the large-amplitude upper-sign mechanism proposed in (674) cannot be supplied by principal hyperdissipation. At the same time, the standard velocity energy identity yields (697) under uniform initial-data and forcing bounds, but this regularity is insufficient by itself to yield the scalar fixed-space estimate (460) near $\alpha = 1$. Hence neither a two-sided invariant strip nor the strong $N \to \infty$ scalar passage can be declared unconditional until the finite-amplitude bracket and the higher-order reconstruction/residual bounds in (709) are verified.*

52 Part 35: Full finite-amplitude bracket and compact upper-sign criterion

212. Start from the exact characteristic scalar equation

The finite-amplitude sign calculation must retain every term whose size changes with the amplitude parameter. Starting from the corrected scalar equation (4), write the exact characteristic equation as

$$\dot{b}_N = \Gamma_N b_N^2 + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}} - \nu \mathcal{L}_{N,\alpha} b_N + \nu \mathcal{C}_{N,\alpha}(u_{x,N}, u_{y,N}), \tag{710}$$

where

$$\mathcal{R}_N^{\mathrm{rec}} := b_N \frac{C_{N,t} + C_{N,z}}{C_N} + 2A_N b_N C_{N,x} + 2\frac{b_N^2}{A_N} C_{N,y}, \tag{711}$$

$\mathcal{L}_{N,\alpha}$ denotes the principal one-phase/geometric fractional operator, and

$$\mathcal{C}_{N,\alpha}(f,g) = \mathcal{R}_{\alpha}(f,g) \tag{712}$$

is the exact fractional product commutator. Equation (710) is only a regrouping of the already established product equation; in particular the commutator is not counted twice.

Use the intrinsic geometric variable

$$\sigma = \sin^2(x - x_*), \qquad \chi = -\sigma^{1/2}, \tag{713}$$

and place a constant-amplitude boundary at

$$\boxed{b_N = P_{N,\alpha}(\sigma)W, \qquad P_{N,\alpha}(\sigma) := p_{N,\alpha}\sigma^{-\rho_\alpha}.} \tag{714}$$

The exact square-root reconstruction gives

$$A_N^2 = -2\chi b_N = 2\sigma^{1/2}P_{N,\alpha}W, \tag{715}$$

so on the positive branch

$$\boxed{A_N = A_{0,N,\alpha}(\sigma)W^{1/2}, \qquad A_{0,N,\alpha} := \sqrt{2\sigma^{1/2}P_{N,\alpha}},} \tag{716}$$

and therefore

$$\boxed{u_{x,N} = U_{0,N,\alpha}(\sigma)W^{1/2}, \qquad U_{0,N,\alpha} := \frac{P_{N,\alpha}}{A_{0,N,\alpha}}.} \tag{717}$$

Thus both reconstructed factors in $b_N = u_{x,N}u_{y,N}$ are exactly homogeneous of degree $1/2$ in W.

213. Exact amplitude degrees of every retained term

At a constant boundary value W, linearity of the fractional operator gives

$$\mathcal{L}_{N,\alpha}(P_{N,\alpha}W) = W\,\mathcal{L}_{N,\alpha}P_{N,\alpha}. \tag{718}$$

Bilinearity of the fractional commutator and (716)–(717) give the equally important identity

$$\boxed{\mathcal{C}_{N,\alpha}(u_{x,N}, u_{y,N}) = W\,\mathcal{C}_{N,\alpha}(U_{0,N,\alpha}, A_{0,N,\alpha}).} \tag{719}$$

Hence the fractional commutator is leading order in the finite-amplitude sign problem, but it is only degree one in W; it cannot create a hidden superlinear damping law.

The three reconstruction terms have the exact amplitude decomposition

$$b_N\frac{C_{N,t} + C_{N,z}}{C_N} = W\,P_{N,\alpha}\frac{C_{N,t} + C_{N,z}}{C_N}, \tag{720}$$

$$2A_N b_N C_{N,x} = 2A_{0,N,\alpha}P_{N,\alpha}C_{N,x}\,W^{3/2}, \tag{721}$$

and

$$2\frac{b_N^2}{A_N}C_{N,y} = 2\frac{P_{N,\alpha}^2}{A_{0,N,\alpha}}C_{N,y}\,W^{3/2}. \tag{722}$$

The Riccati term is degree two,

$$\Gamma_N b_N^2 = \Gamma_N P_{N,\alpha}^2 W^2, \tag{723}$$

whereas the retained forcing is degree zero,

$$2C_N G_N = 2C_N G_N W^0. \tag{724}$$

These identities exhaust the amplitude dependence of the displayed corrected scalar equation provided C_N, its displayed derivatives, Γ_N, and G_N are reconstruction coefficients independent of the boundary parameter W. If a later reconstruction makes any of those quantities depend on W, that dependence must be inserted before the sign test.

214. The geometric-coordinate drift is also degree $3/2$

Because $b_N = P_{N,\alpha}(\sigma)W$, differentiation along the exact geometric characteristic gives

$$\dot{b}_N = P_{N,\alpha}\dot{W} + P'_{N,\alpha}(\sigma)W\dot{\sigma}. \tag{725}$$

From (574) and (714),

$$\dot{\sigma} = -2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}P_{N,\alpha}^{1/2}C_N^2\,W^{1/2}. \tag{726}$$

Since $P_{N,\alpha} = p_{N,\alpha}\sigma^{-\rho_\alpha}$,

$$P'_{N,\alpha} = -\frac{\rho_\alpha}{\sigma}P_{N,\alpha}, \tag{727}$$

and therefore moving the second term in (725) to the right produces

$$\boxed{-P'_{N,\alpha}W\dot{\sigma} = -2\sqrt{2}\,\rho_\alpha(1-\sigma)^{1/2}\sigma^{-1/4}P_{N,\alpha}^{3/2}C_N^2\,W^{3/2}.} \tag{728}$$

Thus the intrinsic geometric reparametrization contributes another explicit $W^{3/2}$ term. It cannot be hidden in an amplitude-independent remainder when the upper sign is tested.

215. Entire finite-N amplitude field

Substituting (718)–(728) into (710) and solving (725) for $\dot{W}$ gives

$$\boxed{\dot{W} = a_{2,N,\alpha}(\sigma)W^2 + a_{3/2,N,\alpha}(\sigma)W^{3/2} + a_{1,N,\alpha}(\sigma)W + a_{0,N,\alpha}(\sigma),} \tag{729}$$

with the exact coefficients

$$\boxed{a_{2,N,\alpha} = \Gamma_N P_{N,\alpha},} \tag{730}$$

$$\boxed{\begin{aligned} a_{3/2,N,\alpha} = & -2A_{0,N,\alpha}C_{N,x} - 2\frac{P_{N,\alpha}}{A_{0,N,\alpha}}C_{N,y} \\ & - 2\sqrt{2}\,\rho_\alpha(1-\sigma)^{1/2}\sigma^{-1/4}P_{N,\alpha}^{1/2}C_N^2, \end{aligned}} \tag{731}$$

$$\boxed{\begin{aligned} a_{1,N,\alpha} = & -\frac{C_{N,t}+C_{N,z}}{C_N} - \nu\frac{\mathcal{L}_{N,\alpha}P_{N,\alpha}}{P_{N,\alpha}} \\ & + \nu\frac{\mathcal{C}_{N,\alpha}(U_{0,N,\alpha},A_{0,N,\alpha})}{P_{N,\alpha}}, \end{aligned}} \tag{732}$$

and

$$\boxed{a_{0,N,\alpha} = \frac{2C_N G_N}{P_{N,\alpha}}.} \tag{733}$$

Equations (729)–(733) are the requested full finite-amplitude decomposition. They retain the reconstruction terms, the geometric drift, the principal fractional operator, and the fractional product commutator simultaneously. No one of these terms may be discarded solely because it is lower order in a large-W expansion when the desired barrier lies at finite W.

Whenever the coefficient limits exist uniformly on a terminal overlap window, write

$$a_{j,N,\alpha}(\sigma) \longrightarrow a_{j,\alpha}(\sigma), \qquad j \in \{2, 3/2, 1, 0\}. \tag{734}$$

Factoring the positive quantity W from (729) gives the entire limiting normalized bracket

$$\boxed{\mathscr{B}_\alpha(W;\sigma) = a_{2,\alpha}(\sigma)W + a_{3/2,\alpha}(\sigma)W^{1/2} + a_{1,\alpha}(\sigma) + \frac{a_{0,\alpha}(\sigma)}{W}.} \tag{735}$$

This replaces the overly compressed notation $D_\alpha W - K_\alpha(W)$. In particular, the fractional commutator enters $a_{1,\alpha}$, while the two nontrivial reconstruction derivatives and the geometric-coordinate drift enter $a_{3/2,\alpha}$.

216. Compact finite-amplitude sign problem becomes a quartic problem

Set

$$y = \sqrt{W} > 0. \tag{736}$$

Since multiplication by $W = y^2$ preserves sign, (735) is negative exactly when

$$\boxed{\mathcal{P}_\alpha(y;\sigma) := a_{2,\alpha}y^4 + a_{3/2,\alpha}y^3 + a_{1,\alpha}y^2 + a_{0,\alpha} < 0.} \tag{737}$$

Thus the finite-amplitude upper-boundary problem is not an asymptotic large-amplitude question. It is the compact minimization problem

$$\boxed{\inf_{\substack{\alpha\in I_\delta,\ 0<\sigma\le\sigma_* \\ y\in[y_-,y_+]}} \mathcal{P}_\alpha(y;\sigma) < 0.} \tag{738}$$

For a *uniform* invariant boundary one needs a common $y^* > 0$ and a strict margin $\zeta_* > 0$ such that

$$\boxed{\sup_{\substack{\alpha\in I_\delta\\ 0<\sigma\le\sigma_*}} \mathcal{P}_\alpha(y^*;\sigma) \le -4\zeta_* < 0.} \tag{739}$$

Then $W^* = (y^*)^2$, and uniform convergence (734), together with the already proved terminal operator remainders, transfers the sign to all sufficiently large finite N.

For fixed (α, σ), every interior candidate is found from

$$\partial_y \mathcal{P}_\alpha = 4a_{2,\alpha}y^3 + 3a_{3/2,\alpha}y^2 + 2a_{1,\alpha}y = 0, \tag{740}$$

or, for $y > 0$,

$$\boxed{4a_{2,\alpha}y^2 + 3a_{3/2,\alpha}y + 2a_{1,\alpha} = 0.} \tag{741}$$

Hence the sign test requires only the endpoints of the chosen compact y-interval and the positive roots of the quadratic (741). This is the exact algebraic reduction of the finite-amplitude problem.

217. What can and cannot be concluded from the coefficients presently proved

The current manuscript proves convergence and smallness for several terminal coefficient defects, but it does not yet provide uniform signed limits for $C_{N,x}$ and $C_{N,y}$ in the geometric reconstruction sufficient to evaluate (731), nor a signed evaluation of the commutator quotient in (732) on the whole compact (α, σ)-set. Therefore it would be incorrect to assert (739) from the existing estimates alone.

There is also an architectural distinction which must be preserved. In the strict affine one-phase specialization

$$C_1 = C_2 = 0, \qquad C_3 = C_4 = 1, \tag{742}$$

one has $C_x = C_y = 0$ and, for $C = C(t - z - \theta_*)$, $C_t + C_z = 0$. Then the three reconstruction terms (720)–(722) vanish. But the intrinsic geometric route uses $\sigma = \sin^2(x - x_*)$ and the active characteristic $\dot{x} = AC^2$. These two reductions cannot be combined merely to set the geometric reconstruction coefficients to zero. The finite-amplitude sign must be evaluated in one consistent architecture.

Consequently the requested bracket has now been derived, while the requested strict inequality is reduced to the following explicit coefficient theorem:

$$\boxed{\text{prove the signed uniform limits in (731)–(733) and verify (739).}} \tag{743}$$

If (739) fails, no constant upper boundary in this normalization follows from the present vector field; if it holds, the finite-N upper boundary follows by the transfer argument of Part 33.

Theorem 25 (Exact finite-amplitude bracket)**.** *Assume the geometric reconstruction (713)–(717), the exact characteristic scalar equation (710), and that C_N, Γ_N, G_N and the displayed spatial derivatives of C_N are independent of the boundary amplitude parameter W. Then*

the constant-boundary vector field has exactly the amplitude decomposition (729)–(733). If the coefficients converge uniformly as in (734), the limiting normalized bracket is (735). A uniform finite upper boundary W^* is equivalent to the strict common sign condition (739), with $W^* = (y^*)^2$. The only candidate interior amplitudes needed for this test are given by the quadratic equation (741).

218. Revised ledger

The abstract function $K_\alpha(W)$ is no longer an unresolved object: its actual amplitude content has been decomposed into the linear fractional and commutator coefficient $a_{1,\alpha}$, the square-root reconstruction and geometric-drift coefficient $a_{3/2,\alpha}$, the quadratic amplification coefficient $a_{2,\alpha}$, and the forcing coefficient $a_{0,\alpha}$. The remaining upper-sign task is therefore finite-dimensional but still construction-specific:

(i) prove the signed uniform coefficient limits in (731)–(734) and evaluate the compact quartic criterion (739); (ii) derive the reconstruction-specific higher-order fixed-space bounds sufficient for (460)–(461); (iii) prove the exact residual convergence (706).	(744)

This removes the ambiguity about the amplitude law without claiming a sign which the presently proved coefficient information does not determine.

53 Part 36: Signed uniform coefficient limits — exact audit and maximal theorem

219. The coefficient question must be separated into four independent limits

Part 35 reduced the constant-boundary dynamics to

$$\dot{W} = a_{2,N,\alpha}W^2 + a_{3/2,N,\alpha}W^{3/2} + a_{1,N,\alpha}W + a_{0,N,\alpha}. \tag{745}$$

The present section determines exactly which signed limits follow from the reconstruction already proved and which require additional signed data. This distinction is essential: uniform magnitude bounds do not determine the sign of $C_{N,x}$, $C_{N,y}$, or of the fractional product commutator.

Throughout the terminal overlap let

$$P_{N,\alpha} = p_{N,\alpha}\sigma^{-\rho_\alpha} > 0, \qquad A_{0,N,\alpha} = \sqrt{2\sigma^{1/2}P_{N,\alpha}} > 0, \tag{746}$$

and assume the positive branch $C_N > 0$. We work on a compact terminal annulus

$$\mathcal{K}_\sigma := \{(\alpha,\sigma) : \alpha \in I_\delta,\ \sigma_- \le \sigma \le \sigma_*\}, \qquad 0 < \sigma_- < \sigma_* < 1, \tag{747}$$

before taking the terminal limit $\sigma_- \downarrow 0$. This order avoids confusing uniform convergence on a compact overlap with a singular endpoint asymptotic.

220. The quadratic coefficient a_2

By (730),

$$a_{2,N,\alpha} = \Gamma_N P_{N,\alpha}. \tag{748}$$

The Lambert construction gives $\Gamma_N = \partial_\xi \log \Delta_{N,k}$ and the terminal branch is chosen so that the amplification coefficient is positive. Thus

$$\Gamma_N > 0, \qquad P_{N,\alpha} > 0, \tag{749}$$

and therefore

$$\boxed{a_{2,N,\alpha} > 0.} \tag{750}$$

If, as required by the coefficient convergence theorem already used in the terminal normalization,

$$\sup_{\mathcal{K}_\sigma} |\Gamma_N P_{N,\alpha} - D_\alpha(\sigma)| \longrightarrow 0, \tag{751}$$

then

$$\boxed{a_{2,N,\alpha} \longrightarrow a_{2,\alpha} = D_\alpha > 0 \quad \text{uniformly on } \mathcal{K}_\sigma.} \tag{752}$$

In particular the previously established gap $D_\alpha \geq d_* > 0$ transfers to $a_{2,\alpha}$. Notice that (752) is a positive, not dissipative, contribution to the upper-boundary polynomial.

221. The constant coefficient a_0

Equation (733) gives

$$a_{0,N,\alpha} = \frac{2C_N G_N}{P_{N,\alpha}}. \tag{753}$$

For the retained forcing branch, $C_N > 0$ and $G_N \geq 0$, hence

$$\boxed{a_{0,N,\alpha} \geq 0.} \tag{754}$$

Assume the already used terminal bounds

$$0 < c_- \leq C_N \leq c_+, \qquad 0 \leq G_N \leq G_+, \tag{755}$$

and $\inf_{\mathcal{K}_\sigma} P_{N,\alpha} \to \infty$ along the terminal finite-N sequence. Then

$$0 \leq a_{0,N,\alpha} \leq \frac{2c_+ G_+}{\inf_{\mathcal{K}_\sigma} P_{N,\alpha}} \longrightarrow 0. \tag{756}$$

Consequently

$$\boxed{a_{0,N,\alpha} \longrightarrow a_{0,\alpha} = 0^+ \quad \text{uniformly on } \mathcal{K}_\sigma.} \tag{757}$$

If P_N is not uniformly divergent on the chosen overlap, the weaker correct statement is only $a_{0,N,\alpha} \geq 0$ plus convergence inherited from the separate limits of C_N, G_N, P_N.

222. Exact sign decomposition of $a_{3/2}$

Write (731) as

$$a_{3/2,N,\alpha} = R^x_{N,\alpha} + R^y_{N,\alpha} + J_{N,\alpha}, \tag{758}$$

where

$$R^x_{N,\alpha} := -2A_{0,N,\alpha}C_{N,x}, \qquad R^y_{N,\alpha} := -2\frac{P_{N,\alpha}}{A_{0,N,\alpha}}C_{N,y}, \tag{759}$$

and

$$\boxed{J_{N,\alpha} := -2\sqrt{2}\,\rho_\alpha(1-\sigma)^{1/2}\sigma^{-1/4}P^{1/2}_{N,\alpha}C^2_N < 0.} \tag{760}$$

Thus the intrinsic geometric drift has a strict favorable sign. This is the first unconditional signed component of the finite-amplitude upper field.

However, exact continuity determines a derivative of C but does not fix the signs of both C_x and C_y. In the square-root reconstruction one has schematically

$$C_s = \partial_x(b/A), \tag{761}$$

and the reconstructed formula for C contains an arbitrary periodic integration datum $C_0(x)$. Differentiation therefore leaves a $C_0'(x)$ contribution in C_x. No estimate of the form $|C_x| \le M$ determines its sign. Likewise the general multidimensional reconstruction does not supply a signed C_y limit.

Consequently the strongest theorem available from the present hypotheses is conditional dominance. If

$$R^x_{N,\alpha} \to R^x_\alpha, \qquad R^y_{N,\alpha} \to R^y_\alpha, \qquad J_{N,\alpha} \to J_\alpha \tag{762}$$

uniformly on $\mathcal{K}_\sigma$, then

$$\boxed{a_{3/2,\alpha} = R^x_\alpha + R^y_\alpha + J_\alpha.} \tag{763}$$

A strict negative limit follows precisely from the verifiable margin

$$\boxed{\sup_{\mathcal{K}_\sigma}(R^x_\alpha + R^y_\alpha) < -\sup_{\mathcal{K}_\sigma} J_\alpha - \eta_{3/2}} \qquad (\eta_{3/2} > 0), \tag{764}$$

or, more transparently pointwise,

$$R^x_\alpha + R^y_\alpha \le |J_\alpha| - \eta_{3/2}. \tag{765}$$

The current reconstruction has not proved (764). Therefore a signed uniform limit $a_{3/2,\alpha} < 0$ cannot be asserted without an additional signed reconstruction estimate.

223. Exact sign decomposition of a_1

From (732), define

$$a_{1,N,\alpha} = T_{N,\alpha} + D_{N,\alpha} + M_{N,\alpha}, \tag{766}$$

with

$$T_{N,\alpha} := -\frac{C_{N,t} + C_{N,z}}{C_N}, \tag{767}$$

$$D_{N,\alpha} := -\nu \frac{\mathcal{L}_{N,\alpha} P_{N,\alpha}}{P_{N,\alpha}}, \tag{768}$$

and

$$M_{N,\alpha} := \nu \frac{\mathcal{C}_{N,\alpha}(U_{0,N,\alpha}, A_{0,N,\alpha})}{P_{N,\alpha}}. \tag{769}$$

In the old affine distinguished phase, $C_t + C_z = 0$, so $T = 0$; but Part 35 showed that this identity cannot simply be imported into the intrinsic x-geometric architecture. Hence $T_{N,\alpha}$ has no established sign in the present route.

For the tangent fractional operator and a pure power $P \propto r^{-\lambda}$, $r = \sqrt{\sigma}$, Mellin homogeneity gives

$$\frac{(-\partial_{rr})^\alpha P}{P} = K_\alpha^+(-\lambda) r^{-2\alpha}. \tag{770}$$

Thus even the sign of the fractional quotient is controlled by the Mellin coefficient $K_\alpha^+(-\lambda)$ and must be evaluated at the actual normalization exponent. The geometric-operator remainder proved in Part 32 is small in the terminal overlap, but smallness does not alter the sign unless a strict tangent-operator margin is first established.

Finally the fractional product commutator has no sign for arbitrary factors. Indeed, with the convention used,

$$\mathcal{C}_\alpha(f, g) = (-\Delta)^\alpha (fg) - f(-\Delta)^\alpha g - g(-\Delta)^\alpha f, \tag{771}$$

its Fourier/integral bilinear form changes sign with the relative variation of f and g. Therefore the reconstruction identity $U_0 A_0 = P$ does not by itself imply $M_{N,\alpha} \le 0$ or $M_{N,\alpha} \ge 0$.

If the three terms converge uniformly,

$$T_{N,\alpha} \to T_\alpha, \qquad D_{N,\alpha} \to D_\alpha^{\mathrm{frac}}, \qquad M_{N,\alpha} \to M_\alpha, \tag{772}$$

then

$$\boxed{a_{1,\alpha} = T_\alpha + D_\alpha^{\mathrm{frac}} + M_\alpha.} \tag{773}$$

A strict negative limit is equivalent to the signed margin

$$\boxed{\sup_{\mathcal{K}_\sigma}(T_\alpha + M_\alpha) < -\sup_{\mathcal{K}_\sigma} D_\alpha^{\mathrm{frac}} - \eta_1} \tag{774}$$

with the obvious pointwise interpretation. This margin has not yet been proved by the reconstruction.

224. Uniform-limit theorem and obstruction to the requested four-sign claim

Theorem 26 (Maximal signed coefficient theorem)**.** *Assume the coefficient convergence hypotheses already established for the Lambert/profile state, positivity of P_N, C_N, bounded nonnegative forcing, and the terminal geometric operator estimate of Part 32. Then on every fixed compact overlap $\mathcal{K}_\sigma$:*

1. *if (751) holds, $a_{2,N,\alpha} \to D_\alpha$ uniformly and $D_\alpha \ge d_* > 0$;*

2. *if $P_N \to \infty$ uniformly while C_N, G_N remain uniformly bounded and nonnegative, then $a_{0,N,\alpha} \to 0$ uniformly from above;*

3. *the geometric part $J_{N,\alpha}$ of $a_{3/2,N,\alpha}$ is strictly negative, but the sign of the full $a_{3/2}$ is undetermined until signed limits for $C_{N,x}$ and $C_{N,y}$ are proved;*

4. *the sign of a_1 is undetermined until the transport quotient and the normalized fractional commutator are evaluated with sign, together with the actual Mellin coefficient of the chosen power.*

Therefore the current hypotheses do not imply a prescribed sign for all four limiting coefficients. In particular, no rigorous proof of $a_{3/2} < 0$ or $a_1 < 0$ follows from the presently available magnitude bounds.

Proof. Items (1) and (2) are (748)–(757). Item (3) follows from the exact splitting (758)–(760): $J_N < 0$, whereas the signs of R_N^x, R_N^y are the opposite of unknown derivative signs. Item (4) is the exact splitting (766)–(769), for which neither the general geometric transport term nor the bilinear commutator has a sign theorem in the stated hypotheses. Uniform convergence preserves a strict sign only after a nonzero signed margin is supplied. This proves the claim. □

225. Consequence for the quartic upper-boundary problem

The limiting quartic is

$$\mathcal{P}_\alpha(y;\sigma) = a_{2,\alpha}y^4 + a_{3/2,\alpha}y^3 + a_{1,\alpha}y^2 + a_{0,\alpha}. \tag{775}$$

The newly proved signs give

$$a_{2,\alpha} > 0, \qquad a_{0,\alpha} \geq 0 \quad (\text{and } a_{0,\alpha} = 0 \text{ under (756)}). \tag{776}$$

Hence negativity on a finite interval can only be created by the middle coefficients. A useful sufficient criterion is obtained by selecting a common $y^* > 0$ and proving

$$a_{3/2,\alpha}(y^*)^3 + a_{1,\alpha}(y^*)^2 \leq -a_{2,\alpha}(y^*)^4 - a_{0,\alpha} - 4\zeta_*. \tag{777}$$

Equivalently,

$$\boxed{a_{3/2,\alpha}y^* + a_{1,\alpha} \leq -a_{2,\alpha}(y^*)^2 - \frac{a_{0,\alpha}}{(y^*)^2} - \frac{4\zeta_*}{(y^*)^2}.} \tag{778}$$

Thus the unknown signs isolated above are not cosmetic: at least one of the middle coefficients must supply enough negative contribution to overcome the positive Riccati coefficient and nonnegative forcing.

There is also a necessary condition. If

$$a_{3/2,\alpha} \geq 0, \qquad a_{1,\alpha} \geq 0, \tag{779}$$

for some (α, σ), then by (776)

$$\mathcal{P}_\alpha(y;\sigma) \geq 0 \qquad \forall y > 0, \tag{780}$$

so no constant upper boundary exists at that parameter value. Consequently a uniform upper barrier requires a uniform negative contribution from $a_{3/2}$ or a_1 (or both).

226. Revised proof ledger

The requested signed-limit calculation therefore produces a partial theorem and an obstruction, rather than four automatically signed limits. The ledger is now

$$\boxed{\begin{array}{c}\text{(i) } a_2 \text{ is uniformly positive and } a_0 \to 0^+ \text{ under the terminal normalization;}\\ \text{(ii) prove the signed reconstruction derivative margin (764) for } C_x, C_y,\\ \text{or prove instead a signed fractional/transport margin (774) for } a_1;\\ \text{(iii) once either middle-coefficient margin is available, evaluate the explicit compact criterion (777)–(778);}\\ \text{(iv) retain the higher-order fixed-space and residual limits from Part 34.}\end{array}} \tag{781}$$

This is the maximal conclusion supported by the current reconstruction. It would be mathematically incorrect to label the signs of $a_{3/2}$ and a_1 as proved before the missing signed derivative/commutator information is supplied.

Part 37. Exact Signed Transverse-Derivative Calculation and the Intrinsic-Geometry Compatibility Test

The preceding signed-coefficient audit reduced the unresolved middle coefficient to the transverse reconstruction combination

$$R^x_{N,\alpha} + R^y_{N,\alpha} = -2\left(A_{0,N,\alpha}C_{N,x} + \frac{P_{N,\alpha}}{A_{0,N,\alpha}}C_{N,y}\right). \tag{782}$$

The purpose of this part is to compute the sign supplied by the actual square-root reconstruction and, equally importantly, to determine whether that computation can be imported unchanged into the intrinsic $\sigma = s_{\mathrm{geo}} = \chi^2$ similarity architecture of Parts 30–36.

227. Exact directional derivative in the periodic square-root core

In the revised square-root reconstruction one has

$$A^2 = -2\chi(x)b, \qquad \mathcal{X} = A\partial_x + \frac{b}{A}\partial_y. \tag{783}$$

In the local distinguished-phase core used in the reconstruction,

$$b = b(s), \qquad s = t - z - \theta_{*,N}, \qquad b_x = b_y = 0, \tag{784}$$

and exact incompressibility gives

$$C_s = \partial_x\left(\frac{b}{A}\right) = \frac{A\chi'}{4\chi^2}. \tag{785}$$

Writing

$$b = B_N s^{-\beta}F_N(s), \qquad I_N(s) = \int_0^s r^{-\beta/2}F_N(r)^{1/2}\,dr, \tag{786}$$

the explicit reconstruction is

$$C(x,s) = c_0 + \frac{\chi'(x)\sqrt{-2B_N\chi(x)}}{4\chi(x)^2}I_N(s). \tag{787}$$

Here $B_N > 0$, $F_N > 0$, $I_N > 0$ on the selected positive branch. A direct differentiation, already implicit in the revised reconstruction, gives

$$\boxed{\mathcal{X}C = B_N s^{-\beta/2} F_N(s)^{1/2} I_N(s) \left[\frac{3}{4}\frac{(\chi')^2}{\chi^2} - \frac{1}{2}\frac{\chi''}{\chi}\right].} \tag{788}$$

For the genuine periodic geometry

$$\chi(x) = \sin(x - x_*), \qquad \chi'' = -\chi, \tag{789}$$

so

$$\frac{3}{4}\frac{(\chi')^2}{\chi^2} - \frac{1}{2}\frac{\chi''}{\chi} = \frac{3}{4}\cot^2(x - x_*) + \frac{1}{2} \geq \frac{1}{2}. \tag{790}$$

Consequently

$$\boxed{\mathcal{X}C > 0} \tag{791}$$

throughout every one-sided core on which $\chi \neq 0$. Therefore the factor-2 reconstruction contribution to the scalar equation satisfies

$$\boxed{-2b\,\mathcal{X}C < 0.} \tag{792}$$

This is a genuine signed statement, stronger than the magnitude estimate used earlier.

If the finite-amplitude normalization is performed while retaining the independent affine phase variable s, then at fixed amplitude W

$$A = A_0\sqrt{W}, \qquad \frac{b}{A} = \frac{P}{A_0}\sqrt{W}, \tag{793}$$

and hence

$$A_0 C_x + \frac{P}{A_0} C_y = \frac{1}{\sqrt{W}}\,\mathcal{X}C. \tag{794}$$

Thus the reconstruction part of the $W^{3/2}$ coefficient has the strict sign

$$\boxed{R^x_{N,\alpha} + R^y_{N,\alpha} = -\frac{2}{\sqrt{W}}\mathcal{X}C < 0.} \tag{795}$$

The precise coefficient obtained after collecting the normalization factors is therefore non-positive, and is strictly negative wherever the reconstructed core is nontrivial. Together with the already negative geometric drift $J_{N,\alpha} < 0$, this would give a strict negative $a_{3/2}$ in that architecture once a uniform lower margin is proved.

228. Why (795) does not yet prove the intrinsic-σ coefficient limit

The calculation above uses the Eulerian reconstruction hypotheses (784): $s = t - z - \theta_{*,N}$ is independent of x, while $\chi = \chi(x)$ is an independent geometric variable. The intrinsic Route-B normalization instead sets

$$\sigma = \chi(x)^2 \tag{796}$$

and uses σ itself as the terminal similarity coordinate. If one now writes an Eulerian ansatz

$$b = P(\sigma)W, \tag{797}$$

then, unless P is constant,

$$b_x = P_\sigma(\sigma)\sigma_x W \neq 0. \tag{798}$$

Therefore the derivation of (785)–(788), which used $b_x = 0$, cannot be imported unchanged.

There is an even sharper periodicity test. If $b = b(\sigma(x))$ and $A = A(\sigma(x))$ have no z-dependence, exact incompressibility reads

$$C_z = -\partial_x\left(\frac{b}{A}\right) - A_y. \tag{799}$$

In the one-dimensional intrinsic geometric ansatz $A_y = 0$, the right-hand side is independent of z. A periodic C can then exist only if

$$\boxed{\partial_x\left(\frac{b}{A}\right) = 0.} \tag{800}$$

But from $A^2 = -2\chi b$,

$$\frac{b}{A} = \sqrt{\frac{-b}{2\chi}}, \tag{801}$$

which is not constant for the terminal power profiles under consideration. Thus a pure Eulerian replacement of the affine phase by $\sigma = \chi^2(x)$ is incompatible with periodic incompressibility unless additional y, z-dependence is introduced.

Theorem 27 (Signed derivative theorem with compatibility obstruction)**.** *For the revised square-root core with independent variables $s = t - z - \theta_{*,N}$ and $\chi(x) = \sin(x - x_*)$, the exact continuity reconstruction satisfies*

$$\mathcal{X}C > 0, \qquad -2b\mathcal{X}C < 0. \tag{802}$$

Hence the transverse reconstruction contribution has the favorable negative sign. However this signed formula does not, by itself, establish the coefficient $a_{3/2}$ of the intrinsic-σ finite-amplitude bracket, because the latter changes the Eulerian dependence from $b = b(s)$ to a profile parametrized by $\sigma = \chi^2$. A pure $b = b(\sigma(x))$ periodic ansatz fails the incompressibility test (800) unless an additional multidimensional correction is supplied.

229. Correct interpretation of σ along characteristics

There is no contradiction in using

$$\sigma_N(t) = \chi(x_N(t))^2 \tag{803}$$

as a *characteristic parameter*. Along one trajectory it is a valid monotone coordinate whenever $\dot\sigma_N \neq 0$, and Parts 30–32 correctly use it to identify the tangent fractional homogeneity.

What fails is the stronger, unstated identification of this Lagrangian parameter with a global Eulerian one-variable ansatz $b = b(\sigma(x))$.

Accordingly the rigorous Route-B architecture must retain two levels:

$$\boxed{\begin{array}{c}\text{Eulerian reconstruction: } (x,s) \text{ with exact periodic continuity,}\\ \text{Lagrangian terminal parameter: } \sigma_N(t) = \chi(x_N(t))^2.\end{array}} \tag{804}$$

The coefficient limit in the σ-bracket must therefore be obtained by pulling the already reconstructed Eulerian coefficients along the characteristic and then reparametrizing by σ_N, rather than by replacing the Eulerian phase everywhere by σ.

230. What is now proved for the signed $a_{3/2}$ problem

Let

$$\mathfrak{R}_N(t) := -2\mathcal{X}C(x_N(t), s_N(t)). \tag{805}$$

Then (791) gives

$$\boxed{\mathfrak{R}_N(t) < 0.} \tag{806}$$

exactly on the selected sine branch. Reparametrize by $\sigma = \sigma_N(t)$:

$$\widehat{\mathfrak{R}}_N(\sigma) := \mathfrak{R}_N(t_N(\sigma)). \tag{807}$$

The sign is preserved:

$$\boxed{\widehat{\mathfrak{R}}_N(\sigma) < 0.} \tag{808}$$

Thus the missing issue is no longer the sign of the reconstruction derivative combination. It is the *uniform normalized lower margin*: one must prove for some $\eta_R > 0$

$$\boxed{\inf_{N\ge N_0}\inf_{\alpha\in I_\delta}\inf_{\sigma\in[\sigma_-,\sigma_+]}\left[-\widehat{\mathfrak{R}}_{N,\alpha}(\sigma)/\mathcal{N}_{N,\alpha}(\sigma)\right] \ge \eta_R,} \tag{809}$$

where $\mathcal{N}_{N,\alpha}$ is the exact positive normalization used in (731). Pointwise negativity alone is insufficient for a uniform invariant boundary.

The source gives the useful exact lower estimate

$$\mathcal{X}C \ge \frac{1}{2}B_N s^{-\beta/2}F_N(s)^{1/2}I_N(s), \tag{810}$$

but converting (810) into (809) requires the explicit overlap relation $s_N = s_N(\sigma)$ along the actual characteristic. Importantly, this is not the discarded assumption $\chi^2 \asymp s$; it is the dynamically computed characteristic reparametrization.

231. Updated ledger

The attempted signed-derivative closure therefore yields a genuine advance and a precise remaining bridge:

$$\boxed{\begin{array}{c}\text{(i) the exact transverse reconstruction sign is favorable: } \mathcal{X}C > 0,\ -2b\mathcal{X}C < 0;\\ \text{(ii) do not impose the inconsistent Eulerian ansatz } b = b(\chi^2(x));\\ \text{(iii) derive the actual characteristic map } s_N \leftrightarrow \sigma_N \text{ and prove the normalized margin (809);}\\ \text{(iv) combine that margin with } J_{N,\alpha} < 0 \text{ to obtain } a_{3/2} \le -\eta_{3/2} < 0;\\ \text{(v) then evaluate the compact quartic upper-boundary criterion, while retaining the fixed-space energy/residual obligations of Part 34.}\end{array}} \tag{811}$$

This replaces the earlier request for separately signed limits of C_x and C_y: the scalar equation only needs their characteristic combination $\mathcal{X}C$, and that combination has now been signed exactly. The remaining task is quantitative uniformity under the Lagrangian s-to-σ reparametrization.

54 Part 38: Exact characteristic map $s_N \leftrightarrow \sigma_N$ and the self-consistent continuity asymptotic

232. Exact differential relation without the discarded identification $s_N \asymp \sigma_N$

We now compute the characteristic relation requested at the end of Part 37. Throughout this part write $s = s_N^{\mathrm{ph}}$ and $\sigma = \sigma_N$. In the large-amplitude terminal region $b_N > 1$, the two exact characteristic laws are

$$\dot{s} = 1 - b_N, \qquad \dot{\sigma} = -2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}b_N^{1/2}C_N^2. \tag{812}$$

Since both derivatives are negative, s and σ decrease together and

$$\boxed{\frac{d\sigma}{ds} = 2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}\frac{b_N^{1/2}}{b_N - 1}C_N^2 > 0.} \tag{813}$$

Equivalently,

$$\boxed{\frac{d}{ds}\sigma^{1/4} = \frac{1}{\sqrt{2}}(1-\sigma)^{1/2}\frac{b_N^{1/2}}{b_N - 1}C_N^2.} \tag{814}$$

Thus, if the terminal endpoint is normalized by $s = \sigma = 0$, the exact implicit characteristic map is

$$\boxed{\sigma(s)^{1/4} = \frac{1}{\sqrt{2}}\int_0^s (1-\sigma(r))^{1/2}\frac{b_N(r)^{1/2}}{b_N(r) - 1}C_N(r)^2\,dr.} \tag{815}$$

This is the actual relation. No comparison $s \asymp \sigma$ has been used.

233. First test with bounded C_N and why it is not self-consistent

Suppose provisionally that

$$b_N(s) = B_N s^{-\beta} F_N(s), \qquad 0 < \beta < 2, \qquad F_N(s) \longrightarrow F_{0,N} > 0, \tag{816}$$

and that $C_N(s) \to c_{0,N} > 0$. Since $b_N \to \infty$,

$$\frac{b_N^{1/2}}{b_N - 1} = b_N^{-1/2}\big(1 + O(b_N^{-1})\big). \tag{817}$$

Equation (815) would then give

$$\sigma(s)^{1/4} \sim \frac{c_{0,N}^2}{\sqrt{2B_N F_{0,N}}} \frac{s^{1+\beta/2}}{1+\beta/2}, \tag{818}$$

so that

$$\boxed{\sigma(s) \sim \left[\frac{c_{0,N}^2}{\sqrt{2B_N F_{0,N}}(1+\beta/2)}\right]^4 s^{4+2\beta}.} \tag{819}$$

This already proves that the old relation $\sigma \asymp s$ is not the characteristic law.

However, (819) cannot be combined with bounded C_N and the exact continuity reconstruction. On the negative sine branch,

$$\chi = -\sqrt{\sigma}, \qquad \chi' = \sqrt{1-\sigma}, \tag{820}$$

and (787) becomes exactly

$$\boxed{C_N(s,\sigma) = c_{0,N} + \frac{\sqrt{2B_N}}{4}(1-\sigma)^{1/2}\sigma^{-3/4} I_N(s),} \tag{821}$$

where

$$I_N(s) = \int_0^s r^{-\beta/2} F_N(r)^{1/2}\, dr. \tag{822}$$

Under (816),

$$I_N(s) \sim \frac{F_{0,N}^{1/2}}{1-\beta/2} s^{1-\beta/2}. \tag{823}$$

Substitution of (819) into (821) makes the second term diverge. Hence the bounded-C_N calculation (818)–(819) is only a diagnostic contradiction; it is not the self-consistent terminal characteristic asymptotic.

234. Self-consistent dominant balance using exact continuity

The exact reconstruction (821) must instead be inserted into (813). Because $I_N(s) > 0$, the reconstructed term is positive. If it dominates $c_{0,N}$, then, uniformly on a terminal regime in which (816) holds,

$$C_N \sim \frac{\sqrt{2B_N F_{0,N}}}{4(1-\beta/2)} \sigma^{-3/4} s^{1-\beta/2}. \tag{824}$$

Using (817) and (824) in (813) yields

$$\frac{d\sigma}{ds} \sim K_N \sigma^{-3/4} s^{2-\beta/2}, \qquad K_N := \frac{\sqrt{2}}{4} \frac{\sqrt{B_N F_{0,N}}}{(1-\beta/2)^2}. \tag{825}$$

Therefore

$$\frac{4}{7}\sigma^{7/4} \sim \frac{K_N}{3-\beta/2} s^{3-\beta/2}, \tag{826}$$

and hence

$$\boxed{\sigma(s) \sim \mathfrak{K}_N s^{p_\beta}, \qquad p_\beta := \frac{12-2\beta}{7},} \tag{827}$$

with

$$\boxed{\mathfrak{K}_N = \left[\frac{7\sqrt{2}\sqrt{B_N F_{0,N}}}{16(1-\beta/2)^2(3-\beta/2)} \right]^{4/7}.} \tag{828}$$

Equivalently,

$$\boxed{s(\sigma) \sim \mathfrak{K}_N^{-1/p_\beta} \sigma^{1/p_\beta} = \mathfrak{K}_N^{-7/(12-2\beta)} \sigma^{7/(12-2\beta)}.} \tag{829}$$

This is the first self-consistent characteristic relation produced by the simultaneous use of the characteristic equations and exact continuity.

235. Consistency check for the reconstructed longitudinal velocity

Combining (824) and (827) gives

$$C_N(s) \asymp s^{-\frac{2(\beta+1)}{7}}, \tag{830}$$

or, in the intrinsic parameter,

$$\boxed{C_N(\sigma) \asymp \sigma^{-\frac{\beta+1}{6-\beta}}.} \tag{831}$$

Thus the exact characteristic reparametrization has an important consequence: for every $\beta > 0$ in the regime above, C_N does *not* tend to a positive finite constant. It grows. Consequently the collection of assumptions

$$C_N \to c_{0,N} > 0, \qquad \dot{x}_N = A_N C_N^2, \qquad C_s = \frac{A\chi'}{4\chi^2}, \qquad b_N \sim B_N F_{0,N} s^{-\beta} \tag{832}$$

cannot all describe the same deep terminal characteristic.

This does not invalidate exact incompressibility. Rather, it says that exact incompressibility itself forces a different C_N asymptotic once the actual x–characteristic is followed.

236. Substitution into the exact signed lower bound for $\mathcal{X}C$

For the sine geometry the bracket in (788) can be written exactly as

$$\frac{3}{4}\frac{(\chi')^2}{\chi^2}-\frac{1}{2}\frac{\chi''}{\chi}=\frac{3}{4\sigma}-\frac{1}{4}. \tag{833}$$

Hence

$$\boxed{\mathcal{X}C=B_N s^{-\beta/2}F_N(s)^{1/2}I_N(s)\left(\frac{3}{4\sigma}-\frac{1}{4}\right).} \tag{834}$$

This is sharper than the coarse lower bound (810). From (816), (823), and (827),

$$\mathcal{X}C\sim\frac{3B_N F_{0,N}}{4(1-\beta/2)}\frac{s^{1-\beta}}{\sigma(s)}. \tag{835}$$

Therefore

$$\boxed{\mathcal{X}C\asymp s^{-\frac{5(\beta+1)}{7}}\asymp\sigma^{-\frac{5(\beta+1)}{2(6-\beta)}}.} \tag{836}$$

In particular,

$$\boxed{-2\mathcal{X}C\longrightarrow-\infty} \tag{837}$$

along the self-consistent terminal characteristic, provided the uniform version of (816), the dominance in (824), and the coefficient bounds needed for (827) hold.

Thus the transverse reconstruction contribution has not merely a favorable pointwise sign: at the level of this dominant balance it has an increasing negative magnitude.

237. What this says about the uniform $a_{3/2}$ margin

At $W=1$, the reconstruction part of (731) is the characteristic combination

$$R^x_{N,\alpha}+R^y_{N,\alpha}=-2\mathcal{X}C. \tag{838}$$

The geometric drift term already satisfies

$$J_{N,\alpha}<0. \tag{839}$$

Consequently, if the finite-N overlap admits uniform constants

$$0<B_-\le B_N\le B_+,\quad 0<F_-\le F_N\le F_+,\quad 0<\beta_-\le\beta\le\beta_+<2, \tag{840}$$

together with uniform control of the errors in (823)–(827), then for every sufficiently small fixed terminal annulus $0<\sigma_-\le\sigma\le\sigma_*$ the reconstruction term possesses a strict negative lower margin. After dividing by any positive normalization $\mathcal{N}_{N,\alpha}$ which is uniformly comparable to the leading scale in (836), one obtains

$$\boxed{\inf_{N\ge N_0}\inf_{\alpha\in I_\delta}\inf_{\sigma\in[\sigma_-,\sigma_*]}\frac{-\widehat{\mathfrak{R}}_{N,\alpha}(\sigma)}{\mathcal{N}_{N,\alpha}(\sigma)}\ge\eta_R>0.} \tag{841}$$

Hence the sign mechanism required by (809) is compatible with the actual characteristic map.

There is, however, a new and more fundamental issue: (831) contradicts the bounded-C_N terminal hypothesis used in several earlier coefficient and velocity estimates. Therefore (841) cannot yet be promoted to an unconditional closure of $a_{3/2}$ for the full reconstruction. The C_N-dependent coefficients in (731)–(733), the velocity norms, and the fixed-space residual estimates must first be re-audited with the growth law (831).

238. Revised characteristic-map theorem

Theorem 28 (Self-consistent s–σ characteristic asymptotic)**.** *Assume the exact characteristic laws (812), the exact continuity reconstruction (821), and the terminal scalar profile (816), with $0 < \beta < 2$ and $F_N(s) \to F_{0,N} > 0$. Assume that the reconstructed term in (821) dominates its integration constant and that the displayed asymptotics hold uniformly on the terminal overlap. Then*

$$\sigma_N(s) \sim \mathfrak{K}_N s^{(12-2\beta)/7}, \qquad s_N(\sigma) \sim \mathfrak{K}_N^{-7/(12-2\beta)} \sigma^{7/(12-2\beta)}, \tag{842}$$

with $\mathfrak{K}_N$ given by (828). Moreover,

$$C_N(\sigma) \asymp \sigma^{-(\beta+1)/(6-\beta)}, \tag{843}$$

and

$$\mathcal{X}C \asymp \sigma^{-5(\beta+1)/(2(6-\beta))} > 0. \tag{844}$$

Thus the signed reconstruction contribution to $a_{3/2}$ is strictly negative and grows in magnitude toward the terminal endpoint. At the same time, the bounded-C_N deep-core hypothesis is incompatible with this self-consistent characteristic asymptotic.

239. Updated ledger after the exact characteristic calculation

The requested calculation therefore resolves the immediate question in a more informative way than the provisional comparison $s \asymp \sigma$:

(i) The exact map is (815), and the self-consistent power law is (842).

(ii) The reconstruction sign not only survives but strengthens as in (844).

(iii) A uniform normalized negative margin follows conditionally from uniform overlap constants and error control.

(iv) The same calculation forces the new growth law (843), so the earlier bounded-C_N terminal coefficient estimates cannot be reused.

(v) The next decisive audit is to substitute (842)–(843) into every C_N-dependent coefficient in (731)–(733), especially $J_{N,\alpha}$ and $a_{1,N,\alpha}$, and determine whether the compact quartic criterion remains uniformly negative.

(845)

The important logical conclusion is therefore not yet a full closure of the upper invariant boundary. The $a_{3/2}$ reconstruction sign is favorable, but the exact characteristic map changes the longitudinal-velocity scaling and consequently changes the remaining coefficient audit.

55 Part 39: Complete C_N-dependent coefficient hierarchy after the self-consistent characteristic map

240. The intrinsic amplitude exponent must be recomputed

Part 38 changes more than the longitudinal velocity. It also changes the power of the intrinsic boundary profile itself. The exponent ρ_α in (586) was obtained from the provisional bounded-C_N relation and cannot be retained after (842). From

$$b_N(s) \sim B_N F_{0,N} s^{-\beta}, \qquad s(\sigma) \sim \mathfrak{K}_N^{-7/(12-2\beta)} \sigma^{7/(12-2\beta)}, \tag{846}$$

we obtain instead

$$\boxed{b_N(\sigma) \asymp \sigma^{-\widehat{\rho}_\beta}, \qquad \widehat{\rho}_\beta := \frac{7\beta}{2(6-\beta)}.} \tag{847}$$

Accordingly, throughout this audit the self-consistent finite-amplitude normalization is

$$\boxed{P_{N,\alpha}(\sigma) = \widehat{p}_{N,\alpha} \sigma^{-\widehat{\rho}_\beta}.} \tag{848}$$

This is a correction to the use of the old ρ_α in (714) when one passes all the way to the deep characteristic regime of Part 38.

For later use set

$$q_\beta := \frac{\beta+1}{6-\beta}, \qquad e_R := \frac{5(\beta+1)}{2(6-\beta)}, \qquad e_J := \frac{7(\beta+1)}{2(6-\beta)}. \tag{849}$$

Then (843)–(844) read

$$C_N \asymp \sigma^{-q_\beta}, \qquad \mathcal{X}C \asymp \sigma^{-e_R}. \tag{850}$$

241. Complete $a_{3/2}$ hierarchy

The reconstruction part of (731) is the exact characteristic combination

$$-2A_{0,N,\alpha} C_{N,x} - 2\frac{P_{N,\alpha}}{A_{0,N,\alpha}} C_{N,y} = -2\mathcal{X}C \tag{851}$$

at unit boundary amplitude. Hence, by (850),

$$\boxed{R^{\mathrm{rec}}_{N,\alpha} \asymp -c_{R,N,\alpha} \sigma^{-e_R}, \qquad c_{R,N,\alpha} > 0.} \tag{852}$$

The geometric drift in (731) is

$$J_{N,\alpha} = -2\sqrt{2}\, \widehat{\rho}_\beta (1-\sigma)^{1/2} \sigma^{-1/4} P_{N,\alpha}^{1/2} C_N^2. \tag{853}$$

Using (848) and (850),

$$|J_{N,\alpha}| \asymp \sigma^{-\left(\frac{1}{4} + \frac{\widehat{\rho}_\beta}{2} + 2q_\beta\right)}. \tag{854}$$

A direct simplification gives

$$\boxed{\frac{1}{4}+\frac{\widehat{\rho}_\beta}{2}+2q_\beta=\frac{7(\beta+1)}{2(6-\beta)}=e_J.} \tag{855}$$

Moreover,

$$e_J-e_R=q_\beta>0. \tag{856}$$

Therefore the negative geometric drift is more singular than the already negative transverse reconstruction term:

$$\boxed{a_{3/2,N,\alpha}(\sigma)=-c_{J,N,\alpha}\sigma^{-e_J}\left(1+O(\sigma^{q_\beta})\right),\qquad c_{J,N,\alpha}>0.} \tag{857}$$

Under uniform two-sided bounds on the coefficients entering (848)–(850), $c_{J,N,\alpha}$ is bounded below by a positive constant. Thus the full $a_{3/2}$ coefficient, not merely one of its pieces, has a strict negative terminal sign.

242. The transport part of a_1 cancels exactly in the active Eulerian reconstruction

The exact continuity reconstruction has the form $C_N=C_N(x,s)$ with

$$s=t-z-\theta_{*,N}. \tag{858}$$

At fixed Eulerian x,

$$C_{N,t}=C_{N,s},\qquad C_{N,z}=-C_{N,s}. \tag{859}$$

Consequently

$$\boxed{C_{N,t}+C_{N,z}=0,} \tag{860}$$

and the first term in (732) vanishes identically. This cancellation is a partial-derivative identity; it is not the total derivative of C_N along the x-moving characteristic.

243. Fractional principal term and commutator in a_1

Write $r=\sqrt{\sigma}$. From (848),

$$P_{N,\alpha}=\widehat{p}_{N,\alpha}r^{-2\widehat{\rho}_\beta}. \tag{861}$$

The two square-root factors in (716)–(717) have the tangent powers

$$A_{0,N,\alpha}\asymp r^{\lambda_A},\qquad U_{0,N,\alpha}\asymp r^{\lambda_U}, \tag{862}$$

where

$$\boxed{\lambda_A=\frac{1}{2}-\widehat{\rho}_\beta,\qquad \lambda_U=-\frac{1}{2}-\widehat{\rho}_\beta,\qquad \lambda_A+\lambda_U=-2\widehat{\rho}_\beta.} \tag{863}$$

For the terminal Mellin tangent operator,

$$(-\partial_{rr})^\alpha r^\lambda=K_\alpha^+(\lambda)r^{\lambda-2\alpha}. \tag{864}$$

Hence

$$\frac{\mathcal{L}_{N,\alpha}P_{N,\alpha}}{P_{N,\alpha}} = K_\alpha^+(-2\widehat{\rho}_\beta)\sigma^{-\alpha} + o(\sigma^{-\alpha}), \tag{865}$$

while the exact commutator definition gives

$$\frac{\mathcal{C}_{N,\alpha}(U_{0,N,\alpha}, A_{0,N,\alpha})}{P_{N,\alpha}} = \Big[K_\alpha^+(-2\widehat{\rho}_\beta) - K_\alpha^+(\lambda_U) - K_\alpha^+(\lambda_A)\Big]\sigma^{-\alpha} + o(\sigma^{-\alpha}). \tag{866}$$

The $K_\alpha^+(-2\widehat{\rho}_\beta)$ terms cancel between the two pieces of (732). Therefore

$$\boxed{a_{1,N,\alpha}(\sigma) = -\nu\Big[K_\alpha^+(\lambda_U) + K_\alpha^+(\lambda_A)\Big]\sigma^{-\alpha} + o(\sigma^{-\alpha}).} \tag{867}$$

This is the complete leading tangent coefficient of a_1. Its sign is not needed for the final dominance statement below, provided the Mellin coefficients remain uniformly finite on the continuation interval.

At the anchor $\alpha = 1$, $K_1^+(\lambda) = \lambda(1-\lambda)$, so (867) can be checked directly. The sign may be positive for the self-consistent powers; this does not destroy the upper sign because the negative $a_{3/2}$ term has a strictly larger terminal exponent in the profile scaling considered below.

244. The C_N-dependent forcing coefficient a_0

Equation (733), together with (848) and (850), gives

$$a_{0,N,\alpha} = \frac{2C_N G_N}{P_{N,\alpha}} \asymp 2\frac{G_N}{\widehat{p}_{N,\alpha}}\sigma^{\widehat{\rho}_\beta - q_\beta}. \tag{868}$$

Since

$$\boxed{\widehat{\rho}_\beta - q_\beta = \frac{5\beta - 2}{2(6-\beta)},} \tag{869}$$

we obtain, for uniformly bounded G_N,

$$\boxed{a_{0,N,\alpha} = \begin{cases} o(1), & \beta > 2/5, \\ O(1), & \beta = 2/5, \\ O\big(\sigma^{-(2-5\beta)/(2(6-\beta))}\big), & 0 < \beta < 2/5. \end{cases}} \tag{870}$$

In the similarity regime relevant to the continuation from $\alpha = 1$, where $\beta = 2\alpha - 1$, one has $\beta \geq 1$ for $\alpha \geq 1$, and therefore

$$\boxed{a_{0,N,\alpha} = o(1)} \tag{871}$$

uniformly if G_N and the prefactors are uniformly controlled.

245. a_2 and the complete hierarchy

The coefficient a_2 contains no explicit C_N:

$$a_{2,N,\alpha} = \Gamma_N P_{N,\alpha}. \tag{872}$$

Thus the previous positive-gap conclusion survives only under the same renormalized coefficient hypothesis, now with the corrected P from (848):

$$\boxed{a_{2,N,\alpha} \longrightarrow D_\alpha, \qquad 0 < d_* \le D_\alpha \le D^*.} \tag{873}$$

It is important that (873) be reverified with $\widehat{\rho}_\beta$; it does not follow merely by copying the old bounded-C_N normalization.

Combining (857), (867), (870), and (873), the terminal hierarchy is

$$\boxed{\begin{aligned} a_2 &= O(1),\\ a_{3/2} &\sim -c_J \sigma^{-e_J},\\ a_1 &= c_1(\alpha,\beta)\sigma^{-\alpha} + o(\sigma^{-\alpha}),\\ a_0 &= O\big(\sigma^{\widehat{\rho}_\beta - q_\beta}\big) \quad (G_N = O(1)). \end{aligned}} \tag{874}$$

246. Comparison in the continuation scaling $\beta = 2\alpha - 1$

For the profile scaling used in the general-α continuation,

$$\beta = r_\alpha = 2\alpha - 1, \qquad 1 \le \alpha < 1 + \delta, \tag{875}$$

we have

$$\widehat{\rho}_\alpha = \frac{7(2\alpha-1)}{2(7-2\alpha)}, \qquad e_J = \frac{7\alpha}{7-2\alpha}. \tag{876}$$

The decisive exponent comparison is

$$\boxed{e_J - \alpha = \frac{2\alpha^2}{7-2\alpha} > 0} \tag{877}$$

for $\alpha < 7/2$, in particular throughout every sufficiently small interval $1 \le \alpha < 1 + \delta$. Therefore, for every fixed compact amplitude interval

$$0 < W_- \le W \le W_+ < \infty, \tag{878}$$

the negative term

$$a_{3/2} W^{3/2} \tag{879}$$

dominates the entire $a_1 W$ contribution as $\sigma \downarrow 0$. It also dominates the bounded $a_2 W^2$ term under (873), while $a_0 = o(1)$ by (871). Thus

$$\boxed{\dot{W} = -c_J W^{3/2} \sigma^{-7\alpha/(7-2\alpha)} \big(1 + o(1)\big) < 0} \tag{880}$$

uniformly on compact positive W-intervals, provided all asymptotic errors and coefficient bounds above are uniform in N and α.

Equivalently, the quartic

$$\mathcal{P}_\alpha(y;\sigma) = a_2 y^4 + a_{3/2} y^3 + a_1 y^2 + a_0, \qquad y = \sqrt{W}, \tag{881}$$

satisfies

$$\boxed{\mathcal{P}_\alpha(y;\sigma) = -c_J y^3 \sigma^{-7\alpha/(7-2\alpha)}\big(1+o(1)\big) < 0} \tag{882}$$

uniformly for $y \in [y_-, y_+] \Subset (0,\infty)$.

247. What has been proved and the remaining uniformity obligation

Theorem 29 (Conditional terminal dominance of the full finite-amplitude bracket)**.** *Assume the exact reconstruction and characteristic hypotheses of Part 38, the corrected intrinsic scaling (848), the continuation relation $\beta = 2\alpha - 1$, the tangent Mellin realization used in Parts 31–32, bounded G_N, the renormalized positive-gap estimate (873), and uniform versions of the asymptotic expansions (850), (865), and (866). Then on every compact positive amplitude interval $[W_-, W_+]$ there exists a sufficiently small terminal $\sigma_* > 0$ such that, for all sufficiently large N and all $\alpha \in [1, 1+\delta]$,*

$$\boxed{\mathscr{B}_{N,\alpha}(W;\sigma) < 0, \qquad 0 < \sigma \le \sigma_*, \quad W_- \le W \le W_+.} \tag{883}$$

The mechanism is the strict exponent gap (877): the negative geometric-drift part of $a_{3/2}$ is more singular than the complete fractional a_1 block, while a_2 remains bounded and a_0 vanishes.

The theorem is deliberately conditional on uniformity. Part 38 derived the self-consistent powers by dominant balance, but it did not yet prove the uniform error estimates in N required to replace every $\asymp$ by a common two-sided estimate on a shrinking terminal overlap. Nor has (873) yet been reproved with the corrected exponent $\widehat{\rho}_\beta$. Those are now the precise construction-level obligations; the exponent/sign competition itself is resolved.

248. Revised ledger after the complete coefficient hierarchy

The coefficient audit therefore yields

$$\boxed{\begin{array}{c}\text{(i) recompute the intrinsic profile exponent as } \widehat{\rho}_\beta = \dfrac{7\beta}{2(6-\beta)};\\ \text{(ii) the full } a_{3/2} \text{ has a negative leading term of order } \sigma^{-7(\beta+1)/(2(6-\beta))};\\ \text{(iii) } C_t + C_z = 0 \text{ exactly, and the complete tangent fractional block } a_1 \text{ is of order } \sigma^{-\alpha};\\ \text{(iv) for } \beta = 2\alpha - 1,\ e_J > \alpha, \text{ so the negative } a_{3/2} \text{ term dominates the full compact-amplitude quartic;}\\ \text{(v) prove the uniform finite-}N\text{ error bounds behind Parts 38–39 and reverify } a_2 \to D_\alpha \text{ with the corrected } P;\\ \text{(vi) retain the higher-order fixed-space energy and residual convergence obligations of Part 34.}\end{array}} \tag{884}$$

Thus the new C_N growth does not reverse the favorable upper-boundary mechanism. On the contrary, after the normalization is corrected self-consistently, the C_N^2 geometric drift produces the strongest terminal singularity in the finite-amplitude bracket. What remains is to make the dominant-balance estimates uniform in N and compatible with the full velocity/residual reconstruction.

56 Part 40: Uniform finite-N terminal estimates and the corrected $\Gamma_N P_{N,\alpha}$ law

249. Quantified hypotheses and notation

Part 38 obtained the self-consistent powers by dominant balance. We now replace each use of $\sim$ by a uniform finite-N estimate and, at the same time, re-audit the coefficient $a_{2,N,\alpha} = \Gamma_N P_{N,\alpha}$. This second calculation produces an important correction to (873).

Let

$$\beta = 2\alpha - 1, \qquad 1 \le \alpha \le 1 + \delta, \qquad 1 \le \beta \le \beta_+ < 2, \tag{885}$$

and assume

$$b_N(s) = B_N F_N(s) s^{-\beta}, \qquad 0 < B_- \le B_N \le B_+, \tag{886}$$

with

$$0 < F_- \le F_{0,N,\alpha} \le F_+, \qquad F_N(s) = F_{0,N,\alpha}(1 + \varepsilon^F_{N,\alpha}(s)), \tag{887}$$

where

$$\sup_{N \ge N_0} \sup_{\alpha \in I_\delta} |\varepsilon^F_{N,\alpha}(s)| \le \omega_F(s), \qquad \omega_F(s) \downarrow 0. \tag{888}$$

All constants below depend only on the displayed compact parameter bounds, not on N, α, or the terminal point.

Define

$$I_N(s) = \int_0^s r^{-\beta/2} F_N(r)^{1/2}\, dr, \qquad I_N^{(0)}(s) = \frac{F_{0,N,\alpha}^{1/2}}{1 - \beta/2} s^{1-\beta/2}. \tag{889}$$

Since $|\sqrt{1+z} - 1| \le C|z|$ for $|z| \le 1/2$, after decreasing s_0 if necessary,

$$\boxed{\left| \frac{I_N(s)}{I_N^{(0)}(s)} - 1 \right| \le C\, \Omega_F(s), \qquad \Omega_F(s) := \sup_{0 < r \le s} \omega_F(r).} \tag{890}$$

In particular there are uniform constants $0 < I_- < I_+$ such that

$$I_- s^{1-\beta/2} \le I_N(s) \le I_+ s^{1-\beta/2}. \tag{891}$$

250. Uniform control of the exact continuity reconstruction

Write the exact reconstruction (821) as

$$C_N = c_{0,N} + R_N^C, \qquad R_N^C := \frac{\sqrt{2B_N}}{4} (1 - \sigma)^{1/2} \sigma^{-3/4} I_N(s). \tag{892}$$

Assume $0 \le c_{0,N} \le c_0^*$. On a terminal region on which $R_N^C \ge 2c_0^*$, define

$$\varepsilon_N^C := \frac{c_{0,N}}{R_N^C}. \tag{893}$$

Then $0 \le \varepsilon_N^C \le 1/2$ and

$$C_N = R_N^C (1 + \varepsilon_N^C). \tag{894}$$

Using (890) and $(1-\sigma)^{1/2} = 1 + O(\sigma)$ gives

$$\boxed{C_N = \frac{\sqrt{2B_N F_{0,N,\alpha}}}{4(1-\beta/2)}\sigma^{-3/4}s^{1-\beta/2}(1+\theta_N^C),} \tag{895}$$

with the quantified error

$$\boxed{|\theta_N^C| \le C\big(\Omega_F(s) + \sigma + \varepsilon_N^C\big).} \tag{896}$$

Consequently, whenever the right side of (896) is at most $\varepsilon_0 < 1$, C_N satisfies uniform two-sided estimates with factors $1 \pm C\varepsilon_0$.

251. Quantified characteristic ODE and two-sided s–σ law

The exact factor in (813) satisfies

$$\frac{b_N^{1/2}}{b_N - 1} = b_N^{-1/2}(1+\theta_N^b), \qquad |\theta_N^b| \le C b_N^{-1} \le C s^{\beta}, \tag{897}$$

uniformly after decreasing s_0. Substitution of (895)–(897) into (813) yields

$$\boxed{\frac{d\sigma}{ds} = K_{N,\alpha}\sigma^{-3/4}s^{2-\beta/2}(1+\theta_N^\sigma),} \tag{898}$$

where

$$K_{N,\alpha} = \frac{\sqrt{2}}{4}\frac{\sqrt{B_N F_{0,N,\alpha}}}{(1-\beta/2)^2}, \tag{899}$$

and

$$\boxed{|\theta_N^\sigma| \le C\big(\Omega_F(s) + s^{\beta} + \sigma + \varepsilon_N^C\big).} \tag{900}$$

Set

$$\mathcal{E}_N(s) := \sup_{0<r\le s}\big(\Omega_F(r) + r^{\beta} + \sigma_N(r) + \varepsilon_N^C(r)\big). \tag{901}$$

Multiplying (898) by $\sigma^{3/4}$ and integrating gives

$$\frac{4}{7}\sigma(s)^{7/4} = \frac{K_{N,\alpha}}{3-\beta/2}s^{3-\beta/2}(1+\Theta_N(s)), \qquad |\Theta_N(s)| \le C\mathcal{E}_N(s). \tag{902}$$

Therefore, with $p_\beta = (12-2\beta)/7$ and $\mathfrak{K}_{N,\alpha}$ from (828),

$$\boxed{\sigma_N(s) = \mathfrak{K}_{N,\alpha}s^{p_\beta}\big(1+\vartheta_N^\sigma(s)\big), \qquad |\vartheta_N^\sigma(s)| \le C\mathcal{E}_N(s).} \tag{903}$$

In particular, if $C\mathcal{E}_N(s) \le 1/2$,

$$\boxed{\frac{1}{2}\mathfrak{K}_{N,\alpha}s^{p_\beta} \le \sigma_N(s) \le \frac{3}{2}\mathfrak{K}_{N,\alpha}s^{p_\beta}.} \tag{904}$$

Since B_N and $F_{0,N,\alpha}$ have uniform two-sided bounds and $\beta \in [1, \beta_+] \Subset (0,2)$, there exist $0 < \mathfrak{K}_- < \mathfrak{K}_+$ such that

$$\mathfrak{K}_- \le \mathfrak{K}_{N,\alpha} \le \mathfrak{K}_+. \tag{905}$$

Inverting (903),

$$\boxed{s_N(\sigma) = \mathfrak{K}_{N,\alpha}^{-1/p_\beta} \sigma^{1/p_\beta} \big(1 + \vartheta_N^s(\sigma)\big), \qquad |\vartheta_N^s| \le C\mathcal{E}_N.} \tag{906}$$

Equations (903)–(906) are the requested uniform two-sided finite-N replacement of the dominant balance (827)–(829).

252. Uniform P_N, C_N, and reconstruction estimates

From $P_{N,\alpha} = b_N$ at unit boundary amplitude, (886), (887), and (906) give

$$\boxed{P_{N,\alpha}(\sigma) = \widehat{p}_{N,\alpha} \sigma^{-\widehat{\rho}_\beta}(1 + \theta_N^P), \qquad \widehat{\rho}_\beta = \frac{7\beta}{2(6-\beta)},} \tag{907}$$

where

$$\widehat{p}_{N,\alpha} := B_N F_{0,N,\alpha} \mathfrak{K}_{N,\alpha}^{\beta/p_\beta}, \qquad |\theta_N^P| \le C(\mathcal{E}_N + \Omega_F), \tag{908}$$

and

$$0 < p_- \le \widehat{p}_{N,\alpha} \le p_+ < \infty. \tag{909}$$

Similarly,

$$\boxed{C_N(\sigma) = \widehat{c}_{N,\alpha} \sigma^{-q_\beta}(1 + \theta_N^{C,\sigma}), \qquad q_\beta = \frac{\beta+1}{6-\beta},} \tag{910}$$

with

$$0 < c_- \le \widehat{c}_{N,\alpha} \le c_+, \qquad |\theta_N^{C,\sigma}| \le C\mathcal{E}_N. \tag{911}$$

The exact identity (834) likewise yields

$$\boxed{\mathcal{X}C = \widehat{r}_{N,\alpha} \sigma^{-e_R}(1 + \theta_N^R), \qquad e_R = \frac{5(\beta+1)}{2(6-\beta)},} \tag{912}$$

where

$$0 < r_- \le \widehat{r}_{N,\alpha} \le r_+, \qquad |\theta_N^R| \le C\mathcal{E}_N. \tag{913}$$

Finally the geometric drift satisfies

$$\boxed{J_{N,\alpha} = -\widehat{j}_{N,\alpha} \sigma^{-e_J}(1 + \theta_N^J), \qquad e_J = \frac{7(\beta+1)}{2(6-\beta)},} \tag{914}$$

with

$$0 < j_- \le \widehat{j}_{N,\alpha} \le j_+, \qquad |\theta_N^J| \le C\mathcal{E}_N. \tag{915}$$

Thus for $C\mathcal{E}_N \le 1/2$,

$$\boxed{a_{3/2,N,\alpha} \le -\frac{j_-}{2} \sigma^{-e_J} + C_R \sigma^{-e_R}.} \tag{916}$$

Since $e_J - e_R = q_\beta \geq q_- > 0$, there is a uniform $\sigma_1 > 0$ such that

$$\boxed{a_{3/2,N,\alpha} \leq -\frac{j_-}{4}\sigma^{-e_J} < 0, \qquad 0 < \sigma \leq \sigma_1.} \tag{917}$$

This upgrades the sign part of Part 39 to a genuine two-sided finite-N estimate, conditional only on the quantified profile modulus (888) and the terminal reconstruction bootstrap $\mathcal{E}_N \ll 1$.

253. Exact re-audit of $\Gamma_N P_{N,\alpha}$

We now address the second requested point. The earlier statement (873) cannot be retained unchanged. By definition of the Euler logarithmic sensitivity,

$$S_{N,k} = s\,\partial_s \log \Delta_{N,k}. \tag{918}$$

Up to the fixed orientation sign between s and ξ, the positive terminal coefficient is therefore

$$\boxed{\Gamma_N(s) = \frac{g_{N,\alpha}(s)}{s}, \qquad g_{N,\alpha}(s) := |S_{N,k(N)}(s)|.} \tag{919}$$

The exact Lambert theorem gives the unconditional upper estimate

$$0 \leq g_{N,\alpha}(s) \leq 2^{-k(N)}|S_{N,0}(s)|. \tag{920}$$

It does *not* by itself give a positive lower bound comparable to $2^{-k(N)}$; such a lower bound is an additional nondegeneracy statement.

Using (906)–(908),

$$\frac{1}{s_N(\sigma)}P_{N,\alpha}(\sigma) = \widehat{d}_{N,\alpha}\sigma^{-\left(1/p_\beta + \widehat{\rho}_\beta\right)}(1 + \theta_N^D), \tag{921}$$

with $0 < d_- \leq \widehat{d}_{N,\alpha} \leq d_+$ and $|\theta_N^D| \leq C\mathcal{E}_N$. But

$$\boxed{\frac{1}{p_\beta} + \widehat{\rho}_\beta = \frac{7(\beta+1)}{2(6-\beta)} = e_J.} \tag{922}$$

Consequently the correct law is

$$\boxed{a_{2,N,\alpha}(\sigma) = \Gamma_N P_{N,\alpha} = g_{N,\alpha}(\sigma)\widehat{d}_{N,\alpha}\sigma^{-e_J}(1 + \theta_N^D).} \tag{923}$$

This has the *same terminal power* as the negative geometric-drift part of $a_{3/2}$. In particular, for fixed finite N with nonzero sensitivity, a_2 is not $O(1)$ as $\sigma \downarrow 0$.

Equation (923) also shows why the asserted convergence $\Gamma_N P_{N,\alpha} \to D_\alpha > 0$ cannot hold uniformly on a fixed σ-annulus merely from the increasing-depth Lambert theorem. If $k(N) \to \infty$ and $S_{N,0}$ is uniformly bounded, then (920) gives $g_{N,\alpha} \to 0$ for every fixed $\sigma > 0$, and hence

$$a_{2,N,\alpha}(\sigma) \longrightarrow 0 \tag{924}$$

on compact annuli where the remaining prefactors are bounded. Conversely, for fixed N and $g_{N,\alpha} > 0$, (923) diverges as $\sigma \downarrow 0$. Thus the limits $N \to \infty$ and $\sigma \downarrow 0$ do not commute for this coefficient.

254. The only consistent positive-gap limit is a coupled overlap limit

Suppose, in addition to the upper Lambert estimate, that on the selected coupled overlap there is a positive sensitivity scale $g_N^* > 0$ and a relative modulus

$$g_{N,\alpha}(\sigma) = g_N^*\,\mathfrak{g}_\alpha(1+\theta_N^g), \qquad 0 < g_- \le \mathfrak{g}_\alpha \le g_+, \qquad |\theta_N^g| \le \omega_g(N), \tag{925}$$

with $\omega_g(N) \to 0$. Define the terminal matching scale

$$\boxed{\sigma_N^* := (g_N^*)^{1/e_J}.} \tag{926}$$

For the rescaled coordinate $\zeta = \sigma/\sigma_N^*$, (923) becomes

$$\boxed{a_{2,N,\alpha}(\sigma_N^*\zeta) = D_{N,\alpha}\,\zeta^{-e_J}\big(1 + O(\mathcal{E}_N + \omega_g(N))\big),} \tag{927}$$

where

$$D_{N,\alpha} := \mathfrak{g}_\alpha \widehat{d}_{N,\alpha}. \tag{928}$$

Therefore a nonzero finite limit exists if and only if the prefactors $D_{N,\alpha}$ converge. If

$$\sup_{\alpha\in I_\delta} |D_{N,\alpha} - D_\alpha| \to 0, \qquad 0 < d_* \le D_\alpha \le D^*, \tag{929}$$

then on every compact $0 < \zeta_- \le \zeta \le \zeta_+ < \infty$,

$$\boxed{a_{2,N,\alpha}(\sigma_N^*\zeta) \longrightarrow D_\alpha \zeta^{-e_J}} \tag{930}$$

uniformly in α and ζ. Notice carefully that the limit is $D_\alpha\zeta^{-e_J}$, not the σ-independent constant D_α of (873). The old statement is recovered only on the single normalized slice $\zeta = 1$.

255. Corrected finite-N quartic competition

The correction (923) changes the leading quartic hierarchy of Part 39. Both a_2W^2 and the negative geometric part of $a_{3/2}W^{3/2}$ have order σ^{-e_J}. Using (914), (923), and the already proved exponent gap $e_J > \alpha$, while $a_0 = o(\sigma^{-e_J})$, gives

$$\boxed{\dot{W} = \sigma^{-e_J} W^{3/2}\left[g_{N,\alpha}(\sigma)\widehat{d}_{N,\alpha}\sqrt{W} - \widehat{j}_{N,\alpha} + \mathfrak{r}_{N,\alpha}(\sigma,W)\right],} \tag{931}$$

where, uniformly for $W \in [W_-, W_+] \Subset (0,\infty)$,

$$\boxed{|\mathfrak{r}_{N,\alpha}(\sigma,W)| \le C\left(\mathcal{E}_N + \sigma^{q_-} + \sigma^{e_J-\alpha} + \sigma^{e_J}|a_{0,N,\alpha}|\right).} \tag{932}$$

Thus the negative upper sign is not a consequence of exponents alone. The correct finite-amplitude condition is the coefficient inequality

$$\boxed{g_{N,\alpha}(\sigma)\widehat{d}_{N,\alpha}\sqrt{W^*} < \widehat{j}_{N,\alpha}.} \tag{933}$$

A convenient uniform sufficient condition is: for some $\eta_* > 0$,

$$\boxed{\sup_{N,\alpha,\sigma\le\sigma_*} g_{N,\alpha}(\sigma)\widehat{d}_{N,\alpha}\sqrt{W^*} \le j_- - 2\eta_*.} \tag{934}$$

If in addition the right side of (932) is at most η_*, then

$$\boxed{\dot{W}\big|_{W=W^*} \le -\eta_*\sigma^{-e_J}(W^*)^{3/2} < 0.} \tag{935}$$

This is a genuine uniform finite-N inward-pointing theorem on any overlap where the quantified errors close.

In the increasing-depth regime, (920) makes (934) especially natural: if $S_{N,0}$ is uniformly bounded and $k(N) \to \infty$, then the positive quadratic coefficient is suppressed by the exact Lambert sensitivity contraction, whereas $\widehat{j}_{N,\alpha}$ has a positive geometric lower bound. No false $a_2 = O(1)$ assumption is required.

256. Uniform finite-N theorem and precise status

Theorem 30 (Quantified terminal estimates and corrected invariant-boundary criterion)**.** *Assume (885)–(888), the exact characteristic equations (812), the exact continuity reconstruction (892), uniform bounds on $c_{0,N}$, and a terminal overlap on which $\mathcal{E}_N \to 0$ uniformly. Then the two-sided estimates (903)–(917) hold uniformly in N and α. In particular,*

$$P_{N,\alpha}(\sigma) = \widehat{p}_{N,\alpha}\sigma^{-7(2\alpha-1)/(2(7-2\alpha))}(1 + O(\mathcal{E}_N)), \tag{936}$$

with uniform positive upper and lower bounds on $\widehat{p}_{N,\alpha}$, and $a_{3/2,N,\alpha}$ has the strict negative estimate (917).

Moreover, the exact Lambert identity implies (923), not the old bounded law (873). Hence $\Gamma_N P_{N,\alpha} \to D_\alpha > 0$ can only be interpreted on a coupled terminal matching scale. Under the additional relative nondegeneracy/convergence assumptions (925) and (929), the coupled limit is (930). Finally, if the strict coefficient margin (934) and the error bound following (934) hold, then the finite-N vector field points strictly inward at $W = W^*$ as in (935).

The theorem separates what follows from the exact reconstruction from what still requires a Lambert lower/nondegeneracy input. The contraction theorem (442) supplies the needed upper sensitivity bound, but it cannot be converted into the relative lower estimate (925) without additional information about the initial sensitivity and the product of the exact multipliers $A(\Delta_{N,j})$.

257. Revised ledger after the uniform finite-N audit

The requested upgrade therefore closes the dominant-balance error estimates but corrects the proposed a_2 limit:

(i) the laws $s_N \leftrightarrow \sigma_N$, P_N, C_N, $\mathcal{X}C$, and J_N admit uniform two-sided estimates with error $O(\mathcal{E}_N)$;
(ii) $P_{N,\alpha} \asymp \sigma^{-7(2\alpha-1)/(2(7-2\alpha))}$ is rigorous on the quantified overlap;
(iii) $\Gamma_N P_{N,\alpha} = g_{N,\alpha}\widehat{d}_{N,\alpha}\sigma^{-e_J}(1 + O(\mathcal{E}_N))$, so (873) is not a uniform fixed-σ limit;
(iv) the positive $a_2 W^2$ and negative geometric $a_{3/2}W^{3/2}$ have the same terminal power;
(v) the actual upper-boundary criterion is the coefficient margin (934), with finite-N error controlled by (932);
(vi) to obtain a nonzero coupled limit D_α one must prove the relative Lambert sensitivity statement (925) and prefactor convergence (929);
(vii) the fixed-space energy and residual convergence obligations of Part 34 remain.

(937)

Thus the finite-N asymptotic hierarchy is now quantitatively controlled, but the calculation also shows that the previously requested conclusion $\Gamma_N P_{N,\alpha} \to D_\alpha$ cannot be proved in its old σ-independent form. The mathematically consistent replacement is the coupled law (930) together with the explicit invariant-boundary margin (934)–(935).

57 Part 41: Two-sided Lambert sensitivity and closure of the finite-N upper boundary

258. Exact product representation and the fixed-point multiplier

We now close the Lambert nondegeneracy issue left open in Part 40 as far as is needed for the upper invariant boundary. Recall the exact recurrence

$$\Delta_{N,j+1} = \mathcal{F}(\Delta_{N,j}), \qquad S_{N,j+1} = A(\Delta_{N,j}) S_{N,j}, \tag{938}$$

where

$$A(\Delta) = \frac{\Delta(1-q)}{q^2}, \qquad q = \mathcal{F}(\Delta), \qquad 0 < A(\Delta) < \frac{1}{2}. \tag{939}$$

Therefore, for every $k > j_c$,

$$\boxed{S_{N,k} = S_{N,j_c} \prod_{j=j_c}^{k-1} A(\Delta_{N,j}).} \tag{940}$$

At the positive attracting fixed point Δ_*, $q = \Delta_*$ and hence

$$\boxed{A_* := A(\Delta_*) = \frac{1-\Delta_*}{\Delta_*} \in (0, 1/2).} \tag{941}$$

Numerically $A_* \approx 0.2550$, but no numerical value is needed below.

259. Uniform tail product lemma

Let $J \Subset (0,1)$ be a compact trapping interval containing Δ_*. Since A is positive and C^1 on J, there are constants

$$0 < A_- \le A(\Delta) \le A_+ < \frac{1}{2}, \qquad |\log A(\Delta) - \log A_*| \le L_A |\Delta - \Delta_*|, \tag{942}$$

for $\Delta \in J$. Assume that after the capture index $j_c = j_c(N,\alpha)$,

$$\Delta_{N,j} \in J, \qquad |\Delta_{N,j} - \Delta_*| \le C_* \rho_*^{j-j_c} \varepsilon_{N,\alpha}^c, \qquad j \ge j_c, \tag{943}$$

where $0 < \rho_* < 1$ and $0 \le \varepsilon_{N,\alpha}^c \le \varepsilon_c$. Then

$$\left| \sum_{j=j_c}^{k-1} \log \frac{A(\Delta_{N,j})}{A_*} \right| \le \frac{L_A C_*}{1-\rho_*} \varepsilon_{N,\alpha}^c =: C_A \varepsilon_{N,\alpha}^c. \tag{944}$$

Exponentiation yields the uniform two-sided product estimate

$$\boxed{e^{-C_A\varepsilon^c_{N,\alpha}} \le \frac{\prod_{j=j_c}^{k-1} A(\Delta_{N,j})}{A_*^{k-j_c}} \le e^{C_A\varepsilon^c_{N,\alpha}}.} \tag{945}$$

In particular,

$$\boxed{|S_{N,j_c}|A_*^{k-j_c}e^{-C_A\varepsilon^c_{N,\alpha}} \le |S_{N,k}| \le |S_{N,j_c}|A_*^{k-j_c}e^{C_A\varepsilon^c_{N,\alpha}}.} \tag{946}$$

This is the required two-sided finite-depth sensitivity estimate. Notice that it does not manufacture a lower bound independent of the entrance sensitivity $|S_{N,j_c}|$: if that quantity vanishes, the whole sensitivity vanishes.

If, more strongly,

$$\sup_{\alpha\in I_\delta} \varepsilon^c_{N,\alpha} \longrightarrow 0, \tag{947}$$

then (945) gives the relative asymptotic

$$\boxed{S_{N,k} = S_{N,j_c}A_*^{k-j_c}\left(1+O(\varepsilon^c_{N,\alpha})\right)} \tag{948}$$

uniformly in α and in all $k \ge j_c$.

260. What is needed for a positive coupled D_α limit

Define the exact entrance scale

$$g_N^* := s_N^* \left|\partial_s \log \Delta_{N,j_c}\right| A_*^{k(N)-j_c}, \tag{949}$$

with the harmless normalization s_N^* chosen consistently with the overlap. Equivalently, at the Euler-sensitivity level one may absorb $|S_{N,j_c}|$ into g_N^*. If the entrance sensitivity has the uniform relative factorization

$$|S_{N,j_c(N,\alpha)}| = s_N^{\mathrm{ent}}\,\mathfrak{s}_\alpha(1+\vartheta_{N,\alpha}), \qquad 0 < s_- \le \mathfrak{s}_\alpha \le s_+, \qquad \sup_\alpha |\vartheta_{N,\alpha}| \to 0, \tag{950}$$

then (948) proves, rather than assumes, the Part 40 form

$$\boxed{g_{N,\alpha} = g_N^*\,\mathfrak{g}_\alpha\left(1+O(\omega_g(N))\right), \qquad \omega_g(N) = \sup_\alpha\big(|\vartheta_{N,\alpha}| + \varepsilon^c_{N,\alpha}\big),} \tag{951}$$

with $0 < g_- \le \mathfrak{g}_\alpha \le g_+$. Thus the only genuinely independent nondegeneracy datum is the entrance sensitivity (950); the entire long Lambert tail contributes the explicit factor $A_*^{k-j_c}$ and a quantified relative error.

Combining (951) with the already proved prefactor convergence of Part 40 gives, on $\sigma = \sigma_N^*\zeta$,

$$\boxed{\Gamma_N P_{N,\alpha} = D_\alpha\zeta^{-e_J} + O\left(\omega_g(N) + \mathcal{E}_N + \sup_\alpha |D_{N,\alpha} - D_\alpha|\right)\zeta^{-e_J}.} \tag{952}$$

Hence the coupled D_α law is fully quantified once (950) and the prefactor convergence are verified. No lower estimate of order 2^{-k} is correct: the true tail rate is $A_*^{k-j_c}$ after capture.

261. Upper-boundary closure does not require the positive D_α limit

For the upper invariant boundary, however, the relative lower estimate (950) is unnecessary. The exact global contraction already gives

$$g_{N,\alpha} \le M_S 2^{-k(N)}, \tag{953}$$

whenever $|S_{N,0}| \le M_S$. Let

$$\widehat{d}_{N,\alpha} \le d_+, \qquad \widehat{j}_{N,\alpha} \ge j_- > 0 \tag{954}$$

be the uniform bounds proved in Part 40. Fix any finite $W^* > 0$. Choose N_1 so large that

$$\boxed{M_S d_+ 2^{-k(N)} \sqrt{W^*} \le \frac{j_-}{4}, \qquad N \ge N_1.} \tag{955}$$

Next choose the terminal overlap and $N_2 \ge N_1$ so that the remainder in (932) satisfies

$$\boxed{\sup_{\alpha,\, W \in [W_-, W_+]} |\mathfrak{r}_{N,\alpha}(\sigma, W)| \le \frac{j_-}{4}, \qquad N \ge N_2.} \tag{956}$$

Substitution into the exact leading field (931) gives at $W = W^*$

$$\begin{aligned} \dot{W} &\le \sigma^{-e_J} (W^*)^{3/2} \left(\frac{j_-}{4} - j_- + \frac{j_-}{4} \right) \\ &= -\frac{j_-}{2} \sigma^{-e_J} (W^*)^{3/2} < 0. \end{aligned} \tag{957}$$

Therefore

$$\boxed{\dot{W}\big|_{W=W^*} \le -\frac{j_-}{2} \sigma^{-e_J} (W^*)^{3/2} < 0} \tag{958}$$

uniformly for $N \ge N_2$, $\alpha \in I_\delta$, and the quantified terminal overlap.

Theorem 31 (Uniform finite-N upper invariant boundary)**.** *Assume the hypotheses of the quantified Part 40 theorem, the uniform initial sensitivity bound $|S_{N,0}| \le M_S$, $k(N) \to \infty$, the positive geometric coefficient bound $\widehat{j}_{N,\alpha} \ge j_- > 0$, and the remainder closure (956). Then every fixed finite $W^* > 0$ lying in the admissible compact amplitude window is, for all sufficiently large N, a strict upper invariant boundary for the normalized terminal field. More precisely, (958) holds uniformly in α and in the terminal overlap.*

This theorem is stronger than the conditional criterion (934): the positive quadratic coefficient is suppressed by increasing Lambert depth, whereas the negative geometric coefficient retains a uniform positive prefactor. The proof uses only the one-sided contraction for the sign closure. The two-sided product estimate (946) is needed only when one also wants a nonzero coupled coefficient limit D_α.

262. Revised ledger after the exact multiplier-product audit

The terminal invariant-boundary ledger is now

$$\boxed{\begin{array}{c}\text{(i) the exact Lambert tail satisfies the two-sided estimate (946);}\\ \text{(ii) under shrinking capture error it has the relative asymptotic (948);}\\ \text{(iii) a positive coupled } D_\alpha \text{ limit reduces precisely to the entrance sensitivity nondegeneracy (950) and prefactor convergence;}\\ \text{(iv) the strict finite-}N\text{ upper invariant-boundary sign does not require (950): the global contraction, } k(N)\to\infty,\ \text{and the geometric lower bound imply (958);}\\ \text{(v) the remaining global obligations are the fixed-space higher-order energy estimates and vanishing scalar/full momentum residuals for the strong } N\to\infty \text{ passage.}\end{array}} \tag{959}$$

Thus the upper-sign problem is closed on the quantified terminal overlap, subject to the Part 40 reconstruction-error hypotheses. What remains is no longer an invariant-boundary sign problem; it is the compactness and residual passage needed to convert the finite-N terminal construction into a limiting solution statement.

58 Part 42: Reconstruction-specific fixed-space energy audit and the global square-root obstruction

263. The precise question left by Part 41

Part 41 closed the finite-N upper invariant-boundary sign on the quantified terminal overlap. The remaining compactness argument in Part 24, however, requires the global fixed-space estimates

$$\sup_{N,\alpha}\|b_{N,\alpha}\|_{L^2(0,T;H^{1+\delta+\varepsilon})}\le C_T, \tag{960}$$

$$\sup_{N,\alpha}\|\partial_t b_{N,\alpha}\|_{L^2(0,T;H^{-1-\delta-\varepsilon})}\le C_T \tag{961}$$

for every fixed $T<T_*$. The stronger sufficient velocity package proposed in Part 34 was

$$\sup_N\|u_N\|_{L^\infty(0,T;H^{s_u})}+\sup_N\|u_N\|_{L^2(0,T;H^{s_u+\alpha})}\le C_T,\qquad s_u>\frac{3}{2}. \tag{962}$$

We now test (962) against the *actual* square-root reconstruction rather than treating it as an abstract higher-order estimate.

264. Exact component formulas and a necessary local integrability condition

The active reconstruction is

$$b=u_xu_y,\qquad A=u_y,\qquad C=u_z,\qquad A^2=-2\chi(x)b, \tag{963}$$

so on the selected sign branch

$$\boxed{u_y=\sqrt{-2\chi b},\qquad u_x=\sqrt{-\frac{b}{2\chi}},\qquad u_z=C.} \tag{964}$$

Let x_* be a simple zero of the periodic geometric coefficient,

$$\chi(x_*) = 0, \qquad \chi'(x_*) \neq 0. \tag{965}$$

Then

$$\chi(x) = \chi'(x_*)(x - x_*) + O((x - x_*)^2). \tag{966}$$

Suppose that, at some fixed (y, z, t), the scalar has a nonzero trace

$$b(x, y, z, t) \longrightarrow b_* > 0 \qquad (x \to x_* \text{ through the selected branch}). \tag{967}$$

Substitution into (964) gives

$$|u_x(x, y, z, t)|^2 = \frac{b(x, y, z, t)}{2|\chi(x)|} \sim \frac{b_*}{2|\chi'(x_*)|} \frac{1}{|x - x_*|}. \tag{968}$$

Consequently

$$\boxed{\int_{x_*-\eta}^{x_*} |u_x|^2 \, dx = +\infty} \tag{969}$$

for every $\eta > 0$ on which the branch is defined. Hence a nonzero scalar trace at a simple zero of χ is incompatible even with $u_x \in L^2$, and therefore a fortiori with (962).

Theorem 32 (Necessary vanishing for finite-energy square-root reconstruction)**.** *Let χ have a simple zero at x_* and let $b \geq 0$ on a one-sided branch. If u_x is reconstructed by (964) and $u_x \in L^2_{\mathrm{loc}}$ near x_*, then*

$$\boxed{\frac{b}{|\chi|} \in L^1_{\mathrm{loc}}.} \tag{970}$$

In particular a continuous b must satisfy $b(x_*, y, z, t) = 0$. More generally, if

$$b(x, y, z, t) = O(|\chi(x)|^{1+\vartheta}) \qquad \text{for some } \vartheta > 0, \tag{971}$$

then $u_x^2 = O(|\chi|^{\vartheta})$ and the zeroth-order kinetic-energy obstruction is removed.

265. Higher Sobolev compatibility at the geometric zero

The obstruction becomes stronger at higher derivative order. Write locally $r = |x - x_*|$ and suppose

$$b = O(r^m), \qquad m > 0. \tag{972}$$

Then (964) gives

$$u_x = O(r^{(m-1)/2}), \qquad u_y = O(r^{(m+1)/2}). \tag{973}$$

For an integer $j \geq 0$,

$$\partial_x^j u_x = O(r^{(m-1)/2-j}). \tag{974}$$

The one-dimensional transverse integral of its square is finite only if

$$m - 1 - 2j > -1, \quad \text{i.e.} \quad \boxed{m > 2j.} \tag{975}$$

Thus H^j control through a simple zero requires increasingly high vanishing of b. For non-integer $s_u > 3/2$, the corresponding fractional compatibility is at least as restrictive as the integer conditions below s_u; in particular, a scalar that stays bounded below by a positive constant at the geometric zero cannot generate a global H^{s_u} velocity through (964).

266. Local tame estimates away from the zero set

The failure at $\chi = 0$ does not prevent a rigorous fixed-space estimate on a preterminal chart separated from the zero set. Let $\Omega_T \Subset \mathbb{T}^3$ be a chart on which, uniformly in N and α,

$$|\chi| \geq c_{\chi,T} > 0, \qquad b_{N,\alpha} \geq c_{b,T} > 0, \tag{976}$$

and suppose for an integer $m > 3/2$

$$\sup_{N,\alpha} \big(\|b_{N,\alpha}\|_{H^m(\Omega_T)} + \|C_{N,\alpha}\|_{H^m(\Omega_T)} \big) \leq M_T. \tag{977}$$

Since H^m is an algebra and smooth composition is tame on sets separated from zero, the maps

$$b \mapsto (-2\chi b)^{1/2}, \qquad b \mapsto \left(-\frac{b}{2\chi} \right)^{1/2} \tag{978}$$

are locally Lipschitz from H^m to H^m. Therefore

$$\boxed{\|u_{x,N}\|_{H^m(\Omega_T)} + \|u_{y,N}\|_{H^m(\Omega_T)} + \|u_{z,N}\|_{H^m(\Omega_T)} \leq C_{T,m}(1 + M_T).} \tag{979}$$

The same argument at order $m + \alpha$ gives

$$\boxed{\|u_N\|_{H^{m+\alpha}(\Omega_T)} \leq C_{T,m,\alpha}\big(1 + \|b_N\|_{H^{m+\alpha}(\Omega_T)} + \|C_N\|_{H^{m+\alpha}(\Omega_T)}\big).} \tag{980}$$

Thus the reconstruction itself is tame away from the geometric zero. The problem is not an algebraic derivative loss there; it is the denominator $\chi^{-1/2}$ at the global periodic zero set.

267. Higher-order velocity energy inequality on a regular reconstruction chart

For completeness, suppose the reconstructed velocity actually satisfies the finite-N hyperdissipative momentum equation on such a regular chart (or on a global periodic reconstruction for which the compatibility conditions remove the zero-set singularity). Applying Λ^{s_u}, pairing with $\Lambda^{s_u} u_N$, and using the standard commutator estimate gives

$$\frac{1}{2}\frac{d}{dt}\|u_N\|_{H^{s_u}}^2 + \nu \|u_N\|_{H^{s_u+\alpha}}^2 \leq C\|\nabla u_N\|_{L^\infty}\|u_N\|_{H^{s_u}}^2 + \|f_N\|_{H^{s_u-\alpha}}\|u_N\|_{H^{s_u+\alpha}}. \tag{981}$$

Young's inequality yields

$$\frac{d}{dt}\|u_N\|_{H^{s_u}}^2 + \nu \|u_N\|_{H^{s_u+\alpha}}^2 \leq C\|\nabla u_N\|_{L^\infty}\|u_N\|_{H^{s_u}}^2 + C_\nu \|f_N\|_{H^{s_u-\alpha}}^2. \tag{982}$$

Hence, if

$$\sup_N \|u_N(0)\|_{H^{s_u}} \leq M_0, \qquad \sup_N \|f_N\|_{L^2(0,T;H^{s_u-\alpha})} \leq M_f, \tag{983}$$

and the reconstruction supplies the uniform Lipschitz integral

$$\boxed{\sup_N \int_0^T \|\nabla u_N(t)\|_{L^\infty}\, dt \leq L_T,} \tag{984}$$

then Gronwall gives

$$\boxed{\sup_N \|u_N\|^2_{L^\infty(0,T;H^{s_u})} + \nu \sup_N \|u_N\|^2_{L^2(0,T;H^{s_u+\alpha})} \le C(M_0, M_f, L_T, T).} \tag{985}$$

This proves the sufficient package (962) *once* (984) and global reconstruction regularity are established. For $1 < \alpha < 1 + \delta < 5/4$, (984) cannot be deleted from the argument by appealing to a general global hyperdissipative regularity theorem; doing so would make the proof circular.

268. Consequence for the scalar fixed-space estimate

Under (985), choose

$$s_u > \frac{3}{2}, \qquad s_u \ge 1 + \delta + \varepsilon. \tag{986}$$

The algebra property gives

$$\|b_N\|_{H^{s_u}} = \|u_{x,N} u_{y,N}\|_{H^{s_u}} \le C \|u_{x,N}\|_{H^{s_u}} \|u_{y,N}\|_{H^{s_u}}, \tag{987}$$

so

$$\boxed{\sup_N \|b_N\|_{L^2(0,T;H^{1+\delta+\varepsilon})} \le C_T.} \tag{988}$$

Thus (960) follows rigorously from the reconstruction-compatible velocity estimate (985). The time derivative follows from the exact scalar equation provided its retained source/reconstruction block is uniformly bounded in $L^2 H^{-1-\delta-\varepsilon}$ and the exact residual has the same boundedness:

$$\boxed{\sup_N \|\partial_t b_N\|_{L^2(0,T;H^{-1-\delta-\varepsilon})} \le C_T.} \tag{989}$$

The stronger convergence requirement $\mathcal{R}_N \to 0$ remains a separate residual problem and is not implied by (989).

269. The terminal scaling fails the necessary global compatibility test

The terminal construction used in Parts 38–41 has, along the selected characteristic,

$$b_N(\sigma) \asymp \sigma^{-\widehat{\rho}_\alpha}, \qquad \widehat{\rho}_\alpha = \frac{7(2\alpha - 1)}{2(7 - 2\alpha)} > 0, \qquad \sigma = \chi^2 \downarrow 0. \tag{990}$$

Therefore

$$b_N \asymp |\chi|^{-2\widehat{\rho}_\alpha}. \tag{991}$$

Substitution into (964) gives

$$|u_{x,N}|^2 \asymp |\chi|^{-1-2\widehat{\rho}_\alpha}, \tag{992}$$

and hence

$$\boxed{\int |u_{x,N}|^2 \, dx = +\infty} \tag{993}$$

if this characteristic asymptotic is interpreted as an Eulerian profile on a full one-sided neighborhood of a simple zero of χ. The divergence is stronger than the logarithmic obstruction (969).

This does *not* say that a single characteristic value by itself has infinite energy: a characteristic is measure zero. It says that the present manuscript has not supplied the transverse localization/concentration law needed to turn the characteristic amplification into a globally finite-energy periodic velocity. One cannot infer (962), (960), or even a global L^2 velocity merely from the one-characteristic asymptotic (990).

270. Fixed-space theorem and obstruction verdict

Theorem 33 (Reconstruction-specific fixed-space criterion)**.** *Fix $T < T_*$. Suppose the finite-N reconstruction is extended globally on $\mathbb{T}^3$ so that the square-root zero set satisfies the compatibility needed for $u_N \in H^{s_u}$, with $s_u > 3/2$ and $s_u \geq 1 + \delta + \varepsilon$. Assume the uniform initial/forcing bounds (983) and the uniform reconstruction Lipschitz integral (984). Then the higher-order energy estimate (985) holds, and consequently the scalar fixed-space estimate (988) holds. If the exact scalar source and residual also satisfy the uniform negative-Sobolev bound stated before (989), then (989) holds and hence the compactness hypotheses (460)–(461) are proved.*

Theorem 34 (Obstruction for the present unlocalized square-root terminal profile)**.** *For the current reconstruction (963)–(964) with a simple periodic zero of χ, a positive nonvanishing scalar trace at that zero is incompatible with finite kinetic energy. The terminal law (990), if extended with its displayed power to a one-sided Eulerian neighborhood, violates the necessary condition (970) and gives (993). Therefore the present manuscript does not prove the global fixed-space estimates (460)–(461) from the terminal characteristic bounds alone.*

The requested higher-order estimate has therefore been resolved in the strongest form supported by the reconstruction: the exact energy estimate and its tame reconstruction closure are proved under explicit global compatibility, and the current square-root terminal ansatz is shown to fail that compatibility if interpreted without transverse localization.

271. Revised ledger

The global ledger after this audit is

(i) the finite-N upper invariant boundary remains closed on the quantified terminal overlap;
(ii) away from $\chi = 0$, the reconstruction is tame in fixed Sobolev spaces, (979)–(980);
(iii) under global compatibility and the Lipschitz integral (984), the higher-order energy estimate (985) proves the scalar fixed-space bound (988);
(iv) the unlocalized terminal square-root law fails the global finite-energy compatibility test (993);
(v) a global proof must therefore add and quantify a transverse localization/concentration mechanism, or replace the square-root reconstruction near the zero set, before invoking Aubin–Lions;
(vi) after that repair, the remaining analytic task is the vanishing scalar/full momentum residual.

(994)

59 Part 43: Transverse concentration width and the fixed-space scaling obstruction

272. Localization model and exact Sobolev scaling

Part 42 showed that the characteristic terminal law cannot be extended with its full strength across a fixed-width Eulerian neighborhood of the simple zero of χ. We now test whether a shrinking transverse concentration width can repair that obstruction.

Let $d_\perp \in \{1,2,3\}$ denote the number of spatial directions in which a terminal core is localized, let $\ell_N(\sigma) \downarrow 0$, and let $\Phi \in C_c^\infty(\mathbb{R}^{d_\perp})$ be nonzero. For an amplitude $M_N(\sigma)$ define the model concentrated field

$$f_{N,\sigma}(x_\perp) = M_N(\sigma)\Phi\left(\frac{x_\perp - x_{*,\perp}}{\ell_N(\sigma)}\right). \tag{995}$$

A direct Fourier change of variables gives, for every $r \geq 0$,

$$\|f_{N,\sigma}\|^2_{\dot{H}^r(\mathbb{R}^{d_\perp})} = M_N(\sigma)^2 \ell_N(\sigma)^{d_\perp - 2r} \|\Phi\|^2_{\dot{H}^r}, \tag{996}$$

and

$$\|f_{N,\sigma}\|^2_{L^2} = M_N(\sigma)^2 \ell_N(\sigma)^{d_\perp} \|\Phi\|_2^2. \tag{997}$$

Thus the fixed-shape localization cost is exact, not merely an upper bound. The same powers hold on a torus chart for sufficiently small ℓ_N, up to uniform multiplicative constants.

Write

$$\ell_N(\sigma) \asymp \sigma^\lambda, \qquad \lambda > 0, \tag{998}$$

and suppose

$$M_N(\sigma) \asymp \sigma^{-a}, \qquad a > 0. \tag{999}$$

Then

$$\|f_{N,\sigma}\|^2_{\dot{H}^r} \asymp \sigma^{-2a + \lambda(d_\perp - 2r)}. \tag{1000}$$

Consequently a necessary and sufficient power condition for this model norm to remain bounded is

$$\boxed{\lambda(d_\perp - 2r) \geq 2a.} \tag{1001}$$

For $r = 0$ this becomes

$$\boxed{\lambda \geq \frac{2a}{d_\perp}.} \tag{1002}$$

If $r \geq d_\perp/2$ and $a > 0$, however, the left side of (1001) is nonpositive for every shrinking width $\lambda > 0$, whereas the right side is positive. Therefore no shrinking fixed-shape localization can uniformly bound such an H^r norm.

273. Terminal amplitudes of the reconstructed velocity

Retain the corrected characteristic scaling from Parts 38–42,

$$b_N(\sigma) \asymp \sigma^{-\widehat{\rho}_\alpha}, \qquad \widehat{\rho}_\alpha = \frac{7(2\alpha-1)}{2(7-2\alpha)}, \tag{1003}$$

and on the selected square-root branch $|\chi| = \sigma^{1/2}$. The two algebraically reconstructed components therefore have amplitudes

$$|u_{x,N}| \asymp \sigma^{-a_x}, \qquad a_x = \frac{1}{4} + \frac{\widehat{\rho}_\alpha}{2}, \tag{1004}$$

$$|u_{y,N}| \asymp \sigma^{-a_y}, \qquad a_y = \frac{\widehat{\rho}_\alpha}{2} - \frac{1}{4}, \tag{1005}$$

where a_y may have either sign depending on the parameter range. The continuity reconstruction obtained in Part 38 gives

$$|u_{z,N}| = |C_N| \asymp \sigma^{-a_z}, \qquad a_z = \frac{\alpha}{7-2\alpha}, \tag{1006}$$

because $q_\beta = (\beta+1)/(6-\beta)$ and $\beta = 2\alpha - 1$.

For the dominant u_x component, (997) yields

$$\|u_{x,N}\|_2^2 \asymp \sigma^{-2a_x + \lambda d_\perp}. \tag{1007}$$

Hence kinetic-energy control of this component requires

$$\boxed{\lambda \geq \lambda_{L^2,x} := \frac{2a_x}{d_\perp} = \frac{\widehat{\rho}_\alpha + 1/2}{d_\perp}.} \tag{1008}$$

The C_N component requires

$$\boxed{\lambda \geq \lambda_{L^2,z} := \frac{2\alpha}{d_\perp(7-2\alpha)}.} \tag{1009}$$

Thus, at the kinetic-energy level alone, concentration can in principle compensate an amplifying pointwise velocity: one chooses

$$\lambda \geq \max\{\lambda_{L^2,x}, \lambda_{L^2,z}, \lambda_{L^2,y}\}, \tag{1010}$$

with the u_y condition omitted whenever $a_y \leq 0$.

274. Why shrinking width cannot prove the required velocity fixed-space estimate

The fixed-space argument of Part 42 requires $s_u > 3/2$. For any physically available concentration dimension $d_\perp \leq 3$,

$$d_\perp - 2s_u < 0. \tag{1011}$$

Applying (1000) to the amplifying u_x component gives

$$\|u_{x,N}\|_{\dot{H}^{s_u}}^2 \asymp \sigma^{-2a_x+\lambda(d_\perp-2s_u)}. \tag{1012}$$

Both exponents on the right are strictly unfavorable: $a_x > 0$ and $d_\perp - 2s_u < 0$. Hence for every $\lambda > 0$,

$$-2a_x + \lambda(d_\perp - 2s_u) < 0, \tag{1013}$$

so

$$\boxed{\|u_{x,N}\|_{H^{s_u}} \longrightarrow \infty \quad \text{as } \sigma \downarrow 0} \tag{1014}$$

in the fixed-shape shrinking-core model.

Theorem 35 (No fixed-shape shrinking-width repair above the Sobolev threshold)**.** *Let a reconstructed velocity component have terminal amplitude $M_N(\sigma) \asymp \sigma^{-a}$ with $a > 0$, and localize it in at most three spatial directions at a shrinking scale $\ell_N(\sigma) \asymp \sigma^{\lambda}$, $\lambda > 0$, with a nontrivial fixed rescaled profile. Then no such localization can remain uniformly bounded in H^{s_u} for any $s_u > 3/2$. In particular the amplified u_x component (1004) cannot satisfy the fixed-space hypothesis (983)–(985) by transverse width shrinkage alone.*

This is the familiar concentration cost of supercritical Sobolev norms: shrinking support lowers L^2 mass but increases derivatives more rapidly than the support volume can compensate once $s_u > d_\perp/2$.

275. Scalar fixed-space requirement

Let

$$r_b := 1 + \delta + \varepsilon > 1. \tag{1015}$$

The scalar amplitude is $M_b \asymp \sigma^{-\widehat{\rho}_\alpha}$, so

$$\|b_N\|_{\dot{H}^{r_b}}^2 \asymp \sigma^{-2\widehat{\rho}_\alpha+\lambda(d_\perp-2r_b)}. \tag{1016}$$

Thus the exact power criterion is

$$\boxed{\lambda(d_\perp - 2r_b) \geq 2\widehat{\rho}_\alpha.} \tag{1017}$$

For one transverse direction, $d_\perp = 1$, this is impossible because $r_b > 1$ and therefore $1-2r_b < 0$. For two transverse directions it is again impossible because $2 - 2r_b < 0$. For isotropic three-dimensional concentration, a power-law width could control the scalar norm only if

$$r_b < \frac{3}{2} \tag{1018}$$

and

$$\boxed{\lambda \geq \frac{2\widehat{\rho}_\alpha}{3 - 2r_b}.} \tag{1019}$$

Even in this exceptional scalar-only case, the velocity H^{s_u} estimate still fails because $s_u > 3/2$ makes $3 - 2s_u < 0$.

276. Anisotropic widths

The conclusion is unchanged for anisotropic concentration. Let the widths be

$$\ell_{N,j}(\sigma) \asymp \sigma^{\lambda_j}, \qquad j = 1, \dots, d_\perp, \tag{1020}$$

with $\lambda_j > 0$. The L^2 mass has the favorable volume factor

$$\prod_j \ell_{N,j} = \sigma^{\sum_j \lambda_j}, \tag{1021}$$

but a derivative of order s in the narrowest active direction costs the corresponding inverse width. Already for an integer $m > s_u - 1$ one obtains schematically

$$\|\partial_{x_j}^m f\|_2^2 \asymp M_N^2 \left(\prod_i \ell_{N,i} \right) \ell_{N,j}^{-2m}. \tag{1022}$$

Thus anisotropy may redistribute the derivative cost, but it does not provide a general uniform H^{s_u} bound for a genuinely concentrating, amplifying three-dimensional core. In particular, concentration in only the transverse x direction is the least favorable case for the present square-root zero.

277. Consequence for the reconstruction architecture

The width calculation separates two statements which should not be conflated. First, kinetic energy alone can be repaired by a sufficiently rapidly shrinking core, subject to (1008)–(1010). Second, the fixed-space higher-order estimate required for the compactness argument cannot be repaired by a standard shrinking fixed-profile localization, because the derivative cost has the wrong sign once $s_u > 3/2$.

Therefore the route proposed at the end of Part 42 does not close in its simplest form. A successful global continuation must use at least one of the following genuinely different mechanisms: a reconstruction in which the singular factor $|\chi|^{-1/2}$ is cancelled by vanishing of b at the zero set; a non-concentrating regularized geometry with a uniform denominator gap; or a compactness framework weaker than the fixed H^{s_u} velocity bound, with the nonlinear and residual limits proved directly in that weaker framework. Merely multiplying the current amplified profile by a cutoff of shrinking width is insufficient.

Theorem 36 (Transverse-width verdict)**.** *For the corrected terminal amplitudes (1003)–(1006), fixed-shape localization on a scale $\ell_N(\sigma) \to 0$ can make the kinetic energy uniform provided its volume decays at least as rapidly as prescribed by (1008)–(1010). It cannot, however, produce a uniform velocity H^{s_u} bound with $s_u > 3/2$ in spatial dimension three or less. In one or two transverse dimensions it also cannot produce the scalar $H^{1+\delta+\varepsilon}$ bound. Hence transverse concentration width alone does not repair the fixed-space obstruction of Part 42.*

278. Revised global ledger

The global proof ledger is now

(i) the quantified finite-N upper invariant boundary remains available on the terminal overlap;
(ii) the exact reconstruction is tame on charts separated from the zero set of χ;
(iii) a shrinking terminal core can preserve kinetic energy if (1008)–(1010) hold;
(iv) shrinking fixed-profile localization cannot provide the required uniform H^{s_u} velocity bound;
(v) therefore the square-root zero-set reconstruction must be regularized/cancelled, or the limiting compactness argument must be rebuilt in a weaker topology;
(vi) only after that structural repair should the scalar and full momentum residual limits be closed.

(1023)

Part I
Scalar b_3 Viscosity-versus-Riccati Audit

60 Purpose of the scalar-only audit

The preceding fixed-space discussion concerns reconstruction of the velocity. The present part deliberately sets that question aside and asks a narrower question: can the dissipative term in the corrected scalar b_3 equation itself dominate the positive quadratic mechanism strongly enough to rule out the proposed finite-time amplification?

Along the terminal characteristic the scalar equation has the form

$$\dot{b}_N = \Gamma_N b_N^2 + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}} - \nu \mathcal{L}_{N,\alpha} b_N + \nu \mathcal{C}_{N,\alpha}(u_{x,N}, u_{y,N}). \tag{1024}$$

The principal comparison in this part is between the positive quadratic term and the complete fractional block.

61 Critical-width law for the scalar fractional dissipation

Suppose, locally in the active spatial direction, that an amplified scalar core has amplitude B_N and width ℓ_N,

$$b_N(x) \simeq B_N \Phi\left(\frac{x - x_N}{\ell_N}\right), \qquad B_N > 0. \tag{1025}$$

By homogeneity of the fractional Laplacian,

$$(-\Delta)^\alpha b_N \simeq B_N \ell_N^{-2\alpha} (-\Delta)^\alpha \Phi. \tag{1026}$$

Thus the magnitude ratio of principal viscosity to the Riccati source is

$$\boxed{\mathfrak{V}_N := \frac{\nu|(-\Delta)^\alpha b_N|}{\Gamma_N B_N^2} \asymp \frac{\nu}{\Gamma_N B_N \ell_N^{2\alpha}}.} \tag{1027}$$

The critical width is consequently

$$\boxed{\ell_{\mathrm{crit,N}} := \left(\frac{\nu}{\Gamma_N B_N}\right)^{1/(2\alpha)}.} \tag{1028}$$

Up to profile-dependent positive constants, $\ell_N \gg \ell_{\mathrm{crit,N}}$ puts viscosity below the quadratic term, whereas $\ell_N \ll \ell_{\mathrm{crit,N}}$ makes the fractional scale larger. At a positive spatial maximum the fractional Laplacian has the dissipative sign, so the latter regime is precisely the regime in which viscosity could arrest the scalar growth.

62 Insertion of the corrected characteristic scaling

The corrected terminal characteristic laws obtained earlier are

$$b_N(s) \asymp s^{-\beta}, \qquad \Gamma_N(s) = \frac{g_{N,\alpha}(s)}{s}, \qquad \sigma_N(s) \asymp s^{p_\beta}, \qquad p_\beta = \frac{12-2\beta}{7}. \tag{1029}$$

Therefore

$$\Gamma_N B_N \asymp g_{N,\alpha} s^{-(\beta+1)}, \tag{1030}$$

and

$$\boxed{\ell_{\mathrm{crit,N}}(s) \asymp \left(\frac{\nu}{g_{N,\alpha}}\right)^{1/(2\alpha)} s^{(\beta+1)/(2\alpha)}.} \tag{1031}$$

Since

$$s \asymp \sigma^{1/p_\beta} = \sigma^{7/[2(6-\beta)]}, \tag{1032}$$

we obtain

$$\boxed{\ell_{\mathrm{crit,N}}(\sigma) \asymp \left(\frac{\nu}{g_{N,\alpha}}\right)^{1/(2\alpha)} \sigma^{\lambda_{\mathrm{crit}}}, \qquad \lambda_{\mathrm{crit}} = \frac{7(\beta+1)}{4\alpha(6-\beta)}.} \tag{1033}$$

For the continuation scaling $\beta = 2\alpha - 1$ this simplifies exactly to

$$\boxed{\lambda_{\mathrm{crit}}(\alpha) = \frac{7}{2(7-2\alpha)}.} \tag{1034}$$

For every $\alpha > 0$ with $\alpha < 7/2$,

$$\lambda_{\mathrm{crit}}(\alpha) - \frac{1}{2} = \frac{\alpha}{7-2\alpha} > 0. \tag{1035}$$

63 Comparison with the actual geometric width

The intrinsic geometric coordinate is

$$\sigma = \sin^2(x - x_*). \tag{1036}$$

Near the simple zero x_*,

$$\sigma = (x - x_*)^2 + O(|x - x_*|^4), \tag{1037}$$

so the geometric spatial scale associated with a terminal level σ is

$$\boxed{\ell_{\mathrm{geo}}(\sigma) \asymp \sigma^{1/2}.} \tag{1038}$$

Because $\lambda_{\mathrm{crit}} > 1/2$, for fixed nondegenerate prefactors,

$$\frac{\ell_{\mathrm{geo}}}{\ell_{\mathrm{crit,N}}} \asymp \left(\frac{g_{N,\alpha}}{\nu}\right)^{1/(2\alpha)} \sigma^{1/2-\lambda_{\mathrm{crit}}} \longrightarrow +\infty \qquad (\sigma \downarrow 0). \tag{1039}$$

Thus the actual geometric core is asymptotically broader than the width at which fractional viscosity would balance the Riccati term.

Equivalently, substituting $\ell_{\mathrm{geo}} \asymp \sigma^{1/2}$ into (1027) gives

$$\mathfrak{V}_N \asymp \frac{\nu}{g_{N,\alpha}} \sigma^{e_J - \alpha}, \qquad e_J = \frac{7\alpha}{7 - 2\alpha}, \tag{1040}$$

and

$$\boxed{e_J - \alpha = \frac{2\alpha^2}{7 - 2\alpha} > 0.} \tag{1041}$$

Hence, whenever $g_{N,\alpha}$ is held nonzero at the finite-N level,

$$\boxed{\mathfrak{V}_N \longrightarrow 0 \quad (\sigma \downarrow 0).} \tag{1042}$$

The principal scalar viscosity is therefore lower order relative to the positive quadratic term on the corrected geometric terminal scale.

64 The finite-N order of limits

Equation (1042) is a finite-N terminal statement. It must not be confused with a uniform statement under an arbitrary coupled limit $N \to \infty$ and $\sigma \to 0$. Since the Lambert sensitivity $g_{N,\alpha}$ may tend to zero with N, the exact ratio is

$$\mathfrak{V}_N \asymp \nu g_{N,\alpha}^{-1} \sigma^{e_J - \alpha}. \tag{1043}$$

Thus viscosity is lower order along a coupled sequence precisely when

$$\boxed{\frac{\sigma_N^{e_J - \alpha}}{g_{N,\alpha}} \longrightarrow 0.} \tag{1044}$$

It is critical when this ratio tends to a finite nonzero constant and can be larger when the ratio diverges. This is the scalar analogue of the path dependence already encountered in the Lambert limit. Consequently, the finite-N blow-up mechanism is not disproved by viscosity, but a uniform $N \to \infty$ singular limit requires the coupling condition (1044) or an equivalent two-sided estimate.

65 Complete fractional block

The scalar product equation contains not only $-\nu(-\Delta)^{\alpha}b$ but also the fractional product commutator. The previous coefficient audit combined these terms and obtained, on the geometric terminal scale,

$$a_{1,N,\alpha} = O(\sigma^{-\alpha}), \tag{1045}$$

whereas the Riccati/geometric leading scale is

$$\sigma^{-e_J}, \qquad e_J > \alpha. \tag{1046}$$

Therefore the commutator does not change the exponent conclusion: under the same finite-N nondegeneracy and coefficient-control assumptions,

$$\boxed{\frac{\text{complete fractional block}}{\text{leading quadratic/geometric block}} = O(\sigma^{e_J-\alpha}) \to 0.} \tag{1047}$$

The sign of the fractional coefficient matters in the critical coupled regime, but it cannot overturn the strict exponent separation at fixed N.

66 Scalar viscosity verdict

Theorem 37 (Scalar viscosity-versus-Riccati verdict)**.** *Assume the quantified terminal estimates of the preceding parts, the continuation scaling $\beta = 2\alpha - 1$, and a fixed finite N for which $g_{N,\alpha} > 0$. Assume also that the complete fractional block obeys the uniform terminal estimate (1045). Then the geometric width $\ell_{\mathrm{geo}} \asymp \sigma^{1/2}$ is asymptotically larger than the critical viscous width (1033), and*

$$\frac{\nu|(-\Delta)^{\alpha}b_N|}{\Gamma_N b_N^2} \to 0 \qquad (\sigma \downarrow 0). \tag{1048}$$

The same lower-order conclusion holds for the complete fractional block. Thus, within the corrected scalar b_3 terminal asymptotics, fractional viscosity does not by itself eliminate the finite-N Riccati amplification mechanism.

For a coupled $N \to \infty$, $\sigma_N \to 0$ limit, the conclusion is uniform only if (1044) holds. Without such a coupling estimate, the scalar analysis alone permits subcritical, critical, or viscosity-dominant paths and hence does not establish a uniform limiting blow-up theorem.

The theorem is intentionally a statement about the scalar b_3 equation only. It neither uses nor repairs the fixed-space velocity obstruction of Parts 42–43.

279. Revised scalar ledger

The scalar-only conclusion is

$$\boxed{\begin{array}{c}\text{(i) the critical dissipative width is } \ell_{\mathrm{crit,N}} \asymp (\nu/g_{N,\alpha})^{1/(2\alpha)}\sigma^{7/[2(7-2\alpha)]};\\ \text{(ii) the actual geometric width is } \ell_{\mathrm{geo}} \asymp \sigma^{1/2};\\ \text{(iii) since } 7/[2(7-2\alpha)] > 1/2, \text{ viscosity is lower order at fixed finite } N;\\ \text{(iv) the complete fractional block has the same lower-order exponent } \alpha < e_J;\\ \text{(v) a coupled } N \to \infty \text{ conclusion requires } \sigma_N^{e_J-\alpha}/g_{N,\alpha} \to 0;\\ \text{(vi) therefore the present scalar audit does not support the claim that viscosity necessarily prevents finite-time amplification.}\end{array}} \tag{1049}$$

Part II
Exact Critical Coupled Path from the Two-Sided Lambert Product

67 The coupled scalar ratio

Set

$$d_\alpha := e_J - \alpha = \frac{2\alpha^2}{7-2\alpha} > 0. \tag{1050}$$

The scalar viscosity audit gives, up to positive coefficient factors that are uniformly bounded above and below on the terminal overlap,

$$\mathfrak{V}_{N,\alpha}(\sigma) := \frac{\text{complete fractional scale}}{\text{positive Riccati scale}} \asymp \frac{\nu\sigma^{d_\alpha}}{g_{N,\alpha}}. \tag{1051}$$

Thus the coupled problem is determined by the relative size of σ^{d_α} and the Lambert sensitivity $g_{N,\alpha}$.

68 Insertion of the exact Lambert tail

Let $j_c = j_c(N,\alpha)$ be the capture index into a fixed compact trapping interval about the positive Lambert fixed point Δ_*. The exact sensitivity recurrence proved earlier gives

$$S_{N,k} = S_{N,j_c} \prod_{j=j_c}^{k-1} A(\Delta_{N,j}), \tag{1052}$$

where

$$A_* = A(\Delta_*) = \frac{1-\Delta_*}{\Delta_*} \in (0,1/2). \tag{1053}$$

The two-sided product theorem yields

$$e^{-C_A\varepsilon^c_{N,\alpha}} \le \frac{\prod_{j=j_c}^{k(N)-1} A(\Delta_{N,j})}{A_*^{k(N)-j_c}} \le e^{C_A\varepsilon^c_{N,\alpha}}. \tag{1054}$$

Consequently, writing

$$G^*_{N,\alpha} := |S_{N,j_c}| A_*^{k(N)-j_c}, \tag{1055}$$

we have the quantified two-sided law

$$\boxed{G^*_{N,\alpha} e^{-C_A\varepsilon^c_{N,\alpha}} \le g_{N,\alpha} \le G^*_{N,\alpha} e^{C_A\varepsilon^c_{N,\alpha}}.} \tag{1056}$$

If $\varepsilon^c_{N,\alpha} \to 0$ uniformly, then

$$\boxed{g_{N,\alpha} = |S_{N,j_c}| A_*^{k(N)-j_c}\big(1+O(\varepsilon^c_{N,\alpha})\big).} \tag{1057}$$

No lower bound on $|S_{N,j_c}|$ independent of N is asserted here; the entrance sensitivity remains the exact nondegeneracy datum.

69 Exact critical coupled path

To retain the positive constants in the scalar comparison, write

$$\mathfrak{V}_{N,\alpha}(\sigma) = \frac{\nu\,\mathfrak{c}_{N,\alpha}}{g_{N,\alpha}\mathfrak{d}_{N,\alpha}}\sigma^{d_\alpha}(1+\vartheta_{N,\alpha}), \tag{1058}$$

where on the quantified overlap

$$0 < c_- \le \mathfrak{c}_{N,\alpha} \le c_+, \qquad 0 < d_- \le \mathfrak{d}_{N,\alpha} \le d_+, \qquad |\vartheta_{N,\alpha}| \le \omega_N \to 0. \tag{1059}$$

Define the coefficient-exact critical scale by

$$\boxed{\sigma^{\mathrm{crit}}_{N,\alpha} := \left(\frac{g_{N,\alpha}\mathfrak{d}_{N,\alpha}}{\nu\mathfrak{c}_{N,\alpha}}\right)^{1/d_\alpha}.} \tag{1060}$$

Then

$$\boxed{\mathfrak{V}_{N,\alpha}(\sigma) = \left(\frac{\sigma}{\sigma^{\mathrm{crit}}_{N,\alpha}}\right)^{d_\alpha}(1+\vartheta_{N,\alpha}).} \tag{1061}$$

Using

$$\frac{1}{d_\alpha} = \frac{7-2\alpha}{2\alpha^2}, \tag{1062}$$

and the Lambert product (1057), the critical path has the explicit form

$$\boxed{\sigma^{\mathrm{crit}}_{N,\alpha} = \left[\frac{|S_{N,j_c}|\,A_*^{k(N)-j_c}}{\nu}\frac{\mathfrak{d}_{N,\alpha}}{\mathfrak{c}_{N,\alpha}}\right]^{\frac{7-2\alpha}{2\alpha^2}}\left(1+O(\varepsilon^c_{N,\alpha}+\omega_N)\right).} \tag{1063}$$

In the coefficient-free $\asymp$ notation this is

$$\boxed{\sigma^{\mathrm{crit}}_{N,\alpha} \asymp \left(\frac{|S_{N,j_c}|}{\nu}A_*^{k(N)-j_c}\right)^{\frac{7-2\alpha}{2\alpha^2}}.} \tag{1064}$$

At the anchor $\alpha = 1$,

$$d_1 = \frac{2}{5}, \qquad \frac{1}{d_1} = \frac{5}{2}, \tag{1065}$$

so

$$\boxed{\sigma^{\mathrm{crit}}_{N,1} \asymp \left(\frac{|S_{N,j_c}|}{\nu}A_*^{k(N)-j_c}\right)^{5/2}.} \tag{1066}$$

70 Classification of coupled paths

Let $\sigma_N \downarrow 0$ while $N \to \infty$. Equation (1061) gives the complete scalar classification.

If

$$\boxed{\frac{\sigma_N}{\sigma_{N,\alpha}^{\mathrm{crit}}} \longrightarrow 0,} \tag{1067}$$

then

$$\mathfrak{V}_{N,\alpha}(\sigma_N) \longrightarrow 0, \tag{1068}$$

so the complete fractional block is asymptotically lower order than the positive Riccati scale. This is the viscosity-subcritical regime and is the coupled regime compatible with persistence of the scalar amplification mechanism.

If

$$\boxed{\frac{\sigma_N}{\sigma_{N,\alpha}^{\mathrm{crit}}} \longrightarrow L \in (0,\infty),} \tag{1069}$$

then

$$\mathfrak{V}_{N,\alpha}(\sigma_N) \longrightarrow L^{d_\alpha} \tag{1070}$$

up to the limiting coefficient normalization. This is the critical regime. Exponent counting alone cannot decide growth versus damping here: the signed fractional coefficient and the remaining leading scalar coefficients must be compared.

Finally, if

$$\boxed{\frac{\sigma_N}{\sigma_{N,\alpha}^{\mathrm{crit}}} \longrightarrow \infty,} \tag{1071}$$

then

$$\mathfrak{V}_{N,\alpha}(\sigma_N) \longrightarrow \infty. \tag{1072}$$

This is the viscosity-dominant magnitude regime. If the complete fractional block has the dissipative sign on this overlap, it overwhelms the positive Riccati contribution; if its signed coefficient changes sign or undergoes a leading cancellation, the magnitude comparison alone is not sufficient to conclude suppression.

Equivalently, using (1055), the three regimes are classified by

$$\boxed{\mathfrak{C}_{N,\alpha} := \frac{\nu \sigma_N^{d_\alpha}}{|S_{N,j_c}| A_*^{k(N)-j_c}}.} \tag{1073}$$

Namely,

$$\mathfrak{C}_{N,\alpha} \to 0 \quad \Longleftrightarrow \quad \text{fractional block lower order,} \tag{1074}$$

$$\mathfrak{C}_{N,\alpha} \to c \in (0,\infty) \quad \Longleftrightarrow \quad \text{critical competition,} \tag{1075}$$

and

$$\mathfrak{C}_{N,\alpha} \to \infty \quad \Longleftrightarrow \quad \text{fractional block dominant in magnitude.} \tag{1076}$$

71 Depth law and the noncommutation of limits

Because $0 < A_* < 1$, increasing Lambert depth makes the critical terminal scale exponentially small in $k(N) - j_c$. If the entrance sensitivity is bounded above and below and $k(N) - j_c \to \infty$, then

$$\boxed{\sigma_{N,\alpha}^{\mathrm{crit}} \asymp A_*^{\frac{7-2\alpha}{2\alpha^2}(k(N)-j_c)}.} \tag{1077}$$

Thus the terminal point must be approached at least this rapidly, and in fact strictly faster for (1067), if the fractional block is to remain negligible uniformly in the coupled limit.

For example, if

$$k(N) - j_c \sim c \log N, \tag{1078}$$

then

$$\sigma_{N,\alpha}^{\mathrm{crit}} \asymp N^{-\mu_\alpha}, \qquad \boxed{\mu_\alpha = c\,|\log A_*| \frac{7-2\alpha}{2\alpha^2},} \tag{1079}$$

up to the entrance-sensitivity and coefficient factors. If instead $k(N) - j_c$ grows like $\log \log N$, the corresponding critical scale is a power of $\log N$. These formulas make the noncommutation of the terminal and increasing-depth limits explicit.

72 Coupled-path theorem for the scalar b_3 equation

Theorem 38 (Critical coupled path for scalar viscosity)**.** *Assume the quantified terminal estimates of Parts 40–44, the two-sided Lambert capture estimate (1054), and positive coefficient bounds (1059). Let $d_\alpha = 2\alpha^2/(7-2\alpha)$ and define $\sigma_{N,\alpha}^{\mathrm{crit}}$ by (1060). Then uniformly for α in a sufficiently small interval $[1, 1+\delta]$,*

$$\mathfrak{V}_{N,\alpha}(\sigma_N) = \left(\frac{\sigma_N}{\sigma_{N,\alpha}^{\mathrm{crit}}}\right)^{d_\alpha} (1 + o(1)). \tag{1080}$$

Consequently, $\sigma_N = o(\sigma_{N,\alpha}^{\mathrm{crit}})$ is the precise coupled condition under which the complete fractional block is lower order than the positive Riccati scale. Comparable scales produce critical competition, whereas $\sigma_{N,\alpha}^{\mathrm{crit}} = o(\sigma_N)$ produces fractional dominance in magnitude.

The last alternative is a no-blow-up conclusion only after the complete fractional block is shown to have the damping sign with a nonvanishing coefficient. Thus the present theorem determines the exact critical path; it does not replace the remaining signed critical-regime calculation.

280. Revised scalar coupled-limit ledger

The scalar coupled-limit audit now gives

$$\boxed{\begin{aligned} g_{N,\alpha} &\asymp |S_{N,j_c}| A_*^{k(N)-j_c}, \\ d_\alpha = e_J - \alpha &= \frac{2\alpha^2}{7-2\alpha}, \\ \sigma_{N,\alpha}^{\mathrm{crit}} &\asymp \left(\frac{|S_{N,j_c}|}{\nu} A_*^{k(N)-j_c} \right)^{(7-2\alpha)/(2\alpha^2)}, \\ \sigma_N = o(\sigma_{N,\alpha}^{\mathrm{crit}}) &\Longrightarrow \text{fractional viscosity is lower order}, \\ \sigma_N \asymp \sigma_{N,\alpha}^{\mathrm{crit}} &\Longrightarrow \text{signed critical competition}, \\ \sigma_{N,\alpha}^{\mathrm{crit}} = o(\sigma_N) &\Longrightarrow \text{fractional dominance in magnitude}. \end{aligned}} \tag{1081}$$

The remaining scalar question is therefore no longer an exponent question: on the critical and viscosity-dominant coupled paths one must compute the sign and nondegeneracy of the complete fractional coefficient itself.

Part III
Signed Complete Fractional Coefficient on the Critical Coupled Scale

73 The sign question after the critical-path reduction

The preceding scalar audit reduced the complete fractional contribution to

$$\mathfrak{F}_{N,\alpha} := -\nu \mathcal{L}_{N,\alpha} b_N + \nu \mathcal{C}_{N,\alpha}(u_{x,N}, u_{y,N}). \tag{1082}$$

On the terminal Mellin tangent, with $r = \sqrt{\sigma}$ and

$$P_{N,\alpha} \sim \widehat{p}_{N,\alpha} r^{-2\widehat{\rho}_\alpha}, \qquad \widehat{\rho}_\alpha = \frac{7(2\alpha-1)}{2(7-2\alpha)}, \tag{1083}$$

the square-root reconstruction has powers

$$\lambda_U(\alpha) = -\frac{1}{2} - \widehat{\rho}_\alpha, \qquad \lambda_A(\alpha) = \frac{1}{2} - \widehat{\rho}_\alpha. \tag{1084}$$

The already-derived commutator cancellation therefore gives

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{P_{N,\alpha}} = -\nu H_\alpha \, \sigma^{-\alpha} + o(\sigma^{-\alpha}),} \tag{1085}$$

where

$$\boxed{H_\alpha := K_\alpha^+(\lambda_U(\alpha)) + K_\alpha^+(\lambda_A(\alpha)).} \tag{1086}$$

Thus the desired damping sign would require $H_\alpha > 0$. The sign can be checked exactly at the anchor $\alpha = 1$.

74 Exact anchor calculation at $\alpha = 1$

At $\alpha = 1$ one has

$$\widehat{\rho}_1 = \frac{7}{10}, \qquad \lambda_U(1) = -\frac{6}{5}, \qquad \lambda_A(1) = -\frac{1}{5}. \tag{1087}$$

For the local anchor operator $(-\partial_{rr}) = -\partial_{rr}$,

$$K_1^+(\lambda) = -\lambda(\lambda - 1) \tag{1088}$$

under the same analytic continuation used by the Mellin tangent formula. Consequently

$$K_1^+\left(-\frac{6}{5}\right) = -\frac{66}{25}, \qquad K_1^+\left(-\frac{1}{5}\right) = -\frac{6}{25}, \tag{1089}$$

and hence

$$\boxed{H_1 = -\frac{72}{25} < 0.} \tag{1090}$$

Substitution into (1085) gives

$$\boxed{\frac{\mathfrak{F}_{N,1}}{P_{N,1}} = +\frac{72}{25}\nu\,\sigma^{-1} + o(\sigma^{-1}).} \tag{1091}$$

Therefore the complete fractional block has the *positive*, not the negative, tangent sign at the $\alpha = 1$ anchor.

The same conclusion follows directly from the local product identity. If $u_x = c_U r^{\lambda_U}$ and $u_y = c_A r^{\lambda_A}$ with $c_U c_A > 0$, then

$$-\nu(-\partial_{rr})b + \nu\mathcal{C}_1(u_x, u_y) = \nu\partial_{rr}b - 2\nu(\partial_r u_x)(\partial_r u_y). \tag{1092}$$

Since $\lambda_U = -6/5$ and $\lambda_A = -1/5$, the coefficient obtained after factoring $b\,r^{-2}$ is exactly $72\nu/25$. This independently verifies the sign in (1091).

75 Uniform persistence for $1 \le \alpha < 1 + \delta$

The Mellin coefficient is

$$K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi}\sin\big(\pi(\alpha-\lambda)\big). \tag{1093}$$

The functions $\widehat{\rho}_\alpha$, $\lambda_U(\alpha)$ and $\lambda_A(\alpha)$ are smooth near $\alpha = 1$. On any sufficiently small interval avoiding poles of the analytically continued Gamma factors, H_α is continuous. Since $H_1 = -72/25$, there exist $\delta_{\mathrm{frac}} > 0$ and $h_* > 0$ such that

$$\boxed{H_\alpha \le -h_* < 0, \qquad 1 \le \alpha < 1 + \delta_{\mathrm{frac}}.} \tag{1094}$$

For example one may choose any $h_* < 72/25$ after shrinking δ_{frac} sufficiently; no numerical value of δ_{frac} is asserted without a certified interval bound for the Gamma factors.

If the finite-N geometric/Mellin remainder satisfies uniformly

$$\left| \frac{\mathfrak{F}_{N,\alpha}}{P_{N,\alpha}} + \nu H_\alpha \sigma^{-\alpha} \right| \le \frac{1}{2} \nu h_* \sigma^{-\alpha}, \tag{1095}$$

then

$$\boxed{\mathfrak{F}_{N,\alpha} \ge \frac{1}{2} \nu h_* P_{N,\alpha} \sigma^{-\alpha} > 0.} \tag{1096}$$

Thus the signed finite-N conclusion furnished by the tangent realization is uniformly *anti-damping* on the small continuation interval.

76 Consequences on the critical coupled path

Let $\mathfrak{Q}_{N,\alpha} > 0$ denote the positive Riccati scale and define $\sigma_{N,\alpha}^{\mathrm{crit}}$ using the absolute fractional coefficient, as in the preceding part. Then

$$\frac{\mathfrak{F}_{N,\alpha}}{\mathfrak{Q}_{N,\alpha}} = +c_{N,\alpha} \left(\frac{\sigma_N}{\sigma_{N,\alpha}^{\mathrm{crit}}} \right)^{d_\alpha} (1 + o(1)), \qquad c_{N,\alpha} > 0, \tag{1097}$$

with

$$d_\alpha = \frac{2\alpha^2}{7 - 2\alpha} > 0. \tag{1098}$$

Hence the three coupled regimes have the following signed interpretation within this Mellin tangent model:

$$\boxed{\begin{aligned} \sigma_N/\sigma_N^{\mathrm{crit}} \to 0 \quad &\Longrightarrow \quad \mathfrak{F}_{N,\alpha} = o(\mathfrak{Q}_{N,\alpha}), \\ \sigma_N/\sigma_N^{\mathrm{crit}} \to L \in (0,\infty) \quad &\Longrightarrow \quad \mathfrak{F}_{N,\alpha} \text{ is comparable and positive}, \\ \sigma_N/\sigma_N^{\mathrm{crit}} \to \infty \quad &\Longrightarrow \quad \mathfrak{F}_{N,\alpha} \text{ dominates with positive tangent sign.} \end{aligned}} \tag{1099}$$

Therefore the hoped-for conclusion "viscosity-dominant paths imply scalar damping" is false for the present tangent coefficient: the leading fractional block points in the same sign as the positive scalar growth at the anchor and, by continuity, throughout a sufficiently small interval above $\alpha = 1$.

77 Why this is not yet an unconditional blow-up conclusion

There is a necessary domain qualification. At the anchor $\lambda_U = -6/5 < -1$, so the singular tangent power is outside the naive locally-integrable half-line power class. Equations (1085)–(1094) therefore use the manuscript's analytic/distributional continuation of the Mellin coefficient. The actual finite-N fields are regularized, and the fractional Laplacian is nonlocal. A rigorous finite-N signed theorem must show that the regularized operator converges to this continued tangent coefficient with the relative error required in (1095). Without that step, one may not infer the sign of the exact finite-N nonlocal term merely from (1090).

Accordingly the result of this part is precise but limited:

> The complete Mellin tangent fractional coefficient is not dissipative near $\alpha = 1$; it is strictly positive, with anchor coefficient $72\nu/25$. Hence the tangent calculation supplies no scalar viscosity-prevents-blow-up mechanism. The remaining issue is to certify that this signed tangent coefficient is the actual finite-N limit of the regularized nonlocal fractional block.

(1100)

78 Updated scalar ledger

The scalar-only problem is now reduced to the following certification step:

> prove the uniform finite-N tangent estimate (1095) for the regularized geometric fractional operator and product commutator;
> if (1095) holds, the complete fractional term is positive near $\alpha = 1$;
> therefore viscosity-dominant coupled paths do not yield damping in this model.

(1101)

This statement concerns only the scalar b_3 equation and does not assert a finite-time singularity for the full Navier–Stokes velocity field.

Part IV
Finite-N Certification of the Signed Scalar Fractional Block

79 The certification problem and the missing scale separation

The preceding part identified the Mellin-tangent coefficient

$$H_\alpha = K_\alpha^+(\lambda_U(\alpha)) + K_\alpha^+(\lambda_A(\alpha)), \qquad H_1 = -\frac{72}{25} < 0, \tag{1102}$$

and reduced the exact finite-N sign question to (1095). We now separate what follows from the already proved geometric operator estimate from what requires an additional finite-N core hypothesis.

Let $r = \sqrt{\sigma} = -\sin(x - x_*)$ on the one-sided terminal chart and let $\varepsilon_N > 0$ denote the smallest regularization/core scale on which the finite-N factors cease to be accurately represented by their tangent powers. Write

$$U_{0,N,\alpha}(r) = c_{U,N,\alpha} r^{\lambda_U(\alpha)}(1 + \eta_{U,N,\alpha}(r)), \qquad A_{0,N,\alpha}(r) = c_{A,N,\alpha} r^{\lambda_A(\alpha)}(1 + \eta_{A,N,\alpha}(r)), \quad (1103)$$

for $r \geq R\varepsilon_N$, with $c_{U,N,\alpha} c_{A,N,\alpha} = \widehat{p}_{N,\alpha}$. The analytic continuation used in Part 46 does not, by itself, imply that (1103) holds uniformly down to $r \asymp \varepsilon_N$. In particular, the weighted geometric estimate (661) controls the change from the affine fractional operator to the intrinsic geometric operator on a terminal logarithmic window, but it does not control the inner regularization layer of a tangent power with $\lambda_U < -1$.

Accordingly the correct overlap condition is

$$\boxed{\frac{\varepsilon_N}{r_N} \longrightarrow 0, \qquad r_N = \sqrt{\sigma_N}.} \quad (1104)$$

Only on such an outer terminal overlap can the continued Mellin coefficient be certified from the regularized finite-N fields without resolving the inner core explicitly.

80 A quantified outer-overlap tangent hypothesis

Fix $0 < \vartheta < 1$. Assume that for $j = 0, 1, 2$,

$$\boxed{\sup_{\substack{1 \leq \alpha < 1+\delta \\ R\varepsilon_N \leq r \leq r_*}} \left|(r\partial_r)^j \eta_{U,N,\alpha}(r)\right| + \left|(r\partial_r)^j \eta_{A,N,\alpha}(r)\right| \leq \omega_N(R),} \quad (1105)$$

where

$$\lim_{R \to \infty} \limsup_{N \to \infty} \omega_N(R) = 0. \quad (1106)$$

Assume also uniform positive bounds on the amplitude coefficients and the finite-N coefficient errors already isolated in Parts 39–40. This is the precise regularized tangent statement needed below; it is stronger than the mere pointwise asymptotic notation used in Part 39.

For the intrinsic geometric operator, Part 32 gives

$$\|\mathcal{Q}_\alpha^{\mathrm{geo}}\|_{X_{\eta,\sigma} \to Y_\eta(X_0)} \leq C e^{2X_0} = C r_*^2. \quad (1107)$$

Thus, after shrinking r_*, the geometric change of variables contributes a relative $O(r_*^2)$ error to the tangent coefficient. The profile errors in (1105) contribute $O(\omega_N(R))$. The remaining nonlocal interaction with the inner core is denoted by $\mathcal{T}_{N,\alpha}^{\mathrm{core}}$.

81 Core-tail estimate and the exact finite-N error budget

For a fractional operator of order 2α, points with $r \geq R\varepsilon_N$ see the inner interval $0 < \rho < C\varepsilon_N$ through a kernel separated from the evaluation point by a distance comparable to r. Consequently the inner-core contribution has the schematic bound

$$|\mathcal{T}^{\mathrm{core}}_{N,\alpha}(r)| \leq Cr^{-1-2\alpha} \int_0^{C\varepsilon_N} \big(|U_{0,N,\alpha}(\rho)| + |A_{0,N,\alpha}(\rho)| + |P_{N,\alpha}(\rho)|\big)\, d\rho. \tag{1108}$$

Define the dimensionless core-tail modulus

$$\boxed{\Theta_{N,\alpha}(r) := r^{\alpha} \frac{|\mathcal{T}^{\mathrm{core}}_{N,\alpha}(r)|}{P_{N,\alpha}(r)}.} \tag{1109}$$

The normalization is chosen so that Θ measures the core tail relative to the leading $P\sigma^{-\alpha} = Pr^{-2\alpha}$ fractional scale. A sufficient finite-N separation condition is

$$\boxed{\sup_{\substack{1\leq\alpha<1+\delta \\ R\varepsilon_N\leq r\leq r_*}} \Theta_{N,\alpha}(r) \leq \omega^{\mathrm{core}}_N(R), \qquad \lim_{R\to\infty} \limsup_{N\to\infty} \omega^{\mathrm{core}}_N(R) = 0.} \tag{1110}$$

This condition is not automatic from analytic continuation of K^+_α; it is the exact nonlocal datum that must be verified for the chosen finite-N regularization.

Combining (1105), (1107), and (1110) yields the error budget

$$\boxed{\left|\frac{\mathfrak{F}_{N,\alpha}}{P_{N,\alpha}} + \nu H_\alpha \sigma^{-\alpha}\right| \leq C\nu\sigma^{-\alpha}\left[\omega_N(R) + \omega^{\mathrm{core}}_N(R) + r_*^2 + \epsilon^{\mathrm{coeff}}_N\right].} \tag{1111}$$

Here $\epsilon^{\mathrm{coeff}}_N$ collects the already quantified finite-N coefficient moduli from the Lambert/reconstruction audit.

Theorem 39 (Finite-N signed fractional certification on an outer overlap)**.** *Assume (1105)–(1110), the coefficient bounds of Parts 39–40, and choose $\delta > 0$ so that*

$$H_\alpha \leq -h_* < 0, \qquad 1 \leq \alpha < 1 + \delta. \tag{1112}$$

There exist $R_0 < \infty$, $r_* > 0$, and $N_0 < \infty$ such that, for $R \geq R_0$, $N \geq N_0$, and

$$R\varepsilon_N \leq r \leq r_*, \tag{1113}$$

provided the bracket in (1111) is at most $h_*/(2C)$, one has

$$\boxed{\left|\frac{\mathfrak{F}_{N,\alpha}}{P_{N,\alpha}} + \nu H_\alpha \sigma^{-\alpha}\right| \leq \frac{1}{2}\nu h_* \sigma^{-\alpha},} \tag{1114}$$

and hence

$$\boxed{\mathfrak{F}_{N,\alpha} \geq \frac{1}{2}\nu h_* P_{N,\alpha}\sigma^{-\alpha} > 0.} \tag{1115}$$

Thus the positive tangent sign is a genuine finite-N sign on every regularized outer overlap for which the inner-core tail is negligible.

Proof. The affine tangent contribution is $-\nu H_\alpha P_{N,\alpha}\sigma^{-\alpha}$ by the cancellation already proved in (866)–(867). The intrinsic geometric correction is bounded by (1107), the multiplicative tangent-profile errors by (1105), the nonlocal inner-core interaction by (1110), and the remaining finite-N coefficient errors by $\epsilon_N^{\mathrm{coeff}}$. This gives (1111). Choose the parameters in the order $r_* \downarrow 0$, $R \uparrow \infty$, and then $N \uparrow \infty$ so that the bracket is at most $h_*/(2C)$. Equation (1114) follows, and (1112) then gives (1115). □

82 Exact local check at the anchor $\alpha = 1$

At $\alpha = 1$ no analytic continuation is needed to check the sign at every point of the outer region where the finite-N factors are C^2. Since $L = -\partial_{rr}$ and

$$\mathcal{C}_1(u,v) = L(uv) - vLu - uLv, \tag{1116}$$

the complete fractional block satisfies the exact identity

$$\boxed{-\nu L(uv) + \nu\mathcal{C}_1(u,v) = -\nu\big(vLu + uLv\big).} \tag{1117}$$

For the tangent powers

$$u = c_U r^{-6/5}, \qquad v = c_A r^{-1/5}, \tag{1118}$$

we have

$$Lu = -\frac{66}{25}c_U r^{-16/5}, \qquad Lv = -\frac{6}{25}c_A r^{-11/5}. \tag{1119}$$

Therefore

$$\boxed{-\nu\big(vLu + uLv\big) = \frac{72}{25}\nu c_U c_A r^{-17/5} > 0.} \tag{1120}$$

This verifies directly that the anchor sign found in Part 46 is not a sign artifact of the Gamma-function representation. What remains nontrivial is uniform convergence of the regularized finite-N factors and their nonlocal core tails to this outer tangent behavior.

83 Interaction with the critical coupled path

The critical path of Part 45 is admissible for the signed theorem only if it lies outside the regularization core:

$$\boxed{\frac{\varepsilon_N}{\sqrt{\sigma_{N,\alpha}^{\mathrm{crit}}}} \longrightarrow 0.} \tag{1121}$$

Equivalently,

$$\varepsilon_N^2 = o(\sigma_{N,\alpha}^{\mathrm{crit}}). \tag{1122}$$

If (1121) and the core-tail condition (1110) hold, then the signed classification of the coupled regimes is rigorous on the selected overlap: critical and viscosity-dominant paths carry a positive complete fractional block rather than damping. If instead $r_N \asymp \varepsilon_N$ or $r_N \ll \varepsilon_N$,

the tangent theorem does not apply and the sign must be computed from the explicit inner regularization.

This produces a sharp dichotomy:

$$\boxed{\begin{array}{ll} \varepsilon_N/\sqrt{\sigma_N} \to 0 & \Longrightarrow \text{ outer tangent sign is certifiable under (1110),} \\ \sqrt{\sigma_N} = O(\varepsilon_N) & \Longrightarrow \text{ inner-core sign remains regularization-dependent.} \end{array}} \tag{1123}$$

84 Status of the scalar no-blow-up alternative

The finite-N audit therefore does not establish a viscosity-driven no-blow-up mechanism. On every certified outer overlap the complete fractional block has the positive sign (1115), and at $\alpha = 1$ the sign is confirmed directly by the local identity (1120). However, the manuscript does not yet specify enough about the inner finite-N regularization to prove (1110) on paths entering $r \asymp \varepsilon_N$. Thus no statement about the sign in that inner layer is justified without an explicit regularization formula.

The exact remaining scalar question is

$$\boxed{\text{derive } \varepsilon_N \text{ and the inner profiles from the actual finite-}N \text{ construction,then verify (1110) and (1121), or compute the inner fractional block directly if (1121) fails.}} \tag{1124}$$

This is strictly a statement about the scalar b_3 product equation; it does not assert finite-time singularity of the full velocity field.

Part V
Actual Finite-N Lambert Regularization and Inner-Core Audit

85 Correction of the auxiliary core notation

Part 47 introduced an abstract spatial core radius, denoted there by ε_N, solely to formulate a core–tail estimate for the fractional operator. The revised Lambert construction already contains a quantity with the same symbol, but it has a different geometric meaning. The two quantities must not be identified without calculation.

In the revised finite-N construction the actual base profile is

$$\boxed{R_N(\theta) = \frac{1}{1 - \sin\theta + \varepsilon_N^{\mathrm{L}}}, \qquad \varepsilon_N^{\mathrm{L}} = \frac{\lambda}{L_N^2}, \qquad L_N = \log(1 + 2^N),} \tag{1125}$$

with fixed $\lambda > 0$ (or, more generally, with the real-branch constraint $L_N \varepsilon_N^{\mathrm{L}} < 1$). Hence

$$\boxed{\varepsilon_N^{\mathrm{L}} \sim \frac{\lambda}{(\log 2)^2}\, N^{-2}.} \tag{1126}$$

This is a denominator regularization in the variable $1-\sin\theta$; it is not a cutoff inserted into the reconstructed scalar profile.

Near $\theta=\pi/2$ write $x=\pi/2-\theta$. Then

$$1-\sin\theta=1-\cos x=\frac{x^2}{2}+O(x^4), \tag{1127}$$

so the physical angular radius on which the denominator regularizer itself is comparable with the quadratic geometric term is

$$\boxed{r_{\varepsilon,N}\asymp(\varepsilon_N^{\mathrm{L}})^{1/2}\asymp L_N^{-1}.} \tag{1128}$$

Thus, if one uses a squared geometric coordinate, the corresponding regularization scale is $\sigma_{\varepsilon,N}\asymp\varepsilon_N^{\mathrm{L}}$, not $(\varepsilon_N^{\mathrm{L}})^2$. The condition (1122) therefore cannot be applied to the manuscript's $\varepsilon_N^{\mathrm{L}}$ without this conversion.

86 The branch point lies outside the denominator core

The finite-N branch point is defined by

$$R_N(\theta_{*,N})=L_N. \tag{1129}$$

Putting $\theta_{*,N}=\pi/2-x_{*,N}$ gives

$$1-\cos x_{*,N}+\varepsilon_N^{\mathrm{L}}=\frac{1}{L_N}. \tag{1130}$$

For $\varepsilon_N^{\mathrm{L}}=\lambda L_N^{-2}$,

$$\boxed{x_{*,N}\sim\sqrt{\frac{2}{L_N}}.} \tag{1131}$$

Consequently

$$\frac{x_{*,N}}{r_{\varepsilon,N}}\asymp L_N^{1/2}\longrightarrow\infty. \tag{1132}$$

The Lambert branch point therefore does not sit in the denominator regularization core around $\pi/2$. At the branch point the denominator is exactly L_N^{-1}, whereas $\varepsilon_N^{\mathrm{L}}=O(L_N^{-2})$ is lower order.

This observation changes the interpretation of Part 47: the nonlocal "inner core" relevant to a tangent expansion about $\theta_{*,N}$ is not $r\lesssim\varepsilon_N^{\mathrm{L}}$. One must instead use the actual Lambert branch-layer width generated by $\Delta_{N,0}$.

87 The actual Lambert inner profile

Define

$$\Delta_{N,0}(\theta)=R_N(\theta)-L_N. \tag{1133}$$

At the left branch point,

$$R'_N(\theta_{*,N}) = L_N^2 \cos\theta_{*,N} \sim \sqrt{2}\, L_N^{3/2}. \tag{1134}$$

Hence, with $y = \theta - \theta_{*,N}$,

$$\boxed{\Delta_{N,0}(\theta) = a_N y + O(L_N^2 y^2), \qquad a_N \sim \sqrt{2}\, L_N^{3/2}.} \tag{1135}$$

The normalized Lambert layer used in the revised construction is

$$2^{2N}\Delta_{N,0} = O(1). \tag{1136}$$

It follows that its actual angular width is

$$\boxed{w_N \asymp \frac{2^{-2N}}{\sqrt{2}\, L_N^{3/2}} \asymp 2^{-2N} N^{-3/2}.} \tag{1137}$$

Introduce the genuine inner coordinate

$$\boxed{\xi_N := \frac{y}{w_N}.} \tag{1138}$$

Then on bounded ξ_N-sets,

$$\boxed{\Delta_{N,0}(\theta) = 2^{-2N}\xi_N\,[1 + o(1)].} \tag{1139}$$

For every fixed finite Lambert depth k, the branch expansion

$$\mathcal{F}(\Delta) = \sqrt{2\Delta} - \frac{2}{3}\Delta + O(\Delta^{3/2}) \tag{1140}$$

therefore yields the inner cascade

$$\boxed{\Delta_{N,k}(\theta) = C_k\, 2^{-2N/2^k}\, \xi_N^{1/2^k}\,[1 + o(1)],} \tag{1141}$$

with $C_k > 0$. Equation (1141), rather than an arbitrarily cut off power, is the actual finite-N Lambert inner profile.

88 Comparison with the critical fractional path

Part 45 gave, up to uniformly positive coefficient factors,

$$\sigma_{N,\alpha}^{\mathrm{crit}} \asymp \left[|S_{N,j_c}| A_*^{k(N)-j_c}\right]^{1/d_\alpha}, \qquad d_\alpha = \frac{2\alpha^2}{7-2\alpha}. \tag{1142}$$

The correct comparison now involves two distinct scales:

$$\boxed{\sigma_{\varepsilon,N} \asymp L_N^{-2}} \tag{1143}$$

for the squared angular radius of the denominator regularization about $\pi/2$, and

$$\boxed{\sigma_{\mathrm{Lam},N} \asymp w_N^2 \asymp 2^{-4N} L_N^{-3}} \tag{1144}$$

for the squared local Lambert branch-layer radius about $\theta_{*,N}$. They are exponentially separated:

$$\frac{\sigma_{\mathrm{Lam},N}}{\sigma_{\varepsilon,N}} \asymp 2^{-4N} L_N^{-1} \longrightarrow 0. \tag{1145}$$

If $k(N) - j_c \sim c \log N$ and the entrance sensitivity is nondegenerate, then

$$\sigma_{N,\alpha}^{\mathrm{crit}} \asymp N^{-\mu_\alpha}, \qquad \mu_\alpha = c|\log A_*| \frac{7-2\alpha}{2\alpha^2}. \tag{1146}$$

Therefore every such polynomial critical path satisfies

$$\boxed{\frac{\sigma_{\mathrm{Lam},N}}{\sigma_{N,\alpha}^{\mathrm{crit}}} \longrightarrow 0.} \tag{1147}$$

Thus the critical path is asymptotically far outside the exponentially thin Lambert branch layer. The critical fractional competition does not, under this depth law, force the solution into the actual Lambert inner layer.

The comparison with the denominator radius is different and depends on μ_α:

$$\boxed{\frac{\sigma_{N,\alpha}^{\mathrm{crit}}}{\sigma_{\varepsilon,N}} \asymp N^{2-\mu_\alpha}.} \tag{1148}$$

Hence

$$\begin{aligned} \mu_\alpha < 2 &\implies \sigma_N^{\mathrm{crit}} \gg \sigma_{\varepsilon,N}, \\ \mu_\alpha = 2 &\implies \text{comparable squared scales}, \\ \mu_\alpha > 2 &\implies \sigma_N^{\mathrm{crit}} \ll \sigma_{\varepsilon,N}. \end{aligned} \tag{1149}$$

This comparison must not be interpreted as entry into the Lambert branch layer, because the centers of the two scalings are different: the denominator regularization is centered at $\pi/2$, while the branch tangent is centered at $\theta_{*,N}$ and (1132) separates those centers by many denominator-core radii.

89 Consequence for the inner fractional block

The preceding calculation removes an ambiguity in the proposed inner fractional audit. There is no manuscript-defined cutoff profile of radius $\varepsilon_N^{\mathrm{L}}$ around the branch point whose fractional Laplacian can be computed. The finite-N scalar coefficient is generated instead by the smooth denominator (1125), the translated branch point (1129), and the renormalized inner cascade (1141).

Accordingly, the correct finite-N inner fractional object is obtained by substituting the exact regularized factors reconstructed from $\Delta_{N,k}(\theta)$ into

$$\boxed{\mathfrak{F}_{N,\alpha} = -\nu(-\Delta)^\alpha b_N + \nu \mathcal{C}_{N,\alpha}(u_{x,N}, u_{y,N}),} \tag{1150}$$

and rescaling $\theta = \theta_{*,N} + w_N\xi$. Since $w_N \ll (\varepsilon_N^{\mathrm{L}})^{1/2}$ but the branch point itself is a distance $x_{*,N} \gg (\varepsilon_N^{\mathrm{L}})^{1/2}$ from $\pi/2$, the regularized denominator is analytic on the w_N branch window. Its Taylor coefficients enter through a_N in (1135); there is no additional cutoff singularity at $|\xi| = O(1)$.

For critical paths satisfying (1147), however, the evaluation scale remains outside this w_N layer. Therefore the inner formula (1141) is not the leading profile on the critical path, and the outer signed fractional audit of Parts 46–47 remains the relevant candidate provided its nonlocal tail estimate is proved using the exact denominator profile rather than the auxiliary cutoff model.

90 Revised scalar status

We have therefore identified the actual regularization and corrected the scale comparison:

$$\boxed{\varepsilon_N^{\mathrm{L}} = \lambda L_N^{-2}, \qquad r_{\varepsilon,N} \asymp L_N^{-1}, \qquad w_N \asymp 2^{-2N} L_N^{-3/2}.} \tag{1151}$$

The earlier abstract condition $\varepsilon_N^2 = o(\sigma_N^{\mathrm{crit}})$ is not a theorem about the $\varepsilon_N^{\mathrm{L}}$ and is superseded by the two-center, two-scale comparison (1147)–(1149).

In particular, under the standard logarithmic depth law the critical path is polynomial while the genuine Lambert inner branch layer is exponential, so

$$\boxed{\sigma_{\mathrm{Lam},N} = o(\sigma_{N,\alpha}^{\mathrm{crit}}).} \tag{1152}$$

Thus the critical path does not reach the actual Lambert inner layer. What remains to certify the positive fractional sign is not an arbitrary inner cutoff computation, but a direct core–tail estimate for the exact finite-N denominator/cascade profile in (1125), (1133), and (1141).

This remains a scalar b_3 statement only. It neither proves nor disproves finite-time singularity of the reconstructed velocity field.

Part 49. Exact Nonlocal Core–Tail Audit for the Actual Finite-N Lambert Profile

The preceding part identified the genuine Lambert branch layer without introducing an auxiliary cutoff. We now use that exact layer to replace the schematic core–tail condition of (1108)–(1110) by a quantitative estimate. The conclusion has two levels. For the reduced scalar characteristic the Lambert invariant gives an explicit exponentially small core mass. For the complete reconstructed fractional block the same conclusion follows provided the square-root geometric coefficient stays uniformly separated from zero on the Lambert phase window. This distinction is essential because the Lambert phase variable and the independent periodic geometric variable are not the same Eulerian coordinate.

91 Exact periodic fractional kernel representation

For a 2π-periodic function $f = f(\theta)$, write the one-dimensional fractional operator in singular-integral form

$$(-\partial_{\theta\theta})^{\alpha} f(\theta) = c_\alpha \ \mathrm{PV} \int_{-\pi}^{\pi} \big(f(\theta) - f(\eta)\big) \mathcal{K}_\alpha(\theta - \eta)\, d\eta, \tag{1153}$$

where

$$\mathcal{K}_\alpha(h) = \sum_{m \in \mathbb{Z}} |h + 2\pi m|^{-1-2\alpha}. \tag{1154}$$

For $|h| \le h_0 < \pi$,

$$\mathcal{K}_\alpha(h) = |h|^{-1-2\alpha} + O_{h_0,\delta}(1) \tag{1155}$$

uniformly for $1 \le \alpha < 1 + \delta$.

Let

$$y = \theta - \theta_{*,N}, \qquad w_N \asymp 2^{-2N} L_N^{-3/2}, \tag{1156}$$

and fix $M > 1$. The actual Lambert core is

$$\mathcal{I}_N(M) = \{\eta : 0 < \eta - \theta_{*,N} < M w_N\}. \tag{1157}$$

For an evaluation point $y > 2M w_N$, every $\eta \in \mathcal{I}_N(M)$ satisfies

$$\frac{y}{2} \le |\theta - \eta| \le \frac{3y}{2}, \tag{1158}$$

and hence

$$\sup_{\eta \in \mathcal{I}_N(M)} \mathcal{K}_\alpha(\theta - \eta) \le C_{M,\delta} y^{-1-2\alpha}. \tag{1159}$$

Thus the contribution of the genuine Lambert core to the fractional operator obeys the exact separated-kernel estimate

$$\boxed{|\mathcal{T}_{N,\alpha}^{\mathrm{Lam}}[f](y)| \le C_{M,\delta} y^{-1-2\alpha} \left(M w_N |f(\theta)| + \int_0^{M w_N} |f(\theta_{*,N} + \rho)|\, d\rho \right).} \tag{1160}$$

No artificial cutoff has entered (1160): the interval is the actual branch window selected by the Lambert normalization $2^{2N}\Delta_{N,0} = O(1)$.

92 Exact Lambert invariant and the inner scalar amplitude

The reduced characteristic system possesses the exact invariant

$$\boxed{\Delta_{N,k} = K_N q_N e^{-q_N}, \qquad q_N = \frac{1}{b_N},} \tag{1161}$$

with $K_N > 0$ fixed along a characteristic. Equivalently,

$$q_N = -W_0\!\left(-\frac{\Delta_{N,k}}{K_N}\right), \qquad b_N = -\frac{1}{W_0(-\Delta_{N,k}/K_N)}. \tag{1162}$$

Since $W_0(-x) = -x + O(x^2)$ as $x \downarrow 0$,

$$\boxed{b_N = \frac{K_N}{\Delta_{N,k}}\big(1 + O(\Delta_{N,k})\big).} \tag{1163}$$

For fixed finite k, Part 48 gave

$$\Delta_{N,k}(\theta_{*,N} + w_N\xi) = C_k 2^{-2N/2^k}\xi^{p_k}[1 + o(1)], \qquad p_k = 2^{-k}. \tag{1164}$$

Therefore, uniformly on compact subintervals $0 < \xi \le M$,

$$\boxed{b_N(\theta_{*,N} + w_N\xi) = \frac{K_N}{C_k}2^{2N/2^k}\xi^{-p_k}[1 + o(1)].} \tag{1165}$$

For every $k \ge 1$, $0 < p_k \le 1/2$, so the inner scalar singularity is locally integrable. Changing variables $\rho = w_N\xi$ gives

$$\begin{aligned}\int_0^{Mw_N} b_N(\theta_{*,N} + \rho)\,d\rho &= \frac{K_N}{C_k}2^{2N/2^k}w_N\int_0^M \xi^{-p_k}\,d\xi\,[1 + o(1)] \\ &= \frac{K_N}{C_k}\frac{M^{1-p_k}}{1-p_k}2^{2N/2^k}w_N[1 + o(1)].\end{aligned} \tag{1166}$$

Using $w_N \asymp 2^{-2N}L_N^{-3/2}$,

$$\boxed{\int_0^{Mw_N} b_N\,d\rho \le C_{M,k}K_N 2^{-2N(1-2^{-k})}L_N^{-3/2}.} \tag{1167}$$

Thus the actual finite-N Lambert core has exponentially small scalar L^1 mass for every fixed $k \ge 1$.

93 The coupled depth is even safer

The fixed-depth expansion (1164) must not be extrapolated blindly to $k = k(N) \to \infty$. In the coupled regime the exact nonlinear Lambert map is used. Once the orbit has entered a compact subset of $(0, \Delta_*)$, the renormalized branch distance is bounded above and below by positive constants. The invariant (1161) then keeps q_N, and hence b_N, bounded on that captured portion of the inner window. Consequently

$$\boxed{\int_0^{Mw_N} b_N\,d\rho = O(w_N) = O(2^{-2N}L_N^{-3/2})} \tag{1168}$$

there. Therefore the fixed-k estimate (1167) is the worst Lambert-core mass estimate relevant to the present audit; the captured coupled cascade does not create a larger core mass merely by increasing the Lambert depth.

94 Scalar fractional tail on an outer evaluation scale

Apply (1160) to $f = b_N$. If $y_N/w_N \to \infty$, then the evaluation value in the outer tangent region satisfies

$$b_N(y_N) \asymp B_{N,\alpha} y_N^{-\beta_\alpha} \tag{1169}$$

with the outer exponent used in the scalar terminal profile. The term $w_N b_N(y_N)$ in (1160) is smaller than the local fractional scale by the factor w_N/y_N. The inner-mass term is controlled by (1167). In particular, for fixed $k \geq 1$,

$$\boxed{|\mathcal{T}^{\mathrm{Lam}}_{N,\alpha}[b_N](y_N)| \leq C y_N^{-1-2\alpha} \left[w_N b_N(y_N) + K_N 2^{-2N(1-2^{-k})} L_N^{-3/2} \right].} \tag{1170}$$

Relative to the local outer fractional magnitude $b_N(y_N) y_N^{-2\alpha}$ this yields

$$\boxed{\frac{|\mathcal{T}^{\mathrm{Lam}}_{N,\alpha}[b_N](y_N)|}{b_N(y_N) y_N^{-2\alpha}} \leq C \left[\frac{w_N}{y_N} + \frac{K_N 2^{-2N(1-2^{-k})} L_N^{-3/2}}{y_N b_N(y_N)} \right].} \tag{1171}$$

On every polynomial outer path $y_N \asymp N^{-\gamma}$ for which K_N and the outer normalization grow at most subexponentially, both terms in (1171) vanish exponentially. In particular, the polynomial critical path of Part 48 satisfies $y_N = \sqrt{\sigma^{\mathrm{crit}}_{N,\alpha}} \gg w_N$, and hence

$$\boxed{\mathcal{T}^{\mathrm{Lam}}_{N,\alpha}[b_N] = o\big(b_N(y_N) y_N^{-2\alpha}\big).} \tag{1172}$$

This proves the required core–tail separation for the reduced scalar fractional term under the displayed subexponential normalization condition.

95 Passage to the complete reconstructed fractional block

The complete block is

$$\mathfrak{F}_{N,\alpha} = -\nu(-\Delta)^\alpha b_N + \nu \mathcal{C}_{N,\alpha}(u_{x,N}, u_{y,N}). \tag{1173}$$

The active square-root reconstruction is

$$u_{y,N} = \sqrt{-2\chi b_N}, \qquad u_{x,N} = \sqrt{-\frac{b_N}{2\chi}}. \tag{1174}$$

The Lambert branch window is in the phase variable $\theta = t - z$, whereas $\chi = \chi(x)$ is an independent periodic geometric coefficient. Therefore one must not identify the Lambert distance y with the distance to a zero of χ.

Fix a transverse Eulerian point x for which

$$0 < c_\chi \leq |\chi(x)| \leq C_\chi < \infty. \tag{1175}$$

Then on the Lambert phase window

$$|u_{x,N}| + |u_{y,N}| \le C_\chi b_N^{1/2}. \tag{1176}$$

Using (1165),

$$\int_0^{Mw_N} b_N^{1/2}\, d\rho \le C_{M,k} K_N^{1/2} 2^{-2N(1-2^{-k-1})} L_N^{-3/2}. \tag{1177}$$

Hence the separated nonlocal tails generated by u_x and u_y are also exponentially small on every polynomial critical path. The bilinear commutator kernel can be written schematically as

$$\mathcal{C}_\alpha(f,g)(\theta) = c_\alpha \int (f(\theta)-f(\eta))(g(\theta)-g(\eta))\mathcal{K}_\alpha(\theta-\eta)\, d\eta, \tag{1178}$$

up to the sign convention already fixed in Part 46. On the separated core, (1159), (1167), and (1177), together with Cauchy–Schwarz or the elementary product bound, give

$$\boxed{\mathcal{T}_{N,\alpha}^{\mathrm{Lam}}[\mathcal{C}(u_x,u_y)] = o\big(P_{N,\alpha}(y_N) y_N^{-2\alpha}\big)} \tag{1179}$$

provided (1175) holds uniformly and the characteristic constant K_N and outer coefficient moduli are subexponential in N.

The condition (1175) cannot be deleted. If the Lambert phase window is simultaneously forced onto a zero of χ, the fixed-space obstruction of Part 41 reappears and the preceding one-dimensional Lambert mass estimate is not sufficient to control the reconstructed velocity commutator. That is a separate two-variable concentration problem, not an inner Lambert-tail problem.

96 Finite-N sign certification without an auxiliary cutoff

Combine the outer fractional estimate (1111) with (1172) and (1179). The abstract cutoff modulus $\omega_N^{\mathrm{core}}(R)$ can now be replaced, on the actual Lambert branch window, by

$$\Omega_{N,\alpha}^{\mathrm{Lam}}(y_N) \le C\left[\frac{w_N}{y_N} + \frac{K_N 2^{-2N(1-2^{-k})} L_N^{-3/2}}{y_N b_N(y_N)} + \Omega_{N,\alpha}^{\mathrm{rec}}(y_N)\right], \tag{1180}$$

where Ω^{rec} denotes the square-root component contribution estimated by (1177)–(1179). Under (1175), subexponential coefficient control, and a polynomial critical path,

$$\boxed{\Omega_{N,\alpha}^{\mathrm{Lam}}(y_N) \longrightarrow 0} \tag{1181}$$

uniformly for $1 \le \alpha < 1+\delta$ after decreasing δ if necessary.

Recall that Part 46 proved

$$H_1 = -\frac{72}{25} < 0 \tag{1182}$$

and, by continuity, chose $\delta > 0$ and $h_* > 0$ so that

$$H_\alpha \le -h_* < 0, \qquad 1 \le \alpha < 1 + \delta. \tag{1183}$$

Therefore the finite-N complete fractional block satisfies

$$\frac{\mathfrak{F}_{N,\alpha}}{P_{N,\alpha}} = -\nu H_\alpha \sigma^{-\alpha} + o(\sigma^{-\alpha}), \tag{1184}$$

and hence, for sufficiently large N on the certified outer/critical path,

$$\boxed{\mathfrak{F}_{N,\alpha} \ge \frac{1}{2}\nu h_* P_{N,\alpha}\sigma^{-\alpha} > 0.} \tag{1185}$$

Thus the positive fractional sign found at the Mellin tangent survives the actual exponentially thin finite-N Lambert branch core; no auxiliary cutoff is needed for this scalar core–tail separation.

97 Precise status and remaining obstruction

The result of this part is narrower than a Navier–Stokes blow-up theorem. It proves that the genuine Lambert inner layer does not reverse the positive signed scalar fractional coefficient on polynomial critical paths, subject to the explicit uniform separation (1175) and subexponential coefficient bounds. It does not resolve the independent fixed-space singularity at a zero of χ or the higher Sobolev compactness problem.

The scalar ledger is therefore sharpened to

$$\boxed{\begin{array}{c}\text{actual Lambert width } w_N \asymp 2^{-2N} L_N^{-3/2},\\ \text{actual inner invariant } b_N = -1/W_0(-\Delta_{N,k}/K_N),\\ \text{Lambert core mass exponentially small,}\\ \text{nonlocal Lambert core tail negligible on polynomial critical paths,}\\ \text{positive fractional sign survives at finite } N \text{ under (1175) and subexponential coefficient control.}\end{array}} \tag{1186}$$

The next genuinely new obstruction is therefore no longer the one-dimensional Lambert core tail. It is the compatibility of this certified scalar sign with the full reconstructed velocity geometry, especially if the terminal characteristic approaches the geometric zero set $\chi = 0$.

Part 50. Interaction of the Certified Lambert Fractional Sign with the Full Reconstructed Velocity Geometry

98 Why the remaining issue is genuinely two-variable

Part 49 certifies the finite-N scalar fractional sign on a Lambert phase window provided that the independent square-root coefficient satisfies

$$0 < c_\chi \le |\chi(x)|. \tag{1187}$$

The point of the present part is to audit exactly what happens when this hypothesis is weakened along the physical characteristic. The two small variables are

$$y = \theta - \theta_{*,N}, \qquad \theta = t - z, \qquad r = -\chi(x) > 0, \tag{1188}$$

on the selected branch. They are independent Eulerian variables. In particular, no identity of the form $r \asymp y$ is available unless it is proved from the characteristic equations.

The reconstruction is

$$b = u_x u_y, \qquad u_y = A = \sqrt{2rb}, \qquad u_x = \sqrt{\frac{b}{2r}}, \qquad u_z = C. \tag{1189}$$

Thus the scalar b may have a certified one-dimensional fractional sign while the individual velocity factors become singular as $r \downarrow 0$. This is the precise mechanism that must now be separated from the Lambert core-tail calculation.

99 Exact geometric evolution along the characteristic

For the reconstructed transport field used in the scalar equation,

$$\dot{x} = AC^2, \qquad \dot{z} = b. \tag{1190}$$

Consequently

$$\dot{\theta} = 1 - b. \tag{1191}$$

Since $r = -\chi(x)$,

$$\boxed{\dot{r} = -\chi'(x)AC^2 = -\chi'(x)\sqrt{2rb}\,C^2.} \tag{1192}$$

Equivalently, with

$$\sigma = \chi^2 = r^2, \tag{1193}$$

one obtains

$$\dot{\sigma} = 2\chi\chi'\dot{x} = 2\chi\chi' AC^2. \tag{1194}$$

For the periodic choice $\chi(x) = \sin(x - x_*)$ on the branch $\chi = -\sqrt{\sigma}$, $|\chi'| = \sqrt{1-\sigma}$. Choosing the orientation for which the characteristic moves toward $\sigma = 0$ gives the exact law

$$\boxed{\dot{\sigma} = -2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}b^{1/2}C^2.} \tag{1195}$$

This formula contains no Lambert approximation and no auxiliary cutoff.

Dividing (1195) by (1191), on the terminal branch $b > 1$, gives

$$\boxed{\frac{d\sigma}{d\theta} = 2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}\frac{b^{1/2}}{b-1}C^2.} \tag{1196}$$

Therefore the approach to $\chi = 0$ is not free data: it must be generated by this coupled characteristic equation.

100 Exact loss factors in the reconstructed Lambert core-tail estimate

The scalar estimate of Part 49 used only the L^1 mass of b. The reconstructed factors carry explicit powers of r:

$$|u_x| = \frac{b^{1/2}}{\sqrt{2}\, r^{1/2}}, \qquad |u_y| = \sqrt{2}\, r^{1/2} b^{1/2}. \tag{1197}$$

Let

$$r_{N,\min} := \inf_{|y| \le M w_N} r_N(y) \tag{1198}$$

be the smallest geometric distance reached across the actual Lambert core. Then the exact finite-N inner estimate (1177) implies

$$\boxed{\int_{|y| \le M w_N} |u_{x,N}(y)|\, dy \le C r_{N,\min}^{-1/2} K_N^{1/2} 2^{-2N(1-2^{-k-1})} L_N^{-3/2}.} \tag{1199}$$

Similarly,

$$\boxed{\int_{|y| \le M w_N} |u_{y,N}(y)|\, dy \le C r_{N,\max}^{1/2} K_N^{1/2} 2^{-2N(1-2^{-k-1})} L_N^{-3/2},} \tag{1200}$$

where $r_{N,\max} = \sup_{|y| \le M w_N} r_N(y)$. Thus u_y does not create the dangerous geometric loss; the potentially singular factor is u_x.

Equation (1199) immediately gives a quantitative replacement for the fixed separation assumption (1175). It is sufficient that

$$\boxed{r_{N,\min}^{-1/2} K_N^{1/2} 2^{-2N(1-2^{-k-1})} L_N^{-3/2} = o(\mathcal{S}_N),} \tag{1201}$$

where $\mathcal{S}_N$ denotes the outer normalization against which the commutator tail is compared. In particular, if K_N and $\mathcal{S}_N^{-1}$ are subexponential and

$$r_{N,\min} \ge e^{-o(N)}, \tag{1202}$$

then the exponentially small Lambert mass still wins. More generally, if

$$r_{N,\min} = e^{-\lambda N + o(N)}, \tag{1203}$$

then (1199) retains exponential decay whenever

$$\boxed{\frac{\lambda}{2} < 2 \log 2\, (1 - 2^{-k-1})} \tag{1204}$$

for fixed k, modulo the displayed subexponential coefficient factors. This is a sufficient threshold, not a necessary one, because the exact bilinear commutator may contain additional cancellation.

101 The square-root product cancellation and what it does not prove

Pointwise, the reconstruction has the exact product identity

$$\boxed{u_x u_y = b.} \tag{1205}$$

Hence the most singular algebraic powers $r^{-1/2}$ and $r^{1/2}$ cancel in the undifferentiated product. This is why the scalar equation can remain well organized even as the individual components become geometrically singular.

For the fractional commutator, however,

$$\mathcal{C}_\alpha(u_x, u_y)(X) = c_\alpha \,\mathrm{p.v.} \int \frac{(u_x(X) - u_x(Y))(u_y(X) - u_y(Y))}{|X - Y|^{3+2\alpha}}\, dY \tag{1206}$$

in three dimensions, with the corresponding periodic kernel on $\mathbb{T}^3$. The identity (1205) does not reduce (1206) to a scalar function of b, because the two differences sample the independent x-geometry at the remote point Y. Therefore the scalar core-tail certification cannot be promoted to the full reconstructed commutator merely from $u_x u_y = b$.

This identifies the exact remaining nonlocal datum: one needs a two-variable estimate which simultaneously resolves the Lambert phase distance and the transverse geometric distance to $\chi = 0$.

102 A two-scale core-tail modulus

Introduce the physical Lambert width w_N and a transverse geometric width $\ell_{\chi,N}$ by

$$|y| \lesssim w_N, \qquad |x - x_*| \lesssim \ell_{\chi,N}. \tag{1207}$$

Near a simple zero,

$$r = |\chi(x)| \asymp |x - x_*|. \tag{1208}$$

Define the reconstructed two-scale tail modulus

$$\boxed{\Theta^{\mathrm{rec}}_{N,\alpha}(X) := \frac{\left| \displaystyle\int_{\mathcal{Q}_N} (u_x(X) - u_x(Y))(u_y(X) - u_y(Y)) \mathcal{K}_\alpha(X - Y)\, dY \right|}{P_{N,\alpha}(X)\sigma(X)^{-\alpha}},} \tag{1209}$$

where

$$\mathcal{Q}_N = \{|y(Y)| \le M w_N,\ |x(Y) - x_*| \le M \ell_{\chi,N}\}. \tag{1210}$$

The exact condition required to extend Part 49 through the geometric zero is

$$\boxed{\sup_{X \in \mathcal{O}_N} \Theta^{\mathrm{rec}}_{N,\alpha}(X) \longrightarrow 0} \tag{1211}$$

uniformly for $1 \le \alpha < 1 + \delta$ on the chosen outer evaluation region $\mathcal{O}_N$. Unlike the old auxiliary-cutoff hypothesis, (1211) is formulated directly for the actual reconstructed velocity and the actual finite-N core.

103 Finite-energy obstruction if the geometric singularity fills an Eulerian neighborhood

There is an independent obstruction which no fractional sign calculation can remove. From (1189),

$$|u_x|^2 = \frac{b}{2r}. \tag{1212}$$

At fixed (y, z, t), if $b \to b_* > 0$ as $r \downarrow 0$, then

$$\int_0^\eta |u_x|^2 \, dr \sim \frac{b_*}{2} \int_0^\eta \frac{dr}{r} = +\infty. \tag{1213}$$

Thus an Eulerian neighborhood carrying a nonzero scalar trace at $\chi = 0$ is incompatible with finite kinetic energy. A single characteristic has measure zero, so (1213) does not follow merely from the fact that one trajectory reaches $\chi = 0$. What must be determined is the transverse measure and profile of the set of nearby trajectories.

If locally

$$b(r) = O(r^m), \tag{1214}$$

then

$$u_x = O(r^{(m-1)/2}). \tag{1215}$$

Consequently the zeroth-order energy obstruction is removed when $m > 0$ in a one-dimensional transverse slice, whereas control of higher derivatives requires progressively stronger vanishing. This is a compatibility condition on the full Eulerian reconstruction, not on the scalar characteristic alone.

104 A conditional geometric persistence theorem

Theorem 40 (Persistence of the finite-N positive fractional sign under geometric separation)**.** *Fix a sufficiently small $\delta > 0$ so that*

$$H_\alpha \le -h_* < 0, \qquad 1 \le \alpha < 1 + \delta. \tag{1216}$$

Assume the scalar hypotheses of Part 49, the polynomial outer critical path, and subexponential Lambert/coefficient normalizations. Suppose in addition that either

$$\inf_{\mathcal{Q}_N} |\chi| \ge c_\chi > 0, \tag{1217}$$

or, more generally, the actual reconstructed two-scale modulus satisfies (1211). Then

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{P_{N,\alpha}} = -\nu H_\alpha \sigma^{-\alpha} + o(\sigma^{-\alpha})} \tag{1218}$$

uniformly on the certified outer region, and therefore for sufficiently large N,

$$\boxed{\mathfrak{F}_{N,\alpha} \ge \frac{1}{2} \nu h_* P_{N,\alpha} \sigma^{-\alpha} > 0.} \tag{1219}$$

The theorem deliberately separates two statements. Part 49 proves that the one-dimensional Lambert core itself has too little mass to reverse the sign. The new hypothesis (1211) asks whether the independent square-root geometry can amplify that exponentially small core enough to destroy the same conclusion. No auxiliary cutoff is inserted.

105 What would falsify the present terminal reconstruction

The preceding analysis gives a concrete obstruction test. The present terminal reconstruction would fail to inherit the certified scalar sign if the characteristic geometry forced

$$r_{N,\min} \downarrow 0 \tag{1220}$$

so rapidly, or on so large a transverse set, that the two-scale modulus (1209) remained order one or diverged. Even if the scalar fractional block stayed positive, a nonintegrable u_x neighborhood as in (1213) would also prevent the reconstructed field from belonging to the finite-energy class required by the Navier–Stokes problem.

Conversely, proving (1211), together with the necessary fixed-space energy and Sobolev estimates, would remove this particular geometric obstruction. Pressure compatability also needs to be checked.

106 Updated proof ledger

The current logical status is

Lambert scalar core tail:	certified negligible on the stated paths,
finite-N scalar fractional sign:	positive on the certified overlap,
fixed separation from $\chi = 0$:	sufficient for reconstructed persistence,
approach to $\chi = 0$:	reduced to the exact two-scale modulus (1209),
finite-energy compatibility:	requires transverse Eulerian control.

(1221)

The next calculation should therefore evaluate (1209) using the explicit continuity reconstruction for $C(x, s)$ and the exact characteristic map (1195)–(1196), rather than imposing an artificial relation between the Lambert phase distance and $|\chi|$.

107 Part 51: Can the exact characteristic map prove the reconstructed two-scale modulus?

273. The precise question

Part 50 isolated the remaining reconstructed core–tail quantity

$$\Theta^{\mathrm{rec}}_{N,\alpha}(X) := \frac{\left| \displaystyle\int_{\mathcal{Q}_N} (u_x(X) - u_x(Y))(u_y(X) - u_y(Y)) \mathcal{K}_\alpha(X - Y)\, dY \right|}{P_{N,\alpha}(X)\sigma(X)^{-\alpha}}. \tag{1222}$$

The proposed next step was to prove

$$\boxed{\sup_{X\in\mathcal{O}_N}\Theta^{\mathrm{rec}}_{N,\alpha}(X)\to 0} \tag{1223}$$

from the exact continuity reconstruction and characteristic map. This part performs that audit without adding an auxiliary cutoff and, crucially, separates what follows from the present equations from what would require a new transverse hypothesis.

274. Exact Eulerian reconstruction and exact Lagrangian map

The active Eulerian reconstruction keeps the two variables independent:

$$s=t-z-\theta_{*,N},\qquad \chi=\sin(x-x_*),\qquad \sigma=\chi^2, \tag{1224}$$

with

$$u_y=A=\sqrt{-2\chi b},\qquad u_x=\frac{b}{A}=\sqrt{-\frac{b}{2\chi}},\qquad u_z=C. \tag{1225}$$

On the negative sine branch, $\chi=-\sqrt{\sigma}$, and the exact continuity reconstruction is

$$\boxed{C_N(s,\sigma)=c_{0,N}+\frac{\sqrt{2B_N}}{4}(1-\sigma)^{1/2}\sigma^{-3/4}I_N(s),} \tag{1226}$$

where

$$I_N(s)=\int_0^s r^{-\beta/2}F_N(r)^{1/2}\,dr. \tag{1227}$$

Along a selected characteristic one has exactly

$$\dot s=1-b_N,\qquad \dot\sigma=-2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}b_N^{1/2}C_N^2, \tag{1228}$$

and hence

$$\boxed{\frac{d\sigma}{ds}=2\sqrt{2}(1-\sigma)^{1/2}\sigma^{3/4}\frac{b_N^{1/2}}{b_N-1}C_N^2.} \tag{1229}$$

Under the terminal scalar asymptotic

$$b_N(s)=B_Ns^{-\beta}F_N(s),\qquad F_N(s)\to F_{0,N}>0,\qquad 0<\beta<2, \tag{1230}$$

the self-consistent dominant balance already proved in Part 38 gives

$$\boxed{\sigma_N(s)\sim\mathfrak{K}_Ns^{p_\beta},\qquad p_\beta=\frac{12-2\beta}{7},} \tag{1231}$$

and

$$\boxed{C_N(\sigma)\asymp\sigma^{-(\beta+1)/(6-\beta)}.} \tag{1232}$$

Equations (1226)–(1232) are Lagrangian information about the selected trajectory after the Eulerian field has been reconstructed. They do not identify s with σ throughout an Eulerian neighborhood.

275. Exact algebraic reduction of the commutator numerator

At two points X, Y on the selected sign branch set

$$b_X = b(X), \quad b_Y = b(Y), \quad r_X = |\chi(X)|, \quad r_Y = |\chi(Y)|. \tag{1233}$$

Then

$$u_x = \frac{b^{1/2}}{\sqrt{2}\, r^{1/2}}, \qquad u_y = \sqrt{2}\, b^{1/2} r^{1/2}. \tag{1234}$$

A direct multiplication, with no asymptotic approximation, gives

$$\begin{aligned} &(u_x(X) - u_x(Y))(u_y(X) - u_y(Y)) \\ &\quad = b_X + b_Y - \sqrt{b_X b_Y} \left[\left(\frac{r_Y}{r_X} \right)^{1/2} + \left(\frac{r_X}{r_Y} \right)^{1/2} \right]. \end{aligned} \tag{1235}$$

Equivalently, with $q = (r_Y/r_X)^{1/2} > 0$,

$$\boxed{D(X,Y) := (u_x(X) - u_x(Y))(u_y(X) - u_y(Y)) = b_X + b_Y - \sqrt{b_X b_Y}(q + q^{-1}).} \tag{1236}$$

This identity exposes the exact geometric difficulty. The product identity $u_x u_y = b$ removes the pointwise square-root singularity, but the commutator contains the ratios r_Y/r_X and r_X/r_Y. Therefore control of b along one characteristic does not by itself control the bilinear difference kernel.

A useful absolute bound is

$$\boxed{|D(X,Y)| \le b_X + b_Y + \sqrt{b_X b_Y} \left[\left(\frac{r_Y}{r_X} \right)^{1/2} + \left(\frac{r_X}{r_Y} \right)^{1/2} \right].} \tag{1237}$$

276. What the characteristic map actually controls

Let $X_N(t)$ be the distinguished characteristic. From (1231),

$$r_N(t) := |\chi(x_N(t))| = \sigma_N(t)^{1/2} \sim \mathfrak{K}_N^{1/2} s_N(t)^{p_\beta/2}. \tag{1238}$$

Thus the exact reconstruction and characteristic map determine the rate at which *that trajectory* approaches the geometric zero. They also give C_N on that trajectory through (1232).

However, the supremum in (1223) is Eulerian and the integral in (1222) samples a set of remote points $Y \in \mathcal{Q}_N$. To estimate it one needs, simultaneously,

$$r(Y), \qquad b(Y), \qquad |X - Y|, \tag{1239}$$

for a positive-measure family of points Y, not only for the selected curve. Neither (1228) nor (1229) supplies a Jacobian or a transverse stability estimate for the map from nearby initial characteristics to Y. Consequently the one-trajectory law (1238) cannot by itself imply (1223).

Theorem 41 (No-go theorem for deriving the two-scale modulus from the present one-characteristic data)**.** *Assume only the Eulerian reconstruction (1224)–(1227), the distinguished characteristic laws (1228)–(1229), and the asymptotic relation (1231) along that distinguished characteristic. These data do not imply*

$$\sup_{X\in\mathcal{O}_N}\Theta^{\mathrm{rec}}_{N,\alpha}(X)\to 0. \tag{1240}$$

More precisely, they leave undetermined the transverse distribution of $r(Y)=|\chi(Y)|$, the transverse scalar profile $b(Y)$, and the distance of $\mathcal{Q}_N$ from the evaluation set $\mathcal{O}_N$. Since all three enter (1237) and the singular kernel $\mathcal{K}_\alpha(X-Y)$, two Eulerian fields may agree on the distinguished characteristic and satisfy the same one-curve law (1231), while producing different values of $\Theta^{\mathrm{rec}}_{N,\alpha}$.

Proof. The characteristic equations constrain only the trace of the reconstructed field on $X_N(t)$. Modify the field away from this curve inside a shrinking transverse tube, leaving its trace and all quantities in (1228)–(1231) unchanged. Such a modification can alter r_Y and b_Y in (1236), and can place positive measure arbitrarily close to the singular kernel region $Y=X$. Hence the integral in (1222) is not determined by the one-curve data. Therefore no bound tending to zero can be deduced from those data alone. □

This is a logical obstruction, not a claim that (1223) is false for the actual solution. It identifies the missing estimate needed to prove it.

277. A sufficient two-scale theorem that can be proved

The preceding no-go result suggests the minimal quantitative information. Let

$$d_N:=\operatorname{dist}(\mathcal{O}_N,\mathcal{Q}_N)>0,\qquad V_N:=|\mathcal{Q}_N|, \tag{1241}$$

and suppose that on $\mathcal{O}_N\cup\mathcal{Q}_N$

$$0<r_{-,N}\le r\le r_{+,N},\qquad 0<b\le B_N^{\max}. \tag{1242}$$

For the periodic fractional kernel there is a constant C_δ, uniform for $1\le\alpha<1+\delta$, such that whenever $X\in\mathcal{O}_N$ and $Y\in\mathcal{Q}_N$,

$$|\mathcal{K}_\alpha(X-Y)|\le C_\delta d_N^{-3-2\alpha}. \tag{1243}$$

Using (1237),

$$|D(X,Y)|\le 2B_N^{\max}\left[1+\left(\frac{r_{+,N}}{r_{-,N}}\right)^{1/2}\right]. \tag{1244}$$

Therefore

$$\boxed{\sup_{X\in\mathcal{O}_N}\Theta^{\mathrm{rec}}_{N,\alpha}(X)\le C_\delta\frac{B_N^{\max}V_N}{P_{N,\alpha}^{\min}}\left[1+\left(\frac{r_{+,N}}{r_{-,N}}\right)^{1/2}\right]d_N^{-3-2\alpha}(\sigma_N^{\max})^\alpha.} \tag{1245}$$

Here $P_{N,\alpha}^{\min}=\inf_{\mathcal{O}_N}P_{N,\alpha}$ and $\sigma_N^{\max}=\sup_{\mathcal{O}_N}\sigma$; the last factor follows from $(P\sigma^{-\alpha})^{-1}\le(P_{N,\alpha}^{\min})^{-1}(\sigma_N^{\max})^\alpha$.

Theorem 42 (Sufficient Eulerian two-scale criterion)**.** *If*

$$\boxed{\sup_{1\le\alpha<1+\delta} \frac{B_N^{\max} V_N}{P_{N,\alpha}^{\min}} \left[1+\left(\frac{r_{+,N}}{r_{-,N}}\right)^{1/2}\right] d_N^{-3-2\alpha} (\sigma_N^{\max})^{\alpha} \longrightarrow 0,} \tag{1246}$$

then

$$\boxed{\sup_{1\le\alpha<1+\delta} \sup_{X\in\mathcal{O}_N} \Theta_{N,\alpha}^{\mathrm{rec}}(X) \longrightarrow 0.} \tag{1247}$$

The proof is exactly (1243)–(1245). Unlike an auxiliary cutoff condition, (1246) is a statement about the actual Eulerian core volume, geometric separation, scalar amplitude, and kernel distance.

278. Incorporating the actual Lambert width

Part 49 gives the genuine Lambert phase width

$$w_N \asymp 2^{-2N} L_N^{-3/2}. \tag{1248}$$

If the core occupies a phase interval of width $O(w_N)$ and a transverse geometric region of measure $A_{\chi,N}$, then

$$V_N \lesssim w_N A_{\chi,N}. \tag{1249}$$

Substitution into (1245) yields

$$\boxed{\sup_X \Theta_{N,\alpha}^{\mathrm{rec}}(X) \lesssim \frac{B_N^{\max}}{P_{N,\alpha}^{\min}} 2^{-2N} L_N^{-3/2} A_{\chi,N} \left[1+\left(\frac{r_{+,N}}{r_{-,N}}\right)^{1/2}\right] d_N^{-3-2\alpha} (\sigma_N^{\max})^{\alpha}.} \tag{1250}$$

Thus the exponential Lambert width wins against every collection of factors on the right that grows only subexponentially in N. In particular, a sufficient outer-core regime is

$$\log \frac{B_N^{\max}}{P_{N,\alpha}^{\min}} = o(N), \quad \log A_{\chi,N}^{-1} = o(N), \quad \log r_{-,N}^{-1} = o(N), \quad \log d_N^{-1} = o(N), \tag{1251}$$

with polynomial σ_N. Under (1251), (1247) follows.

This recovers and strengthens the qualitative conclusion of Part 50: the Lambert core is exponentially thin enough to survive subexponential geometric deterioration. What is *not* yet derived from the current PDE is that the actual Eulerian transverse quantities in (1251) obey those bounds.

279. Consequence for the scalar differential inequality

Suppose now that (1246), or the sufficient specialization (1251), has been proved for the actual Eulerian solution. Then Part 50 gives uniformly on the certified overlap

$$\mathfrak{F}_{N,\alpha} \ge \frac{1}{2} \nu h_* P_{N,\alpha} \sigma^{-\alpha} > 0. \tag{1252}$$

This term is favorable for a lower bound on $\dot{b}_N$; it does not need to be discarded as a negative viscous error.

Write the exact scalar characteristic equation in the form

$$\dot{b}_N = \Gamma_N b_N^2 + \mathfrak{F}_{N,\alpha} + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}}. \tag{1253}$$

Therefore a sufficient condition for a Riccati lower inequality is

$$\boxed{\mathcal{R}_N^{\mathrm{rec}} - 2C_N G_N \le (1-c)\Gamma_N b_N^2 \qquad \text{for some } c \in (0,1),} \tag{1254}$$

because (1252) then gives

$$\boxed{\dot{b}_N \ge c\,\Gamma_N b_N^2.} \tag{1255}$$

Integrating (1255) gives

$$\frac{d}{dt}\frac{1}{b_N(t)} = -\frac{\dot{b}_N}{b_N^2} \le -c\Gamma_N(t), \tag{1256}$$

so

$$\boxed{\frac{1}{b_N(t)} \le \frac{1}{b_N(t_e)} - c\int_{t_e}^{t} \Gamma_N(\tau)\,d\tau.} \tag{1257}$$

Consequently, if for some finite $T_N > t_e$

$$\boxed{c\int_{t_e}^{T_N} \Gamma_N(\tau)\,d\tau > \frac{1}{b_N(t_e)},} \tag{1258}$$

then a positive classical solution satisfying (1255) cannot remain finite up to T_N. It must become unbounded at or before the first time at which the right side of (1257) reaches zero.

Theorem 43 (Conditional finite-time scalar blow-up criterion)**.** *Assume on a terminal characteristic that $b_N(t_e) > 0$, $\Gamma_N \ge 0$, the reconstructed two-scale estimate (1247), the positive fractional sign (1252), and the remainder inequality (1254) hold throughout $[t_e, T_N)$. If (1258) holds for a finite T_N, then*

$$\boxed{\limsup_{t\uparrow T_{*,N}} b_N(t) = +\infty} \tag{1259}$$

for some $T_{*,N} \le T_N$.

The proof is the reciprocal comparison (1256)–(1258). This theorem is a rigorous implication, but its hypotheses (1246)/(1251), (1254), and (1258) must still be verified from the actual finite-N reconstruction before it can be called an unconditional blow-up theorem for the scalar PDE.

280. Updated status: what was proved and what remains

The attempted proof of (1223) has therefore produced a sharper result than a formal assertion:

one-characteristic map $s \leftrightarrow \sigma$:	insufficient by itself for the Eulerian supremum,
exact square-root difference identity (1236):	proved,
sufficient two-scale bound (1245):	proved,
Lambert-width specialization (1250):	proved,
subexponential criterion (1251):	sufficient, but not yet derived from the PDE,
Riccati implication (1255)–(1259):	rigorous once the displayed hypotheses are verified.

(1260)

Thus the present manuscript must not state that $\sup_X \Theta^{\mathrm{rec}}_{N,\alpha} \to 0$ follows solely from the exact characteristic map. The missing object is an Eulerian transverse stability or Jacobian estimate for the family of characteristics. Proving such an estimate, together with the signed reconstruction remainder bound (1254), is the next scalar-only task.

108 Part 52: Eulerian transverse Jacobian and signed reconstruction-remainder audit

281. Triangular characteristic structure of the active Eulerian reconstruction

The scalar transport field appearing in the corrected b_3 equation is

$$V_N = (A_N C_N^2,\, b_N C_N^2/A_N,\, b_N). \tag{1261}$$

In the active reconstruction $b_N = b_N(s)$, $A_N = A_N(x,s)$, $C_N = C_N(x,s)$ and $s = t - z - \theta_{*,N}$. Hence the (x,s) subsystem is triangular:

$$\boxed{\dot{x} = A_N(x,s) C_N(x,s)^2, \qquad \dot{s} = 1 - b_N(s).} \tag{1262}$$

In particular, neighboring characteristics having the same initial phase s_e retain the same $s(t)$; their transverse separation is governed by the one-dimensional nonautonomous x flow.

Let a denote the initial transverse coordinate and write $X_N(t;a)$ for the solution of (1262). Its transverse Jacobian

$$J_N(t;a) := \partial_a X_N(t;a) \tag{1263}$$

satisfies the exact variational equation

$$\boxed{\dot{J}_N = \partial_x (A_N C_N^2)(X_N(t;a), s(t)) J_N, \qquad J_N(t_e;a) = 1.} \tag{1264}$$

Thus

$$\boxed{J_N(t;a) = \exp\left(\int_{t_e}^{t} \partial_x (A_N C_N^2)(X_N(\tau;a), s(\tau))\, d\tau \right) > 0.} \tag{1265}$$

This is the missing Eulerian Jacobian formula; it follows from the actual reconstruction and does not impose $b = b(\sigma)$ Eulerianly.

282. Exact terminal form of the transverse velocity

Put $r = -\chi(x) > 0$ on the selected negative sine branch. Then

$$A_N = \sqrt{2rb_N}, \qquad |\chi'| = (1-r^2)^{1/2}. \tag{1266}$$

From the exact continuity reconstruction (1226),

$$C_N = c_{0,N} + D_N(s)(1-r^2)^{1/2}r^{-3/2}, \qquad D_N(s) := \frac{\sqrt{2B_N}}{4} I_N(s). \tag{1267}$$

In the deep terminal overlap in which the reconstructed term dominates $c_{0,N}$, uniformly up to a relative error $\varepsilon_{C,N} = o(1)$,

$$C_N = D_N(s)(1-r^2)^{1/2}r^{-3/2}(1 + O(\varepsilon_{C,N})). \tag{1268}$$

Consequently

$$\boxed{A_N C_N^2 = \sqrt{2b_N(s)} D_N(s)^2 (1-r^2) r^{-5/2}(1 + O(\varepsilon_{C,N})).} \tag{1269}$$

Since $r_x = -\chi' = -(1-r^2)^{1/2}$ in the chosen orientation, differentiation gives, uniformly for $r \downarrow 0$,

$$\boxed{\partial_x(A_N C_N^2) = \frac{5}{2}\sqrt{2b_N} D_N^2 r^{-7/2}(1 + O(r^2 + \varepsilon_{C,N})).} \tag{1270}$$

The same characteristic satisfies

$$\dot r = r_x \dot x = -\sqrt{2b_N} D_N^2 r^{-5/2}(1 + O(r^2 + \varepsilon_{C,N})). \tag{1271}$$

Dividing (1270) by (1271) eliminates every large coefficient:

$$\boxed{\frac{d\log J_N}{dr} = -\frac{5}{2}\frac{1}{r}(1 + O(r^2 + \varepsilon_{C,N})).} \tag{1272}$$

Therefore, between an entry radius r_e and a terminal radius $r < r_e$,

$$\boxed{J_N(r;a) = \left(\frac{r_e}{r}\right)^{5/2} \exp\left(O(r_e^2) + O\left(\int_r^{r_e} \varepsilon_{C,N}(\rho)\frac{d\rho}{\rho} \right) \right).} \tag{1273}$$

Thus the transverse distortion is algebraic in r^{-1}, not exponential in the Lambert depth. On every polynomial critical path $r_N = \sigma_N^{1/2} = N^{-O(1)}$, and under a uniform integrable relative reconstruction error,

$$\boxed{\log J_N = O(\log N) = o(N).} \tag{1274}$$

The inverse Jacobian obeys the corresponding lower bound. Hence an initial transverse set whose measure and distance from the evaluation region have subexponential N dependence remains subexponential after transport through the terminal overlap.

Theorem 44 (Eulerian transverse/Jacobian estimate)**.** *Assume the finite-N reconstruction (1266)–(1268) on a common terminal overlap, with*

$$\sup_N \int_0^{r_e} \varepsilon_{C,N}(r)\frac{dr}{r} < \infty. \tag{1275}$$

Then the transverse flow is orientation preserving and there exist uniform constants $c_J, C_J > 0$ *such that*

$$\boxed{c_J\left(\frac{r_e}{r}\right)^{5/2} \le J_N(r;a) \le C_J\left(\frac{r_e}{r}\right)^{5/2}.} \tag{1276}$$

In particular, on polynomial critical paths all Jacobian losses are subexponential in N*.*

283. Consequence for the two-scale modulus

Part 51 proved that the genuine Lambert width contributes $w_N \asymp 2^{-2N}L_N^{-3/2}$. The only new loss generated by the transported transverse family is algebraic by (1276). Therefore, provided the entry cross-section, the outer/core distance d_N, the scalar amplitude ratio $B_N^{\max}/P_{N,\alpha}^{\min}$, and the geometric ratio $r_{+,N}/r_{-,N}$ have at most polynomial (more generally subexponential) growth, (1250) becomes

$$\sup_X \Theta_{N,\alpha}^{\mathrm{rec}}(X) \le 2^{-2N}\exp(o(N)). \tag{1277}$$

Hence

$$\boxed{\sup_{1\le\alpha<1+\delta}\ \sup_{X\in\mathcal{O}_N} \Theta_{N,\alpha}^{\mathrm{rec}}(X) \longrightarrow 0.} \tag{1278}$$

This closes the missing Jacobian part of the two-scale estimate under the explicit entry-set and separation hypotheses just stated. It does not prove those hypotheses for an arbitrary global Eulerian neighborhood; they are now isolated as finite-dimensional outer-overlap data rather than an unknown characteristic Jacobian.

284. Exact simplification of the reconstruction remainder

Recall

$$\mathcal{R}_N^{\mathrm{rec}} = b_N\frac{C_{N,t}+C_{N,z}}{C_N} + 2A_Nb_NC_{N,x} + 2\frac{b_N^2}{A_N}C_{N,y}. \tag{1279}$$

For the active Eulerian reconstruction $C_N = C_N(x,s)$ with $s = t - z - \theta_{*,N}$,

$$C_{N,t} + C_{N,z} = 0. \tag{1280}$$

Moreover C_N is independent of y in the displayed reconstruction, so $C_{N,y} = 0$. Equivalently, retaining the directional notation of Part 37,

$$\boxed{\mathcal{R}_N^{\mathrm{rec}} = 2b_N\mathcal{X}C_N > 0, \qquad \mathcal{X} = A_N\partial_x + \frac{b_N}{A_N}\partial_y.} \tag{1281}$$

Using (834)–(836),

$$\mathcal{X}C_N \asymp \sigma^{-e_R}, \qquad e_R = \frac{5(\beta+1)}{2(6-\beta)}, \tag{1282}$$

and

$$b_N \asymp \sigma^{-\widehat{\rho}_\beta}, \qquad \widehat{\rho}_\beta = \frac{7\beta}{2(6-\beta)}. \tag{1283}$$

Therefore

$$\boxed{\mathcal{R}_N^{\mathrm{rec}} \asymp c_{\mathrm{rec},N}\, \sigma^{-\frac{12\beta+5}{2(6-\beta)}}, \qquad c_{\mathrm{rec},N} > 0.} \tag{1284}$$

285. Comparison with the Riccati coefficient

The Lambert sensitivity has the form

$$\Gamma_N = \frac{g_{N,\alpha}}{s}, \qquad g_{N,\alpha} > 0, \tag{1285}$$

and the exact characteristic map gives

$$s \asymp \sigma^{7/[2(6-\beta)]}. \tag{1286}$$

Hence

$$\boxed{\Gamma_N b_N^2 \asymp c_{Q,N} g_{N,\alpha} \sigma^{-\frac{14\beta+7}{2(6-\beta)}}.} \tag{1287}$$

Subtracting the powers in (1284) and (1287) gives

$$\frac{14\beta+7-(12\beta+5)}{2(6-\beta)} = \frac{\beta+1}{6-\beta} =: q_\beta > 0. \tag{1288}$$

Consequently

$$\boxed{\frac{\mathcal{R}_N^{\mathrm{rec}}}{\Gamma_N b_N^2} \le C_{\mathrm{rec}} \frac{\sigma^{q_\beta}}{g_{N,\alpha}} (1+o(1)).} \tag{1289}$$

This is the precise coupled-limit condition hidden by a fixed-N exponent comparison. At each fixed finite N, the ratio tends to zero as $\sigma \downarrow 0$. Uniformly in N it tends to zero along every path satisfying

$$\boxed{\frac{\sigma_N^{q_\beta}}{g_{N,\alpha}} \longrightarrow 0.} \tag{1290}$$

286. The forcing term relative to the Riccati scale

Assume, as in the scalar core-tail audit, that the forcing profile is uniformly bounded on the terminal overlap,

$$|G_N| \le G_*. \tag{1291}$$

Since $C_N \asymp \sigma^{-q_\beta}$, (1287) gives

$$\boxed{\frac{2|C_N G_N|}{\Gamma_N b_N^2} \le C_G \frac{\sigma^{m_\beta}}{g_{N,\alpha}} (1+o(1)), \qquad m_\beta := \frac{12\beta+5}{2(6-\beta)} > 0.} \tag{1292}$$

Because

$$m_\beta - q_\beta = \frac{10\beta + 3}{2(6-\beta)} > 0, \tag{1293}$$

the reconstruction remainder is the slower-decaying of these two normalized errors. Thus the single condition (1290) implies both

$$\frac{\mathcal{R}_N^{\mathrm{rec}}}{\Gamma_N b_N^2} \to 0, \qquad \frac{2|C_N G_N|}{\Gamma_N b_N^2} \to 0, \tag{1294}$$

provided $0 < \sigma_N < 1$ and the coefficient prefactors are uniformly bounded. In particular,

$$\boxed{\frac{\mathcal{R}_N^{\mathrm{rec}} - 2C_N G_N}{\Gamma_N b_N^2} \le C \frac{\sigma_N^{q_\beta}}{g_{N,\alpha}} + o(1).} \tag{1295}$$

287. Uniform Riccati lower inequality on the terminal overlap

Choose any $c \in (0,1)$. If the coupled terminal path is chosen so that for all sufficiently large N

$$C \frac{\sigma_N^{q_\beta}}{g_{N,\alpha}} \le 1 - c, \tag{1296}$$

then (1295), after shrinking the terminal overlap if necessary, gives

$$\mathcal{R}_N^{\mathrm{rec}} - 2C_N G_N \le (1-c)\Gamma_N b_N^2. \tag{1297}$$

Together with the positive finite-N fractional block certified in Parts 46–50 and the two-scale persistence (1278), the exact scalar equation therefore yields

$$\boxed{\dot b_N \ge c\,\Gamma_N b_N^2.} \tag{1298}$$

Thus the signed remainder estimate requested after Part 51 closes on every coupled path satisfying (1290), subject to the explicit uniform coefficient, entry-set, and separation assumptions of this part.

288. What is still required for actual finite-time blow-up

The reciprocal comparison now gives

$$\frac{1}{b_N(t)} \le \frac{1}{b_N(t_e)} - c \int_{t_e}^{t} \Gamma_N(\tau)\, d\tau. \tag{1299}$$

Accordingly finite-time blow-up follows if the accumulated Riccati coefficient crosses the reciprocal entry amplitude:

$$\boxed{c \int_{t_e}^{T_N} \Gamma_N(\tau)\, d\tau > \frac{1}{b_N(t_e)}.} \tag{1300}$$

The present calculation has proved the missing Jacobian estimate and the relative reconstruction/forcing estimate only under the displayed terminal overlap hypotheses. It has

not proved (1300). Moreover, because $\Gamma_N = g_{N,\alpha}/s$ and $\dot{s} = 1 - b_N$, the time integral in (1300) must be evaluated along the actual accelerating characteristic; it cannot be replaced by a formal $\int ds/s$ without the factor $dt = ds/(1 - b_N)$. This is the next decisive scalar calculation.

Theorem 45 (Conditional closure of the two missing scalar estimates)**.** *Fix $1 \le \alpha < 1 + \delta$ and $0 < \beta < 2$. Assume the uniform terminal profile bounds used in Parts 38–41, the dominance and error condition (1275), subexponential entry geometry and outer/core separation, bounded forcing (1291), the finite-N positive fractional sign, and the coupled path condition (1290). Then*

$$\boxed{\sup_X \Theta^{\mathrm{rec}}_{N,\alpha}(X) \to 0, \qquad \frac{\mathcal{R}^{\mathrm{rec}}_N - 2C_N G_N}{\Gamma_N b_N^2} \to 0,} \tag{1301}$$

and for every fixed $c \in (0,1)$ the terminal overlap can be chosen so that

$$\boxed{\dot{b}_N \ge c\Gamma_N b_N^2.} \tag{1302}$$

The theorem is a conditional finite-N/coupled-limit closure of the two estimates requested after Part 51; it is not yet an unconditional finite-time blow-up theorem because the integral threshold (1300) remains to be proved.

289. Updated scalar proof ledger

The scalar-only status is now

transverse variational equation:	exactly derived,
terminal Jacobian distortion:	$J_N \asymp (r_e/r)^{5/2}$ under (1275),
two-scale modulus:	closed under subexponential entry/separation data,
reconstruction remainder/Riccati ratio:	$O(\sigma^{q_\beta}/g_{N,\alpha})$,
forcing/Riccati ratio:	$O(\sigma^{m_\beta}/g_{N,\alpha})$,
Riccati lower inequality:	closed under (1290),
finite-time threshold:	still requires evaluation of $\int \Gamma_N dt$.

(1303)

The next scalar calculation should therefore evaluate the accumulated coefficient in (1300) using $\dot{s} = 1 - b_N$ and the already established terminal profile, rather than assuming that the logarithmic singularity of $\Gamma_N = g_{N,\alpha}/s$ is automatically nonintegrable in physical time.

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290. The threshold integral in the phase variable

We now evaluate the quantity left open in (1300). Along the terminal characteristic the two exact identities are

$$\Gamma_N(t) = \frac{g_{N,\alpha}}{s(t)}, \qquad \dot{s}(t) = 1 - b_N(t). \tag{1304}$$

On the large-amplitude branch $b_N > 1$, s decreases as physical time increases. Therefore, if $s_e = s(t_e)$ and $s = s(t) < s_e$, then

$$\boxed{\int_{t_e}^{t} \Gamma_N(\tau)\, d\tau = g_{N,\alpha} \int_{s}^{s_e} \frac{d\xi}{\xi\,[b_N(\xi) - 1]}.} \tag{1305}$$

This is the exact physical-time formula. The factor $(b_N - 1)^{-1}$ is the factor which was absent from the formal logarithmic argument.

291. Two-sided terminal profile bounds make the integral finite

Assume the terminal profile bounds already used in the preceding characteristic analysis: for some $0 < \beta < 2$ and positive constants B_-, B_+ independent of the terminal phase,

$$\boxed{B_-\, \xi^{-\beta} \le b_N(\xi) \le B_+\, \xi^{-\beta}, \qquad 0 < \xi \le s_e.} \tag{1306}$$

Shrink s_e so that $B_- s_e^{-\beta} \ge 2$. Then

$$b_N(\xi) - 1 \ge \frac{1}{2} b_N(\xi) \ge \frac{B_-}{2} \xi^{-\beta}, \tag{1307}$$

and (1305) gives

$$\boxed{0 \le \int_{t_e}^{T_N} \Gamma_N(t)\, dt \le \frac{2 g_{N,\alpha}}{B_- \beta} s_e^{\beta} < \infty.} \tag{1308}$$

Thus the singular factor $1/s$ in Γ_N is integrable in physical time once the accelerating characteristic law is included.

If the sharper asymptotic

$$b_N(s) = B_N F_{0,N} s^{-\beta}(1 + o(1)) \tag{1309}$$

holds, then dominated asymptotics in (1305) yield

$$\boxed{\int_{t_e}^{T_N} \Gamma_N(t)\, dt = \frac{g_{N,\alpha}}{\beta B_N F_{0,N}} s_e^{\beta}(1 + o(1)).} \tag{1310}$$

At entry,

$$\frac{1}{b_N(t_e)} = \frac{1}{B_N F_{0,N}} s_e^{\beta}(1 + o(1)), \tag{1311}$$

so the ratio appearing in the reciprocal threshold is

$$\boxed{\frac{c \displaystyle\int_{t_e}^{T_N} \Gamma_N(t)\, dt}{1/b_N(t_e)} = \frac{c\, g_{N,\alpha}}{\beta}(1 + o(1)).} \tag{1312}$$

Consequently (1300) can hold asymptotically through this terminal interval only if

$$\boxed{c\, g_{N,\alpha} > \beta + o(1).} \tag{1313}$$

292. The Lambert coupled limit gives the opposite regime

The Lambert sensitivity estimates used earlier give $g_{N,\alpha} = |S_{N,k(N)}|$, with the global contraction bound

$$g_{N,\alpha} \le M_S 2^{-k(N)}. \tag{1314}$$

Hence whenever $k(N) \to \infty$,

$$\boxed{g_{N,\alpha} \longrightarrow 0.} \tag{1315}$$

For every fixed $\beta > 0$ and fixed $c \in (0,1)$, (1312) therefore implies

$$\boxed{\frac{c\displaystyle\int_{t_e}^{T_N} \Gamma_N(t)\,dt}{1/b_N(t_e)} \longrightarrow 0.} \tag{1316}$$

In particular, the proposed threshold (1300) is not merely unproved on this coupled terminal regime: under the two-sided profile asymptotic and the Lambert contraction $g_{N,\alpha} \to 0$, it is asymptotically false.

A version requiring only two-sided bounds follows from (1306). Since $1/b_N(t_e) \ge s_e^\beta/B_+$, (1308) gives

$$\frac{c\displaystyle\int_{t_e}^{T_N} \Gamma_N dt}{1/b_N(t_e)} \le \frac{2cB_+}{B_-\beta} g_{N,\alpha}. \tag{1317}$$

Thus (1300) fails for all sufficiently large N whenever $g_{N,\alpha} \to 0$ and the ratio B_+/B_- is uniformly bounded.

293. Compatibility check with the Riccati lower inequality

There is a useful independent check. From

$$\dot b_N \ge c\frac{g_{N,\alpha}}{s} b_N^2, \qquad \dot s = 1 - b_N < 0, \tag{1318}$$

division by the negative quantity $\dot s$ reverses the inequality:

$$\frac{db_N}{ds} \le -c\frac{g_{N,\alpha}}{s}\frac{b_N^2}{b_N - 1}. \tag{1319}$$

For $b_N \to \infty$ this is

$$\frac{d\log b_N}{d\log s} \le -c g_{N,\alpha}(1 + o(1)). \tag{1320}$$

A power profile $b_N \sim B s^{-\beta}$ has left side $-\beta + o(1)$, so compatibility requires

$$\boxed{\beta \ge c g_{N,\alpha} + o(1).} \tag{1321}$$

This is exactly opposite to the strict condition (1313) required for the reciprocal threshold. Thus the terminal power profile and the Riccati lower comparison are mutually consistent, but the lower comparison by itself is too weak to force the reciprocal to cross zero on the same terminal interval.

294. Finite physical time of the terminal power profile

The failure of (1300) must not be confused with a proof that the terminal profile cannot blow up. If the two-sided profile (1306) is independently proved to persist all the way to $s = 0$, then the exact phase law itself gives finite physical terminal time:

$$T_N - t = \int_0^{s(t)} \frac{d\xi}{b_N(\xi) - 1}. \tag{1322}$$

Using (1309),

$$\boxed{T_N - t = \frac{s(t)^{\beta+1}}{(\beta+1)B_N F_{0,N}}(1 + o(1)).} \tag{1323}$$

Therefore

$$s(t) \sim [(\beta+1)B_N F_{0,N}(T_N - t)]^{1/(\beta+1)}, \tag{1324}$$

and

$$\boxed{b_N(t) \sim (B_N F_{0,N})^{1/(\beta+1)}(\beta+1)^{-\beta/(\beta+1)}(T_N - t)^{-\beta/(\beta+1)}.} \tag{1325}$$

For the similarity value $\beta = 2\alpha - 1$, this becomes

$$\boxed{b_N(t) \asymp (T_N - t)^{-(2\alpha-1)/(2\alpha)}.} \tag{1326}$$

Thus a rigorously persistent terminal profile would itself imply finite-time scalar blow-up. What remains to be proved is the persistence/existence of that profile for the actual finite-N scalar PDE up to $s = 0$; it cannot be replaced by the failed integral threshold (1300).

Theorem 46 (Resolution of the integral-threshold question)**.** *Assume the two-sided terminal profile (1306), $b_N > 1$, the exact phase law $\dot{s} = 1 - b_N$, and $\Gamma_N = g_{N,\alpha}/s$. Then the accumulated Riccati coefficient is finite and satisfies (1308). If in addition (1309) holds, its sharp terminal asymptotic is (1310), and threshold (1300) is equivalent at leading order to $cg_{N,\alpha} > \beta$. Consequently, on the Lambert coupled regime $g_{N,\alpha} \to 0$ with fixed $\beta > 0$, threshold (1300) fails for all sufficiently large N. If the terminal profile is independently known to persist to $s = 0$, then nevertheless $s = 0$ is reached in finite physical time and b_N obeys the blow-up law (1325).*

295. Updated scalar proof ledger

The corrected scalar status is

two-scale reconstructed modulus:	closed under the hypotheses of Part 51,
signed reconstruction/forcing ratio:	closed under (1290),
Riccati lower inequality:	$\dot{b}_N \geq c\Gamma_N b_N^2$,
physical-time $\int \Gamma_N dt$:	finite and explicitly evaluated,
threshold (1300):	fails when $g_{N,\alpha} \to 0$ with fixed $\beta > 0$,
finite-time scalar profile law:	follows if terminal-profile persistence is proved,
remaining scalar gap:	prove actual finite-N profile persistence to $s = 0$.

(1327)

The next scalar step is therefore not to seek a larger value of the integral in (1300). It is to close a bootstrap which proves that the actual finite-N solution remains trapped between two positive multiples of $s^{-\beta}$ all the way to the terminal endpoint, with the already audited fractional, reconstruction, forcing, and transverse errors remaining subordinate.

110 Part 53: Two-Sided Terminal Persistence Audit and the Coefficient-Matching Obstruction

296. Exact normalized equation for the proposed terminal profile

Part 52 showed that the reciprocal threshold (1300) cannot close the scalar argument in the Lambert coupled regime. We now test the more decisive proposal

$$b_N(s) = B_N s^{-\beta} F_N(s), \qquad 0 < F_- \le F_N(s) \le F_+ < \infty, \tag{1328}$$

all the way to $s = 0$. This test must be performed from the exact characteristic equation rather than by assuming the desired power in the coefficient estimates.

Along the large-amplitude branch,

$$\dot{s} = 1 - b_N < 0. \tag{1329}$$

Introduce the exact dimensionless physical-time growth coefficient

$$\boxed{H_N(s) := \frac{s\,\dot{b}_N}{b_N^2}.} \tag{1330}$$

Since $db_N/ds = \dot{b}_N/(1-b_N)$, direct differentiation of $F_N = B_N^{-1} s^{\beta} b_N$ gives the exact identity

$$\boxed{\frac{d\log F_N}{d\log s} = \beta - \frac{H_N(s)}{1 - b_N(s)^{-1}}.} \tag{1331}$$

No asymptotic approximation has entered (1331). It is the correct equation for a two-sided terminal bootstrap.

Integrating between s and an entry point s_e gives

$$\boxed{F_N(s) = F_N(s_e) \exp\left\{ - \int_s^{s_e} \left[\beta - \frac{H_N(r)}{1 - b_N(r)^{-1}} \right] \frac{dr}{r} \right\}.} \tag{1332}$$

Thus the persistence question is exactly a logarithmic coefficient-matching problem.

297. A rigorous sufficient persistence theorem

Define the matching defect

$$D_N(s) := \beta - \frac{H_N(s)}{1 - b_N(s)^{-1}}. \tag{1333}$$

Suppose $b_N \geq 2$ on $(0, s_e]$ and there is a nonnegative modulus ω_N such that

$$|D_N(s)| \leq \omega_N(s), \qquad \int_0^{s_e} \omega_N(r)\frac{dr}{r} \leq M < \infty. \tag{1334}$$

Then (1332) immediately yields

$$\boxed{F_N(s_e)e^{-M} \leq F_N(s) \leq F_N(s_e)e^{M}, \qquad 0 < s \leq s_e.} \tag{1335}$$

Consequently

$$\boxed{B_N F_N(s_e)e^{-M}s^{-\beta} \leq b_N(s) \leq B_N F_N(s_e)e^{M}s^{-\beta}.} \tag{1336}$$

If, more strongly,

$$\int_0^{s_e} |D_N(r)|\frac{dr}{r} < \infty, \tag{1337}$$

then the integral in (1332) has a finite limit and

$$\boxed{F_N(s) \longrightarrow F_{0,N} \in (0, \infty).} \tag{1338}$$

This proves a genuine two-sided persistence theorem, but it also identifies the precise estimate that must be verified from the scalar PDE: the matching defect must be integrable in logarithmic phase measure.

A convenient sufficient specialization is

$$\boxed{H_N(s) = \beta + O(s^{\eta}), \qquad b_N(s)^{-1} = O(s^{\beta}), \qquad \eta > 0,} \tag{1339}$$

which gives $D_N(s) = O(s^{\min\{\eta,\beta\}})$ and hence (1334)–(1338).

298. What the presently audited scalar equation actually gives

The exact scalar characteristic equation is

$$\dot{b}_N = \Gamma_N b_N^2 + \mathfrak{F}_{N,\alpha} + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}}, \qquad \Gamma_N = \frac{g_{N,\alpha}(s)}{s}. \tag{1340}$$

Multiplying by s/b_N^2 gives the exact decomposition

$$\boxed{H_N(s) = g_{N,\alpha}(s) + \Phi_{N,\alpha}(s) + E_N(s),} \tag{1341}$$

where

$$\Phi_{N,\alpha}(s) := \frac{s\,\mathfrak{F}_{N,\alpha}}{b_N^2}, \qquad E_N(s) := \frac{s(2C_N G_N - \mathcal{R}_N^{\mathrm{rec}})}{b_N^2}. \tag{1342}$$

The Part 51 estimate

$$\frac{\mathcal{R}_N^{\mathrm{rec}}}{\Gamma_N b_N^2} = O\!\left(\frac{\sigma^{q_\beta}}{g_{N,\alpha}}\right), \qquad \frac{2|C_N G_N|}{\Gamma_N b_N^2} = O\!\left(\frac{\sigma^{m_\beta}}{g_{N,\alpha}}\right) \tag{1343}$$

implies, on the coupled path (1290),

$$\boxed{E_N(s) = o(g_{N,\alpha}(s)).} \tag{1344}$$

Indeed $s\Gamma_N = g_{N,\alpha}$.

For the already certified positive tangent fractional block, the terminal scalings used in Parts 40–50 give, within a bootstrap in which (1328) and the corresponding s–σ law hold,

$$\mathfrak{F}_{N,\alpha} \asymp \nu b_N \sigma^{-\alpha}. \tag{1345}$$

Hence

$$\Phi_{N,\alpha} \asymp \nu \frac{s\sigma^{-\alpha}}{b_N}. \tag{1346}$$

For the continuation exponent $\beta = 2\alpha - 1$ and

$$\sigma \asymp s^{p_\beta}, \qquad p_\beta = \frac{12 - 2\beta}{7} = \frac{2(7 - 2\alpha)}{7}, \tag{1347}$$

one obtains the exact exponent identity

$$\boxed{1 + \beta - \alpha p_\beta = \frac{4\alpha^2}{7} > 0.} \tag{1348}$$

Therefore, subject to the same uniform coefficient bounds used in the fractional audit,

$$\boxed{\Phi_{N,\alpha}(s) = O\Big(s^{4\alpha^2/7}\Big) = o(1).} \tag{1349}$$

The positive fractional sign is favorable for $\dot{b}_N$, but in this normalization its contribution to the exponent-matching quantity H_N is lower order.

Combining (1341), (1344), and (1349) yields

$$\boxed{H_N(s) = g_{N,\alpha}(s) + o(g_{N,\alpha}(s)) + o(1).} \tag{1350}$$

This is the decisive persistence audit.

299. Necessary coefficient matching and failure of the proposed fixed exponent on the Lambert coupled regime

Suppose the desired stronger persistence holds:

$$F_N(s) \longrightarrow F_{0,N} \in (0, \infty). \tag{1351}$$

Then (1331) shows that the logarithmic defect must have zero accumulated leading order. In particular, if $H_N(s)$ has a terminal limit h_N, then necessarily

$$\boxed{h_N = \beta.} \tag{1352}$$

More generally, if for some $\varepsilon > 0$

$$\left| \frac{H_N(s)}{1 - b_N(s)^{-1}} - \beta \right| \geq \varepsilon \tag{1353}$$

with one fixed sign for all sufficiently small s, then (1332) forces $F_N(s)$ either to 0 or to $+\infty$. Thus no two-sided trapping with a positive finite terminal amplitude is possible.

Now impose the Lambert coupled regime used in Parts 40–52. If

$$\sup_{0<s\leq s_e} g_{N,\alpha}(s) \longrightarrow 0 \tag{1354}$$

and the uniform versions of (1344) and (1349) hold, then

$$\sup_{0<s\leq s_e} H_N(s) \longrightarrow 0. \tag{1355}$$

For every fixed $\beta = 2\alpha - 1 > 0$, the defect in (1333) therefore satisfies

$$D_N(s) = \beta + o(1) \tag{1356}$$

on the coupled terminal overlap. Its logarithmic integral is not integrable toward $s = 0$:

$$\int_0^{s_e} |D_N(s)| \frac{ds}{s} = \infty. \tag{1357}$$

Consequently the persistence criterion (1334) fails, and the proposed fixed-β profile cannot be proved from the present subordinate-error regime. Under the uniform limit assumptions just stated, it is in fact incompatible with that regime.

This conclusion is consistent with Part 52. There the reciprocal threshold would have required $cg_{N,\alpha} > \beta$, whereas the present exact normalized equation shows that persistence of the exponent β requires the stronger coefficient matching $H_N \to \beta$. If all non-Riccati terms are lower order and $g_{N,\alpha} \to 0$, that matching is absent.

300. The exponent selected by a Riccati-dominant finite-N regime

The obstruction does not say that every finite-N scalar trajectory is bounded. It says that the previously imposed exponent $\beta = 2\alpha - 1$ is not selected by the currently audited Riccati-dominant coefficient balance when $g_{N,\alpha}$ is small.

For example, suppose at one fixed finite N that

$$g_{N,\alpha}(s) \longrightarrow g_{*,N} > 0, \qquad \Phi_{N,\alpha}(s) + E_N(s) \longrightarrow 0. \tag{1358}$$

Then (1331) suggests the self-consistent exponent

$$\boxed{\gamma_N = g_{*,N},} \tag{1359}$$

not the externally imposed value $2\alpha - 1$. Indeed, defining $\widetilde{F}_N = s^{\gamma_N} b_N$, the exact analogue of (1331) is

$$\frac{d \log \widetilde{F}_N}{d \log s} = \gamma_N - \frac{H_N(s)}{1 - b_N^{-1}}. \tag{1360}$$

If the right side is logarithmically integrable, then

$$b_N(s) \sim \widetilde{B}_N s^{-\gamma_N}. \tag{1361}$$

Any rigorously established $\gamma_N > 0$ profile persisting to $s = 0$ would still reach $s = 0$ in finite physical time, since

$$T_N - t \asymp s^{1+\gamma_N}, \qquad b_N(t) \asymp (T_N - t)^{-\gamma_N/(1+\gamma_N)}. \tag{1362}$$

But (1358), the positive lower limit $g_{*,N} > 0$, and the logarithmic integrability needed for (1360) are new statements; they have not yet been proved uniformly from the finite-N Lambert construction.

Theorem 47 (Terminal persistence criterion and obstruction for the current fixed exponent)**.** *Let $b_N > 1$ solve the exact scalar characteristic equation (1340) on $(0, s_e]$, and define H_N by (1330). For any $\beta > 0$, the profile $b_N = B_N s^{-\beta} F_N$ satisfies the exact identity (1331). If the matching defect (1333) obeys (1334), then the two-sided trapping estimate (1336) holds; if (1337) holds, then F_N converges to a positive finite limit. Conversely, if the defect is eventually separated from zero with one sign, then such a positive finite terminal amplitude is impossible. In particular, under the uniform Lambert coupled assumptions (1354), (1344), and (1349), one has $H_N \to 0$, so a fixed exponent $\beta = 2\alpha - 1 > 0$ cannot satisfy the required coefficient matching.*

301. Corrected scalar proof ledger and the next non-circular problem

The attempted persistence proof therefore produces a structural correction:

exact normalized profile equation:	proved in (1331),
two-sided persistence criterion:	proved in (1334)–(1338),
fractional exponent contribution:	$O(s^{4\alpha^2/7})$ in the current bootstrap,
reconstruction/forcing contribution:	$o(g_{N,\alpha})$ under (1290),
required fixed-β matching:	$H_N \to \beta$,
current Lambert coupled balance:	$H_N \to 0$ under (1354),
fixed $\beta = 2\alpha - 1$ persistence:	not closed; incompatible with the stated uniform subordinate-error assumptions,
next scalar problem:	derive the exponent from the exact finite-N coefficient balance.

(1363)

Thus one must not use the power law $b_N \asymp s^{-(2\alpha-1)}$ as an input to prove the same power law. The next non-circular route is to analyze $H_N(s)$ directly from the exact finite-N Lambert coefficients, without assuming a terminal exponent, and determine whether it has a positive limit or a positive logarithmic average. Only after that coefficient is derived can a terminal power and the corresponding finite-time scalar blow-up law be claimed.

111 Part 54: Exact Fixed-N Terminal Lambert Sensitivity and the Exponent Selected by the Scalar Balance

302. Fixed finite N must be taken before the coupled limit

Part 53 reduced the non-circular scalar problem to the terminal behavior of

$$S_{N,k}(s) = s\,\partial_s \log \Delta_{N,k}(s), \qquad g_{N,\alpha}(s) = |S_{N,k(N)}(s)|. \tag{1364}$$

We now keep $N < \infty$ fixed. Then $k_N := k(N)$ is one fixed finite integer. The actual branch coordinate from Part 48 satisfies, on the positive side of the finite-N Lambert branch point,

$$\Delta_{N,0}(s) = a_N s\big(1 + O_N(s)\big), \qquad a_N = R'_N(\theta_{*,N}) > 0, \tag{1365}$$

where $s = \theta - \theta_{*,N}$ after choosing the orientation for which $\Delta_{N,0} > 0$. In the revised regularization,

$$a_N \sim \sqrt{2}\, L_N^{3/2}. \tag{1366}$$

Consequently

$$\boxed{S_{N,0}(s) = 1 + O_N(s).} \tag{1367}$$

This fixed-N statement is different from the later coupled estimate $|S_{N,k(N)}| \le 2^{-k(N)}|S_{N,0}|$: at fixed N there is no limit in which k_N is sent to infinity while $s \downarrow 0$.

303. Exact recurrence and its branch-point limit

Recall the exact renormalized Lambert map

$$\Delta_{N,j+1} = \mathcal{F}(\Delta_{N,j}), \qquad \mathcal{F}(\Delta) = 1 + W_0(-e^{-1-\Delta}), \tag{1368}$$

and the exact sensitivity recurrence

$$S_{N,j+1} = A(\Delta_{N,j}) S_{N,j}, \qquad A(\Delta) = \frac{\Delta(1-q)}{q^2}, \quad q = \mathcal{F}(\Delta). \tag{1369}$$

The branch expansion

$$q = \sqrt{2\Delta} - \frac{2}{3}\Delta + O(\Delta^{3/2}) \tag{1370}$$

gives

$$\boxed{A(\Delta) = \frac{1}{2} + O(\Delta^{1/2}) \qquad (\Delta \downarrow 0).} \tag{1371}$$

For every fixed j the same expansion gives

$$\Delta_{N,j}(s) = C_j (a_N s)^{2^{-j}} \left(1 + O_N\big(s^{2^{-j}}\big)\right), \qquad C_j = 2^{1-2^{-j}}, \tag{1372}$$

with the convention $C_0 = 1$. Hence

$$A(\Delta_{N,j}(s)) = \frac{1}{2} + O_N\big(s^{2^{-(j+1)}}\big). \tag{1373}$$

Iterating (1369) only a fixed number k_N of times and using (1367) yields

$$\boxed{S_{N,k_N}(s) = 2^{-k_N}\left[1 + O_N\big(s^{2^{-k_N}}\big)\right].} \tag{1374}$$

In particular the terminal limit exists and is strictly positive:

$$\boxed{g_{*,N} := \lim_{s\downarrow 0} g_{N,\alpha}(s) = 2^{-k_N} > 0.} \tag{1375}$$

Moreover the convergence is logarithmically integrable:

$$\boxed{\int_0^{s_e} |g_{N,\alpha}(s) - 2^{-k_N}|\,\frac{ds}{s} < \infty} \tag{1376}$$

after decreasing the fixed-N branch neighborhood if necessary. Thus the positive lower limit requested in (1358) is not an additional Lambert nondegeneracy hypothesis at fixed finite N; it follows from the actual branch expansion and the exact sensitivity recurrence.

Theorem 48 (Exact fixed-N terminal Lambert sensitivity)**.** *Fix $N < \infty$ and a finite Lambert depth k_N. On the real positive side of a simple branch point for which (1365) holds, the exact renormalized Lambert cascade satisfies (1374)–(1376). In particular*

$$\lim_{s\downarrow 0} s\,\partial_s \log \Delta_{N,k_N}(s) = 2^{-k_N}. \tag{1377}$$

304. What this proves for the Riccati part of the scalar equation

The exact scalar normalized coefficient from Part 53 is

$$H_N(s) = g_{N,\alpha}(s) + \Phi_{N,\alpha}(s) + E_N(s). \tag{1378}$$

If the last two terms were logarithmically integrable errors, then (1374) would give

$$H_N(s) = 2^{-k_N} + O_N(s^{\eta_N}) \tag{1379}$$

for some $\eta_N > 0$, and the persistence theorem of Part 53 would select

$$\boxed{\gamma_N = 2^{-k_N}.} \tag{1380}$$

Indeed

$$b_N(s) = \widetilde{B}_N s^{-\gamma_N}(1 + o(1)) \tag{1381}$$

would then be the exponent generated by the Riccati/Lambert coefficient itself, rather than an exponent inserted in advance. If (1381) persisted to $s = 0$, the exact characteristic law would give

$$T_N - t \asymp s^{1+\gamma_N}, \qquad b_N(t) \asymp (T_N - t)^{-\gamma_N/(1+\gamma_N)}. \tag{1382}$$

Thus the Lambert part alone does select a positive finite-N candidate exponent.

305. Re-audit of the fractional term at the newly selected exponent

It is essential, however, not to declare (1381) proved before checking the fractional term with this new exponent. The geometric characteristic calculation for a general tentative power $b_N \asymp s^{-\beta}$ gives

$$\sigma \asymp s^{p_\beta}, \qquad p_\beta = \frac{12-2\beta}{7}. \tag{1383}$$

The normalized fractional contribution has the scale

$$\Phi_{N,\alpha} = \frac{s\mathfrak{F}_{N,\alpha}}{b_N^2} \asymp \frac{\nu\, s\sigma^{-\alpha}}{b_N} \asymp s^{E_\alpha(\beta)}, \tag{1384}$$

where

$$\boxed{E_\alpha(\beta) := 1+\beta-\alpha p_\beta = \frac{7-12\alpha+(7+2\alpha)\beta}{7}.} \tag{1385}$$

Therefore the fractional term is lower order in the normalized coefficient only if

$$\boxed{\beta > \beta_{\mathrm{frac}}(\alpha), \qquad \beta_{\mathrm{frac}}(\alpha) := \frac{12\alpha-7}{7+2\alpha}.} \tag{1386}$$

At the anchor

$$\boxed{\beta_{\mathrm{frac}}(1) = \frac{5}{9}.} \tag{1387}$$

For every $k_N \geq 1$,

$$\gamma_N = 2^{-k_N} \leq \frac{1}{2} < \frac{5}{9} = \beta_{\mathrm{frac}}(1), \tag{1388}$$

and the strict inequality persists for α in a sufficiently small right neighborhood of 1. Hence, in the depth regime relevant to the renormalized construction,

$$\boxed{E_\alpha(\gamma_N) < 0 \qquad (k_N \geq 1,\ \alpha \approx 1).} \tag{1389}$$

Thus $\Phi_{N,\alpha}$ is not a subordinate error at the Riccati-selected power $s^{-2^{-k_N}}$; according to the currently audited scaling it becomes larger in the normalization H_N. This prevents us from promoting (1380) to a proved terminal exponent for the complete scalar equation.

306. The fractional-critical candidate and the amplitude matching equation

Equation (1385) identifies the only power at which the leading fractional block can remain of the same normalized order as the Riccati coefficient:

$$\boxed{\beta = \beta_{\mathrm{frac}}(\alpha) = \frac{12\alpha-7}{7+2\alpha}.} \tag{1390}$$

At this exponent $E_\alpha(\beta) = 0$. If the complete finite-N fractional coefficient has a positive terminal constant

$$\mathfrak{F}_{N,\alpha} = \nu\, \mathfrak{h}_{N,\alpha}\, b_N \sigma^{-\alpha}(1+o(1)), \qquad \mathfrak{h}_{N,\alpha} > 0, \tag{1391}$$

and

$$\sigma = \mathfrak{K}_N s^{p_\beta}(1+o(1)), \qquad b_N = B_N s^{-\beta}(1+o(1)), \tag{1392}$$

then

$$\Phi_{N,\alpha}(s) \longrightarrow \frac{\nu \mathfrak{h}_{N,\alpha} \mathfrak{K}_N^{-\alpha}}{B_N}. \tag{1393}$$

The exact coefficient-matching condition $H_N \to \beta$ would consequently become

$$\boxed{\beta_{\mathrm{frac}}(\alpha) = 2^{-k_N} + \frac{\nu \mathfrak{h}_{N,\alpha} \mathfrak{K}_N^{-\alpha}}{B_N},} \tag{1394}$$

up to any other nonvanishing normalized terminal terms. Formally this gives

$$\boxed{B_N = \frac{\nu \mathfrak{h}_{N,\alpha} \mathfrak{K}_N^{-\alpha}}{\beta_{\mathrm{frac}}(\alpha) - 2^{-k_N}}.} \tag{1395}$$

For $k_N \geq 1$ and α sufficiently close to 1, the denominator is positive by (1388). Thus the complete balance has a plausible positive amplitude rather than a sign contradiction.

But (1391) has not yet been certified for the new exponent (1390). The signed fractional audit in Parts 46–50 was carried out for the previous continuation exponent, so its Mellin coefficient and geometric powers must be recomputed at $\beta_{\mathrm{frac}}(\alpha)$ before (1394)–(1395) can be used as a theorem. This is the next non-circular scalar calculation.

307. Order of limits and reconciliation with the coupled contraction

There is no contradiction between (1375) and the coupled result $g_{N,\alpha} \to 0$. The two limits are

$$\boxed{\lim_{N\to\infty} \left[\lim_{s\downarrow 0} g_{N,\alpha}(s) \right] = \lim_{N\to\infty} 2^{-k(N)} = 0,} \tag{1396}$$

when $k(N) \to \infty$. At every fixed finite N, however,

$$\boxed{\lim_{s\downarrow 0} g_{N,\alpha}(s) = 2^{-k(N)} > 0.} \tag{1397}$$

The fixed-N terminal coefficient is therefore positive but degenerates as $N \to \infty$. This precisely explains why Part 53's uniform coupled limit lost the positive exponent while the fixed-N calculation recovers one.

Theorem 49 (Fixed-N coefficient selection and remaining fractional obstruction)**.** *For every fixed finite N and finite k_N, the actual Lambert sensitivity has the positive terminal limit $g_{*,N} = 2^{-k_N}$ with logarithmically integrable error. Hence $\gamma_N = 2^{-k_N}$ is the exponent selected by the Riccati/Lambert part of the scalar equation. For $k_N \geq 1$ and α in a sufficiently small right neighborhood of 1, however, the presently audited fractional scaling is not lower order at that exponent. The complete scalar balance instead points to the fractional-critical candidate $\beta_{\mathrm{frac}}(\alpha) = (12\alpha - 7)/(7 + 2\alpha)$ and the amplitude matching condition (1394). A finite-time scalar blow-up theorem therefore requires a new signed fractional coefficient calculation at this exponent and a logarithmically integrable persistence estimate for the resulting matched profile.*

308. Updated scalar proof ledger

The fixed-N audit gives the following sharpened status:

$S_{N,0}$ at the simple branch point:	$1 + O_N(s)$,
fixed-N Lambert sensitivity:	$S_{N,k_N} = 2^{-k_N}[1 + O_N(s^{2^{-k_N}})]$,
positive fixed-N limit:	$g_{*,N} = 2^{-k_N} > 0$,
Riccati-selected candidate exponent:	$\gamma_N = 2^{-k_N}$,
fractional term at γ_N:	not subordinate for $k_N \geq 1$, $\alpha \approx 1$,
fractional-critical exponent:	$\beta_{\mathrm{frac}} = (12\alpha - 7)/(7 + 2\alpha)$,
complete amplitude matching:	conditional formula (1394)–(1395),
finite-time scalar blow-up:	not yet proved; requires signed fractional certification and persistence at β_{frac}.

(1398)

The next calculation is therefore sharply determined: recompute the complete signed finite-N fractional coefficient for the general exponent β, evaluate it at $\beta_{\mathrm{frac}}(\alpha)$, and test whether the amplitude in (1395) is positive and whether the remaining matching defect is integrable in ds/s. That calculation, rather than the Lambert sensitivity itself, is now the decisive scalar obstruction.

112 Part 55: Complete Signed Fractional Coefficient at the Derived Critical Exponent

309. Recompute the complete tangent coefficient for arbitrary β

We now perform the calculation requested at the end of Part 54 without reusing the specialization $\beta = 2\alpha - 1$. Assume only the self-consistent characteristic scaling of Parts 38–39,

$$b_N(\sigma) \sim \widehat{p}_N \sigma^{-\widehat{\rho}_\beta}, \qquad \widehat{\rho}_\beta = \frac{7\beta}{2(6-\beta)}, \qquad 0 < \beta < 2. \tag{1399}$$

Put $r = \sqrt{\sigma}$. The square-root reconstruction

$$u_{y,N} = \sqrt{-2\chi b_N}, \qquad u_{x,N} = \sqrt{-\frac{b_N}{2\chi}}$$

has tangent powers

$$u_{y,N} \sim c_{A,N} r^{\lambda_A(\beta)}, \qquad u_{x,N} \sim c_{U,N} r^{\lambda_U(\beta)}, \tag{1400}$$

where

$$\boxed{\lambda_A(\beta) = \frac{1}{2} - \widehat{\rho}_\beta, \qquad \lambda_U(\beta) = -\frac{1}{2} - \widehat{\rho}_\beta.} \tag{1401}$$

For

$$K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi} \sin\big(\pi(\alpha-\lambda)\big), \tag{1402}$$

the product/commutator cancellation from (865)–(867) is algebraic and does not require the old value of β. Consequently the complete leading fractional block is

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{b_N} = -\nu H_\alpha(\beta)\sigma^{-\alpha} + o(\sigma^{-\alpha}),} \tag{1403}$$

with

$$\boxed{H_\alpha(\beta) := K_\alpha^+\left(-\frac{1}{2} - \widehat{\rho}_\beta\right) + K_\alpha^+\left(\frac{1}{2} - \widehat{\rho}_\beta\right).} \tag{1404}$$

Thus the signed positive fractional coefficient is

$$\boxed{\mathfrak{h}_\alpha(\beta) := -H_\alpha(\beta).} \tag{1405}$$

The finite-N certification requires, as before, that the actual geometric/nonlocal remainder be $o(\sigma^{-\alpha})$ relative to b_N. Equation (1403) is the exact Mellin tangent coefficient; that remainder condition is logically separate.

310. Exact simplification at the fractional-critical exponent

Set

$$\beta_c(\alpha) := \frac{12\alpha - 7}{7 + 2\alpha}. \tag{1406}$$

A useful cancellation occurs:

$$6 - \beta_c = \frac{49}{7 + 2\alpha}, \qquad \boxed{\widehat{\rho}_{\beta_c} = \frac{12\alpha - 7}{14}.} \tag{1407}$$

Therefore

$$\boxed{\lambda_U(\alpha, \beta_c) = -\frac{6\alpha}{7}, \qquad \lambda_A(\alpha, \beta_c) = 1 - \frac{6\alpha}{7}.} \tag{1408}$$

The complete coefficient at the derived exponent is consequently

$$\boxed{H_\alpha^{\mathrm{crit}} = K_\alpha^+\left(-\frac{6\alpha}{7}\right) + K_\alpha^+\left(1 - \frac{6\alpha}{7}\right), \qquad \mathfrak{h}_\alpha^{\mathrm{crit}} = -H_\alpha^{\mathrm{crit}}.} \tag{1409}$$

This is the requested general-α signed coefficient.

At $\alpha = 1$, $K_1^+(\lambda) = -\lambda(\lambda - 1)$. Hence

$$K_1^+\left(-\frac{6}{7}\right) = -\frac{78}{49}, \qquad K_1^+\left(\frac{1}{7}\right) = \frac{6}{49}, \tag{1410}$$

and therefore

$$\boxed{H_1^{\mathrm{crit}} = -\frac{72}{49} < 0, \qquad \mathfrak{h}_1^{\mathrm{crit}} = \frac{72}{49} > 0.} \tag{1411}$$

Thus the complete tangent fractional block is strictly positive at the critical exponent:

$$\boxed{\frac{\mathfrak{F}_{N,1}}{b_N} = \frac{72}{49}\nu\,\sigma^{-1} + o(\sigma^{-1}).} \tag{1412}$$

Because the two arguments in (1409) and the analytically continued Mellin coefficient vary continuously with α away from Gamma poles, there are $\delta_{\mathrm{crit}} > 0$ and $h_{\mathrm{crit}} > 0$ such that, after shrinking the continuation interval if necessary,

$$\boxed{\mathfrak{h}_{\alpha}^{\mathrm{crit}} \geq h_{\mathrm{crit}} > 0, \qquad 1 \leq \alpha < 1 + \delta_{\mathrm{crit}}.} \tag{1413}$$

No explicit decimal value of δ_{crit} is asserted without a certified interval evaluation of the Gamma factors.

311. The amplitude equation must include the dependence of $\mathfrak{K}_N$ on amplitude

There is a correction to the formal formula (1395). The characteristic constant $\mathfrak{K}_N$ is not independent of the terminal amplitude. Write the physical leading amplitude as

$$b_N(s) = \mathcal{B}_N s^{-\beta_c}(1 + o(1)). \tag{1414}$$

Equation (828) gives

$$\boxed{\mathfrak{K}_N = C_{\beta_c}\mathcal{B}_N^{2/7}, \qquad C_\beta := \left[\frac{7\sqrt{2}}{16(1-\beta/2)^2(3-\beta/2)}\right]^{4/7}.} \tag{1415}$$

At the critical exponent the normalized fractional term therefore has the limit

$$\Phi_{N,\alpha} \longrightarrow \nu\mathfrak{h}_{\alpha}^{\mathrm{crit}} C_{\beta_c}^{-\alpha}\mathcal{B}_N^{-(1+2\alpha/7)}. \tag{1416}$$

Together with the fixed-N Lambert limit $g_{*,N} = 2^{-k_N}$, coefficient matching $H_N \to \beta_c$ becomes

$$\boxed{\beta_c - 2^{-k_N} = \nu\mathfrak{h}_{\alpha}^{\mathrm{crit}} C_{\beta_c}^{-\alpha}\mathcal{B}_N^{-(1+2\alpha/7)}.} \tag{1417}$$

Thus, whenever

$$\boxed{\beta_c(\alpha) > 2^{-k_N},} \tag{1418}$$

there is one and only one positive matched amplitude,

$$\boxed{\mathcal{B}_N = \left[\frac{\nu\mathfrak{h}_{\alpha}^{\mathrm{crit}} C_{\beta_c}^{-\alpha}}{\beta_c(\alpha) - 2^{-k_N}}\right]^{7/(7+2\alpha)}.} \tag{1419}$$

At $\alpha = 1$, $\beta_c = 5/9 > 1/2$, so (1418) holds for every $k_N \geq 1$; by continuity it continues to hold on a sufficiently small right neighborhood of $\alpha = 1$.

312. Logarithmic persistence of the matched profile

Return to the exact identity from Part 53,

$$\frac{d\log F_N}{d\log s} = \beta_c - \frac{H_N(s)}{1 - b_N(s)^{-1}}. \tag{1420}$$

The fixed-N Lambert calculation gives

$$g_{N,\alpha}(s) = 2^{-k_N} + O_N(s^{2^{-k_N}}). \tag{1421}$$

Suppose now that the finite-N fractional certification at the new exponent is sharpened from $o(1)$ to a positive-power remainder,

$$\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu \mathfrak{h}_\alpha^{\mathrm{crit}} \sigma^{-\alpha}\big(1 + O_N(s^{\eta_F})\big), \qquad \eta_F > 0, \tag{1422}$$

and that the characteristic reconstruction and the normalized reconstruction/forcing errors satisfy

$$\sigma = C_{\beta_c} \mathcal{B}_N^{2/7} s^{p_{\beta_c}} \big(1 + O_N(s^{\eta_\sigma})\big), \qquad E_N(s) = O_N(s^{\eta_E}), \tag{1423}$$

for some $\eta_\sigma, \eta_E > 0$. Since $b_N^{-1} = O(s^{\beta_c})$, equations (1417) and (1420)–(1423) give

$$\boxed{\frac{d\log F_N}{d\log s} = O_N(s^{\eta_N}), \qquad \eta_N := \min\{2^{-k_N}, \eta_F, \eta_\sigma, \eta_E, \beta_c\} > 0.} \tag{1424}$$

Therefore

$$\int_0^{s_e} \left| \frac{d\log F_N}{d\log s} \right| \frac{ds}{s} < \infty, \tag{1425}$$

and the persistence lemma of Part 53 yields

$$\boxed{F_N(s) \longrightarrow F_{0,N} \in (0,\infty), \qquad b_N(s) \sim \mathcal{B}_N s^{-\beta_c}.} \tag{1426}$$

This is a genuine conditional persistence theorem. What remains to make it unconditional is the positive-power finite-N remainder estimate (1422)–(1423) for the actual reconstructed fields at the newly selected exponent. A mere $o(1)$ remainder is not sufficient for logarithmic integrability.

313. Consequence for finite physical time

If the hypotheses of the preceding persistence result are established, then $\beta_c > 0$ for $\alpha > 7/12$, in particular throughout the continuation interval near $\alpha = 1$. Since $\dot{s} = 1 - b_N$,

$$T_N - t = \int_0^{s(t)} \frac{d\xi}{b_N(\xi) - 1} \sim \frac{s(t)^{\beta_c+1}}{(\beta_c+1)\mathcal{B}_N}. \tag{1427}$$

Hence the terminal point occurs in finite physical time and

$$\boxed{b_N(t) \sim \mathcal{B}_N^{1/(\beta_c+1)} (\beta_c+1)^{-\beta_c/(\beta_c+1)} (T_N - t)^{-\beta_c/(\beta_c+1)}.} \tag{1428}$$

Using (1406), the time exponent simplifies to

$$\boxed{\frac{\beta_c}{1+\beta_c} = \frac{12\alpha - 7}{14\alpha}.} \tag{1429}$$

Thus the complete signed tangent coefficient and the nonlinear amplitude matching are favorable. The remaining obstruction is no longer a sign or amplitude obstruction: it is the quantitative positive-power remainder needed to justify (1422)–(1423) for the actual finite-N nonlocal and reconstructed scalar equation.

Theorem 50 (Critical signed coefficient and conditional matched persistence)**.** *Let $k_N \geq 1$ be fixed and finite and let α lie in a sufficiently small right neighborhood of 1. For $\beta_c = (12\alpha - 7)/(7 + 2\alpha)$, the complete Mellin tangent coefficient is given by (1409), with $\mathfrak{h}_1^{\mathrm{crit}} = 72/49 > 0$ and hence $\mathfrak{h}_\alpha^{\mathrm{crit}} > 0$ after shrinking the continuation interval. Accounting for the amplitude dependence of the characteristic constant gives the unique positive amplitude (1419). If, in addition, the actual finite-N fractional, characteristic, reconstruction, and forcing remainders obey the positive-power estimates (1422)–(1423), then the matching defect is integrable in ds/s, the profile persists as (1426), and the scalar b_N becomes unbounded at a finite physical time with rate (1428). Without those positive-power remainder estimates, the last persistence and blow-up conclusion remains conditional.*

314. Updated scalar proof ledger

General-β complete tangent coefficient:	$H_\alpha(\beta)$ in (1404),
critical exponent:	$\beta_c = (12\alpha - 7)/(7 + 2\alpha)$,
critical tangent powers:	$\lambda_U = -6\alpha/7$, $\lambda_A = 1 - 6\alpha/7$,
anchor signed coefficient:	$\mathfrak{h}_1^{\mathrm{crit}} = 72/49 > 0$,
near-anchor sign:	strictly positive after shrinking δ_{crit},
fixed-N Lambert limit:	$g_{*,N} = 2^{-k_N} > 0$,
matched amplitude:	unique positive value (1419) for $\beta_c > 2^{-k_N}$,
logarithmic persistence:	proved under positive-power remainders (1422)–(1423),
finite-time scalar blow-up:	follows conditionally from that persistence,
remaining scalar obstruction:	prove (1422)–(1423) for the actual finite-N fields.

(1430)

113 Part 56: Positive-Power Remainder Audit at the Matched Critical Exponent

315. Purpose and the distinction between what can and cannot be closed

We now test the three positive-power statements left open in (1422)–(1423), with

$$\beta = \beta_c(\alpha) := \frac{12\alpha - 7}{7 + 2\alpha}, \qquad p_c := p_{\beta_c} = \frac{12 - 2\beta_c}{7}. \tag{1431}$$

A direct simplification gives

$$\boxed{p_c = \frac{14}{7 + 2\alpha} > 0.} \tag{1432}$$

The audit must be non-circular: estimates obtained previously after assuming $F_N \to F_{0,N} > 0$ may be used inside a bootstrap, but they cannot by themselves establish that convergence.

The outcome is asymmetric. The reconstruction/forcing error admits an unconditional positive power once the two-sided terminal bootstrap holds. The characteristic law admits a positive-power improvement coupled to the profile modulus. The complete fractional remainder, however, is not yet controlled by the existing estimates all the way to $s = 0$ at fixed N. The obstruction is the order of the fixed-N terminal limit relative to the actual Lambert inner width. This distinction is essential for the proof ledger.

316. The reconstruction/forcing term has an explicit positive power

Recall

$$E_N(s) = \frac{s(2C_N G_N - \mathcal{R}_N^{\mathrm{rec}})}{b_N^2}, \qquad \Gamma_N = \frac{g_{N,\alpha}(s)}{s}. \tag{1433}$$

Equations (1289) and (1292) imply, at every fixed finite N,

$$\frac{\mathcal{R}_N^{\mathrm{rec}}}{\Gamma_N b_N^2} = O_N(\sigma^{q_\beta}), \qquad \frac{2|C_N G_N|}{\Gamma_N b_N^2} = O_N(\sigma^{m_\beta}), \tag{1434}$$

because $g_{N,\alpha}(s) \to 2^{-k_N} > 0$. Multiplying by $s\Gamma_N = g_{N,\alpha}$ therefore cancels the apparent g^{-1} loss and gives

$$\boxed{|E_N(s)| \le C_N\big(\sigma^{q_\beta} + \sigma^{m_\beta}\big).} \tag{1435}$$

Here

$$q_\beta = \frac{\beta + 1}{6 - \beta}, \qquad m_\beta = \frac{12\beta + 5}{2(6 - \beta)}, \qquad m_\beta > q_\beta > 0. \tag{1436}$$

At $\beta = \beta_c$ one obtains the exact simplifications

$$\boxed{q_c = \frac{2\alpha}{7},} \qquad \boxed{p_c q_c = \frac{4\alpha}{7 + 2\alpha}.} \tag{1437}$$

Consequently, on any two-sided characteristic bootstrap $\sigma \asymp s^{p_c}$,

$$\boxed{E_N(s) = O_N(s^{\eta_E}), \qquad \eta_E = \frac{4\alpha}{7+2\alpha} > 0.} \tag{1438}$$

Thus the third estimate requested in (1423) is genuinely available at fixed N; it is not merely an $o(1)$ statement.

317. Quantitative characteristic reconstruction and its bootstrap exponent

The exact finite-N characteristic estimate (900) is

$$|\theta_N^\sigma| \le C_N\big(\Omega_F(s) + s^{\beta_c} + \sigma + \varepsilon_N^C(s)\big). \tag{1439}$$

After integration, (903) gives

$$\sigma(s) = C_{\beta_c}\mathcal{B}_N^{2/7} s^{p_c}\left[1 + O_N\big(\Omega_F(s) + s^{\beta_c} + \sigma + \varepsilon_N^C(s)\big)\right]. \tag{1440}$$

Suppose temporarily that the profile bootstrap has a positive modulus

$$\Omega_F(s) \le M_N s^{\eta} \tag{1441}$$

and that the explicit continuity-reconstruction error satisfies

$$\varepsilon_N^C(s) = O_N(s^{\eta_C}), \qquad \eta_C > 0. \tag{1442}$$

Using $\sigma \asymp s^{p_c}$ on the bootstrap interval, (1440) yields

$$\boxed{\sigma(s) = C_{\beta_c}\mathcal{B}_N^{2/7} s^{p_c}\big(1 + O_N(s^{\eta_\sigma})\big),} \tag{1443}$$

where

$$\boxed{\eta_\sigma = \min\{\eta, \beta_c, p_c, \eta_C\} > 0.} \tag{1444}$$

This is a quantitative improvement, but it is coupled to the profile modulus Ω_F. Hence (1443) is suitable for a bootstrap closure; it is not an independent proof of persistence.

318. Positive powers already present in the local geometric operator

The geometric fractional reduction from Part 35 has a quadratic metric remainder. In the intrinsic variable $r = \sqrt{\sigma}$,

$$\mathcal{L}_\alpha^{\mathrm{geo}}\big(r^{-a}G(\log r)\big) = r^{-a-2\alpha}\big[\mathcal{M}_a G + \mathcal{Q}_{\alpha,r}G\big], \qquad \mathcal{Q}_{\alpha,r} = O(r^2). \tag{1445}$$

Therefore its local relative error is

$$O(r^2) = O(\sigma) = O(s^{p_c}), \tag{1446}$$

which is a genuine positive power. Likewise, the fixed-N Lambert sensitivity has

$$g_{N,\alpha}(s) = 2^{-k_N} + O_N(s^{2^{-k_N}}). \tag{1447}$$

Thus two natural positive exponents are already available:

$$\boxed{\eta_{\mathrm{geo}} = p_c, \qquad \eta_{\mathrm{Lam}} = 2^{-k_N}.} \tag{1448}$$

If the complete finite-N fractional block were represented on one common terminal chart by the Mellin tangent plus only these local corrections, then one would obtain

$$\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu \mathfrak{h}_\alpha^{\mathrm{crit}} \sigma^{-\alpha} \left(1 + O_N(s^{\min\{p_c, 2^{-k_N}, \eta\}})\right). \tag{1449}$$

The remaining question is whether the actual nonlocal reconstructed operator has exactly this common-chart property at fixed N as $s \downarrow 0$.

319. The fixed-N Lambert inner scale prevents promotion of (1449) from the existing core–tail audit

The actual finite-N Lambert layer has phase width

$$w_N \asymp 2^{-2N} L_N^{-3/2} > 0 \tag{1450}$$

for every fixed N. The core–tail estimates of Parts 48–50 certify that this layer is negligible on an outer path satisfying a separation condition of the form

$$s \gg w_N \tag{1451}$$

(or, in the squared geometric comparison, a corresponding separation from $\sigma_{\mathrm{Lam},N}$). At fixed N, however,

$$\boxed{s \downarrow 0 \quad \Longrightarrow \quad s/w_N \downarrow 0.} \tag{1452}$$

Thus the terminal limit required by logarithmic persistence eventually enters the actual Lambert inner layer. Exponential smallness of w_N in N does not remove this issue when N is held fixed first.

Accordingly, the previously proved separated-support estimate does not imply an estimate of the form

$$\mathfrak{F}_{N,\alpha}/b_N = \nu \mathfrak{h}_\alpha^{\mathrm{crit}} \sigma^{-\alpha}(1 + O_N(s^{\eta_F})) \tag{1453}$$

for all sufficiently small s at that fixed N. The order of limits is crucial: an estimate uniform on polynomial N-dependent outer paths is not a fixed-N asymptotic through the inner core.

This is not a sign reversal. It is a missing asymptotic matching theorem for the complete nonlocal operator in the inner Lambert chart.

320. What must be proved in the inner chart

Introduce the genuine inner coordinate

$$\xi = \frac{s}{w_N}. \tag{1454}$$

For fixed N the desired result would follow from an inner expansion of the complete reconstructed fractional block of the form

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu\sigma^{-\alpha}\left[\mathfrak{h}_\alpha^{\mathrm{crit}} + R_{N,\alpha}^{\mathrm{in}}(\xi, s)\right],} \tag{1455}$$

with

$$\boxed{|R_{N,\alpha}^{\mathrm{in}}(\xi, s)| \le C_N s^{\eta_F} \quad \text{uniformly for } 0 \le \xi \le \xi_0,} \tag{1456}$$

and with matching to the outer coefficient as $\xi \to \infty$. Neither the weighted geometric estimate $O(r^2)$ nor the separated core–tail estimate proves (1456), because the complete operator samples both the Lambert phase and the independent χ-geometry.

A rigorous proof therefore requires one of the following equivalent pieces of new information: an exact inner spectral calculation for the finite-N reconstructed fields, or a two-variable pseudodifferential expansion whose remainder is uniform through $s \sim w_N$. No such estimate has yet been derived in the manuscript.

321. A closed bootstrap theorem conditional only on the inner fractional estimate

The preceding audit nevertheless reduces the persistence problem to one precise missing estimate. Assume (1455)–(1456) for some $\eta_F > 0$ and assume the explicit continuity error (1442). Set

$$\eta_* := \min\left\{2^{-k_N}, \eta_F, \eta_C, \beta_c, p_c, \frac{4\alpha}{7+2\alpha}\right\} > 0. \tag{1457}$$

Choose a bootstrap exponent $0 < \eta < \eta_*$. Equations (1438), (1443), (1447), (1455)–(1456), the amplitude identity (1417), and $b_N^{-1} = O(s^{\beta_c})$ then give

$$\left|\frac{d\log F_N}{d\log s}\right| \le C_N s^{\eta}. \tag{1458}$$

Integrating from s to s_e yields

$$|\log F_N(s) - \log F_N(s_e)| \le \frac{C_N}{\eta} s_e^{\eta}, \tag{1459}$$

and, after choosing s_e sufficiently small, improves the assumed bound (1441). Standard continuity of the bootstrap therefore gives

$$\boxed{F_N(s) = F_{0,N}\big(1 + O_N(s^{\eta})\big), \qquad F_{0,N} \in (0, \infty).} \tag{1460}$$

Substitution into (1443) then gives the requested positive-power characteristic law, and (1438) supplies the requested E_N estimate. Thus the persistence proof closes once and only once the inner fractional estimate (1456) is supplied.

Theorem 51 (Positive-power audit at fixed finite N)**.** *Let N and $k_N \geq 1$ be fixed and let α lie in the continuation interval where $\mathfrak{h}_\alpha^{\mathrm{crit}} > 0$ and $\beta_c > 2^{-k_N}$. On the matched terminal bootstrap, the normalized reconstruction/forcing term satisfies the unconditional positive-power estimate*

$$E_N = O_N\left(s^{4\alpha/(7+2\alpha)}\right).$$

The exact characteristic reconstruction has the positive-power form (1443) whenever the profile modulus and explicit continuity error have positive powers. The local geometric fractional remainder contributes the positive power s^{p_c} and the Lambert sensitivity contributes $s^{2^{-k_N}}$. However, the existing separated Lambert core–tail audit does not control the complete nonlocal fractional remainder through the fixed-N inner region $s \lesssim w_N$. Therefore the estimate (1422), and hence unconditional logarithmic persistence and finite-time scalar blow-up, cannot yet be claimed. If the inner estimate (1456) is established with any $\eta_F > 0$, the bootstrap closes and yields (1460), followed by the finite physical-time law (1427)–(1429).

322. Corrected proof ledger and next decisive calculation

E_N positive power:	proved, $\eta_E = 4\alpha/(7+2\alpha)$,
characteristic σ positive power:	proved inside the profile/continuity bootstrap,
local geometric fractional error:	$O(s^{p_c})$, $p_c = 14/(7+2\alpha)$,
fixed-N Lambert sensitivity error:	$O_N(s^{2^{-k_N}})$,
outer Lambert core–tail error:	negligible only while separated from the inner layer,
fixed-N limit $s \downarrow 0$:	eventually enters $s \lesssim w_N$,
complete fractional positive-power estimate:	not yet proved through that inner layer,
logarithmic persistence:	closes if the inner estimate (1456) is proved,
finite-time scalar b_N blow-up:	remains conditional on that single inner fractional matching step.

(1461)

The next calculation is therefore more precise than the statement at the end of Part 55: compute the complete reconstructed fractional operator in the inner variable $\xi = s/w_N$, prove uniform matching of its signed coefficient to $\mathfrak{h}_\alpha^{\mathrm{crit}}$, and obtain a remainder $O_N(s^{\eta_F})$. The reconstruction and characteristic positive powers no longer constitute the principal obstruction.

114 Part 57: Genuine Inner-Lambert Fractional Operator and the Failure of Constant-Coefficient Matching

323. Purpose

Part 56 reduced the fixed-N persistence problem to the inner estimate

$$\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu\sigma^{-\alpha}\left[\mathfrak{h}_\alpha^{\mathrm{crit}} + R_{N,\alpha}^{\mathrm{in}}\right], \qquad R_{N,\alpha}^{\mathrm{in}} = O_N(s^{\eta_F}). \tag{1462}$$

We now compute the leading operator in the genuine Lambert coordinate

$$\xi = \frac{s}{w_N} \tag{1463}$$

at fixed finite N. The calculation shows that (1462) is not the correct inner asymptotic. The sign at the anchor remains favorable, but the inner Lambert singularity generates a different power of s. Consequently the outer coefficient $\mathfrak{h}_\alpha^{\mathrm{crit}}$ cannot be matched to the inner region by a positive-power *relative* error of the form assumed in Part 56.

324. Exact fixed-N inner Lambert power

Let $k_N = k(N) < \infty$ be fixed and put

$$p_N := 2^{-k_N} \in (0, 1/2]. \tag{1464}$$

From (1139)–(1141), for bounded positive ξ,

$$\Delta_{N,k_N}(s) = D_N \xi^{p_N}\big(1 + O_N(\xi^{p_N})\big), \tag{1465}$$

where $D_N > 0$ is independent of s. The exact characteristic invariant used in Part 49 is

$$\Delta_{N,k_N} = K_N q_N e^{-q_N}, \qquad q_N = \frac{1}{b_N}. \tag{1466}$$

Since $q_N \to 0$ as $\Delta_{N,k_N} \to 0$, inversion gives

$$b_N = \frac{K_N}{\Delta_{N,k_N}}\big(1 + O(\Delta_{N,k_N})\big). \tag{1467}$$

Therefore, at every fixed finite N,

$$\boxed{b_N(s) = B_N^{\mathrm{in}} s^{-p_N}\left(1 + O_N(s^{p_N})\right),} \tag{1468}$$

where the factor $w_N^{p_N}$ has been absorbed into the positive constant B_N^{in}. This is the genuine pointwise inner Lambert law. It is not the outer critical power $s^{-\beta_c}$ unless the two exponents happen to coincide.

325. Reconstructed velocities in the inner phase chart

Keep the independent geometric variable x in a compact set on which

$$0 < c_\chi \le |\chi(x)| \le C_\chi < \infty. \tag{1469}$$

This hypothesis is only for the present phase-inner calculation; the separate approach to $\chi = 0$ remains governed by Parts 50–51. The exact reconstruction

$$u_x = \sqrt{-\frac{b_N}{2\chi}}, \qquad u_y = \sqrt{-2\chi b_N} \tag{1470}$$

then gives

$$u_x(s,x) = a_x(x) s^{-p_N/2} \left(1 + O_N(s^{p_N})\right), \qquad u_y(s,x) = a_y(x) s^{-p_N/2} \left(1 + O_N(s^{p_N})\right), \tag{1471}$$

with

$$a_x(x) a_y(x) = B_N^{\mathrm{in}}. \tag{1472}$$

Because $s = t - z - \theta_{*,N}$ is affine in z, the phase-direction part of $(-\Delta)^\alpha$ has exactly the one-dimensional Mellin scaling. For the model power s_+^λ,

$$(-\partial_{ss})^\alpha s_+^\lambda = K_\alpha^+(\lambda) s^{\lambda - 2\alpha}. \tag{1473}$$

Thus the singular phase contribution to the complete fractional block

$$\mathfrak{F}_{N,\alpha} = -\nu \left[u_y (-\Delta)^\alpha u_x + u_x (-\Delta)^\alpha u_y\right] \tag{1474}$$

is

$$\boxed{\mathfrak{F}_{N,\alpha}^{\mathrm{ph}} = -2\nu K_\alpha^+\left(-\frac{p_N}{2}\right) B_N^{\mathrm{in}} s^{-p_N - 2\alpha} \left(1 + O_N(s^{p_N})\right).} \tag{1475}$$

The derivatives falling only on the smooth x-dependent factors in (1471), as well as smooth torus tails separated from the phase singularity, carry at most the amplitude s^{-p_N} and are therefore lower than (1475) by the positive factor $s^{2\alpha}$ on the region (1469). Mixed pseudodifferential corrections have strictly lower phase order and can be absorbed into a positive-power remainder after localization away from $\chi = 0$. Hence the phase singularity determines the leading inner order.

Dividing by (1468) gives the genuine inner normalized law

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu \mathfrak{h}_{N,\alpha}^{\mathrm{in}} s^{-2\alpha} \left(1 + O_N(s^{\eta_{\mathrm{in}}})\right), \qquad \mathfrak{h}_{N,\alpha}^{\mathrm{in}} := -2K_\alpha^+\left(-\frac{p_N}{2}\right),} \tag{1476}$$

for some positive local exponent η_{in} bounded above by p_N and by the localization/pseudodifferential remainder exponent. Formula (1476) is the correct inner replacement for (1455).

326. Signed anchor computation

At $\alpha = 1$,

$$K_1^+(\lambda) = -\lambda(\lambda - 1). \tag{1477}$$

Taking $\lambda = -p_N/2$ gives

$$K_1^+\Big(-\frac{p_N}{2}\Big) = -\frac{p_N}{2}\left(1 + \frac{p_N}{2}\right) < 0, \tag{1478}$$

and therefore

$$\boxed{\mathfrak{h}_{N,1}^{\mathrm{in}} = p_N\left(1 + \frac{p_N}{2}\right) > 0.} \tag{1479}$$

For each fixed finite N, continuity in α consequently yields a number $\delta_N^{\mathrm{in}} > 0$ such that

$$\boxed{\mathfrak{h}_{N,\alpha}^{\mathrm{in}} > 0, \qquad 1 \le \alpha < 1 + \delta_N^{\mathrm{in}}.} \tag{1480}$$

Thus entry into the genuine Lambert layer does not reverse the fractional sign at the Laplacian anchor. What changes is the homogeneity.

327. Why the Part 56 matching ansatz cannot hold

The desired outer law was proportional to $\sigma^{-\alpha}$. If one inserts the characteristic scaling corresponding to the inner Lambert power $\beta = p_N$,

$$\sigma(s) \asymp s^{p_{p_N}}, \qquad p_{p_N} = \frac{12 - 2p_N}{7}, \tag{1481}$$

then

$$\sigma^{-\alpha} \asymp s^{-\alpha(12-2p_N)/7}. \tag{1482}$$

The actual phase-inner operator instead has $s^{-2\alpha}$. Their quotient is

$$\frac{s^{-2\alpha}}{\sigma^{-\alpha}} \asymp s^{-2\alpha+\alpha(12-2p_N)/7} = s^{-2\alpha(1+p_N)/7}. \tag{1483}$$

Since $p_N > 0$, the exponent in (1483) is strictly negative. Hence

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}/b_N}{\nu\mathfrak{h}_\alpha^{\mathrm{crit}}\sigma^{-\alpha}} \not\longrightarrow 1 \qquad (s \downarrow 0,\ N \text{ fixed}).} \tag{1484}$$

In fact its magnitude grows like a negative power of s, provided both coefficients are nonzero. Therefore no estimate

$$R_{N,\alpha}^{\mathrm{in}} = O_N(s^{\eta_F}), \qquad \eta_F > 0, \tag{1485}$$

can repair (1455). The obstruction is not lack of technical precision: the leading inner homogeneity is different.

328. Consequence for the normalized scalar balance

Recall

$$H_N(s) := \frac{s\dot{b}_N}{b_N^2}. \tag{1486}$$

For a tentative power $b_N \asymp s^{-\beta}$, the phase fractional term (1476) contributes to H_N at order

$$\frac{s}{b_N^2}\mathfrak{F}^{\mathrm{ph}}_{N,\alpha} \asymp s^{1+\beta-2\alpha}. \tag{1487}$$

Thus the phase fractional term is lower order only if

$$\beta > 2\alpha - 1, \tag{1488}$$

critical if

$$\boxed{\beta = 2\alpha - 1,} \tag{1489}$$

and dominant if $\beta < 2\alpha - 1$. Since

$$p_N = 2^{-k_N} \le \frac{1}{2} \tag{1490}$$

while $2\alpha - 1 \ge 1$ for $\alpha \ge 1$, the genuine Lambert power $\beta = p_N$ lies in the fractional-dominant regime. Therefore the pointwise invariant law (1468) cannot simultaneously be used as a terminal solution law for the full scalar PDE without a further leading-order balance. The fractional operator feeds back before $s = 0$.

This calculation also explains why the earlier exponent $2\alpha - 1$ reappears naturally: it is the critical exponent selected by the affine phase-direction fractional operator itself. This is logically distinct from assuming that exponent in advance.

329. Corrected status of the outer critical exponent β_c

Part 55 found the geometric critical exponent

$$\beta_c = \frac{12\alpha - 7}{7 + 2\alpha} \tag{1491}$$

by balancing the $\sigma^{-\alpha}$ geometric fractional contribution with the scalar coefficient equation. Part 57 shows that this balance cannot be continued unchanged into the fixed-N phase-inner limit, because the phase-direction contribution has the stronger homogeneity $s^{-2\alpha}$. In particular, near $\alpha = 1$,

$$\beta_c(1) = \frac{5}{9} < 1 = 2\alpha - 1. \tag{1492}$$

Hence a profile $s^{-\beta_c}$ would also enter the phase-fractional dominant regime as $s \downarrow 0$ at fixed N.

Accordingly, the unique positive amplitude derived in Part 55 remains an *outer matched amplitude*; it is not by itself the amplitude of a fixed-N terminal profile extending through the Lambert inner layer.

330. Revised inner problem

The correct next scalar problem is no longer to prove (1455). Instead one must solve the full leading inner balance in $\xi = s/w_N$, retaining the affine phase fractional operator. In normalized form the required equation has the structure

$$H_N(s) = g_{N,\alpha}(s) + \nu \mathfrak{h}^{\mathrm{in}}_{N,\alpha} \frac{s^{1-2\alpha}}{b_N(s)} + E^{\mathrm{in}}_N(s), \tag{1493}$$

with the already determined positive sign at $\alpha = 1$ and with E^{in}_N containing reconstruction, transverse geometry, and lower phase orders. A power-law fixed point of (1493) can occur only after the coefficient at the critical phase exponent

$$\beta_{\mathrm{ph}} = 2\alpha - 1 \tag{1494}$$

is matched. The corresponding amplitude equation and its stability must be derived from the actual finite-N inner equation; they cannot be imported from the outer β_c calculation.

Theorem 52 (Fixed-N inner fractional audit)**.** *Fix $N < \infty$ and $k_N \geq 1$, and work on a transverse compact set satisfying (1469). The actual Lambert cascade gives the inner pointwise power $b_N \sim B^{\mathrm{in}}_N s^{-p_N}$ with $p_N = 2^{-k_N}$. The complete phase-direction contribution to the reconstructed fractional block has leading normalized form (1476). At $\alpha = 1$ its coefficient is strictly positive, namely $p_N(1 + p_N/2)$. Nevertheless its homogeneity is $s^{-2\alpha}$, not $\sigma^{-\alpha}$. Therefore the positive-power matching estimate (1455)–(1456) to the outer coefficient $\mathfrak{h}^{\mathrm{crit}}_\alpha$ is false in the fixed-N inner limit. The phase fractional term is dominant at the Lambert exponent p_N and becomes critical at $\beta = 2\alpha - 1$. Consequently Part 56's conditional persistence theorem cannot be made unconditional by proving (1456); the full inner scalar balance must instead be rebuilt with the affine phase fractional contribution retained at leading order.*

331. Updated proof ledger

actual inner Lambert exponent:	$p_N = 2^{-k_N}$,
inner reconstructed phase coefficient:	$\mathfrak{h}^{\mathrm{in}}_{N,\alpha} = -2K^+_\alpha(-p_N/2)$,
anchor sign:	positive, $\mathfrak{h}^{\mathrm{in}}_{N,1} = p_N(1 + p_N/2)$,
inner normalized homogeneity:	$\mathfrak{F}/b \asymp s^{-2\alpha}$,
matching to $\sigma^{-\alpha}\mathfrak{h}^{\mathrm{crit}}_\alpha$:	fails by the power (1483),
Part 56 estimate (1456):	cannot hold in the stated form,
outer amplitude from Part 55:	valid only as an outer matched amplitude,
phase-fractional critical exponent:	$\beta_{\mathrm{ph}} = 2\alpha - 1$,
finite-time scalar blow-up:	not yet proved; requires the new inner balance (1493).

(1495)

The calculation therefore resolves the specific question posed at the end of Part 56: the inner signed coefficient remains favorable at the anchor, but it does *not* converge to $\mathfrak{h}^{\mathrm{crit}}_\alpha$

with a positive-power relative error. The reason is a genuine change of leading homogeneity. The next rigorous step is to derive and analyze the amplitude fixed point of (1493) at $\beta = 2\alpha - 1$, including the exact finite-N reconstruction errors, before any finite-time scalar blow-up conclusion is made.

Part VI
Fixed-N Inner Amplitude Fixed Point, Stability, and Compatibility Audit

115 Purpose of the audit

Part 57 reduced the terminal scalar problem to

$$H_N(s) = g_{N,\alpha}(s) + \nu \mathfrak{h}^{\mathrm{in}}_{N,\alpha} \frac{s^{1-2\alpha}}{b_N(s)} + E^{\mathrm{in}}_N(s), \tag{1496}$$

where

$$H_N(s) := \frac{s\dot{b}_N}{b_N^2}, \qquad g_{N,\alpha}(s) = 2^{-k_N} + O_N(s^{2^{-k_N}}). \tag{1497}$$

The purpose here is to derive the amplitude fixed point at the phase-critical exponent, prove the stability of that fixed point for the normalized scalar equation, quantify the reconstruction error, and then check compatibility with the exact finite-N Lambert invariant. The last check is essential: a stable fixed point of the scalar balance is useful only if it is compatible with all exact relations imposed by the construction.

116 The fractional coefficient must be recomputed at the candidate exponent

For a general phase power

$$b_N(s) \sim B s^{-\beta}, \tag{1498}$$

the reconstructed velocities on a transverse compact set with $0 < c_\chi \le |\chi| \le C_\chi$ have phase exponent $-\beta/2$. Hence the complete leading phase fractional coefficient is

$$\boxed{\mathfrak{h}^{\mathrm{ph}}_\alpha(\beta) := -2K^+_\alpha\left(-\frac{\beta}{2}\right).} \tag{1499}$$

This corrects a possible ambiguity in (1493): once the scalar exponent changes from the raw Lambert exponent p_N to a candidate PDE-selected exponent, the Mellin coefficient must be evaluated at the new exponent.

The phase-critical exponent is

$$\boxed{\beta_{\mathrm{ph}} = 2\alpha - 1.} \tag{1500}$$

Accordingly define

$$\boxed{h_\alpha^{\mathrm{ph}} := -2K_\alpha^+\left(-\frac{2\alpha-1}{2}\right).} \tag{1501}$$

At $\alpha = 1$,

$$K_1^+(\lambda) = -\lambda(\lambda - 1), \tag{1502}$$

so

$$K_1^+(-1/2) = -\frac{3}{4}, \qquad \boxed{h_1^{\mathrm{ph}} = \frac{3}{2} > 0.} \tag{1503}$$

Therefore, by continuity, there exists $\delta_{\mathrm{ph}} > 0$ such that

$$\boxed{h_\alpha^{\mathrm{ph}} > 0, \qquad 1 \le \alpha < 1 + \delta_{\mathrm{ph}}.} \tag{1504}$$

No numerical value of δ_{ph} is asserted without certified bounds for the Gamma factors in K_α^+.

117 Exact formal amplitude fixed point

Set

$$\beta := 2\alpha - 1, \qquad b_N(s) = B_N s^{-\beta} F_N(s). \tag{1505}$$

For a persistent power law, $H_N \to \beta$. At the critical exponent,

$$\frac{s^{1-2\alpha}}{b_N(s)} = \frac{1}{B_N F_N(s)}. \tag{1506}$$

Let

$$g_{*,N} := \lim_{s\downarrow 0} g_{N,\alpha}(s) = 2^{-k_N}. \tag{1507}$$

Neglecting only terms which tend to zero, (1496) therefore requires

$$\beta = g_{*,N} + \frac{\nu h_\alpha^{\mathrm{ph}}}{B_N}. \tag{1508}$$

Thus the candidate amplitude is uniquely determined:

$$\boxed{B_{*,N} = \frac{\nu h_\alpha^{\mathrm{ph}}}{2\alpha - 1 - 2^{-k_N}}.} \tag{1509}$$

For $k_N \ge 1$ and $\alpha \ge 1$,

$$2\alpha - 1 - 2^{-k_N} \ge \frac{1}{2} > 0, \tag{1510}$$

so (1509) is a unique positive number whenever (1504) holds. In particular, at $\alpha = 1$,

$$\boxed{B_{*,N} = \frac{(3/2)\nu}{1 - 2^{-k_N}} > 0.} \tag{1511}$$

This is the exact positive amplitude fixed point of the *formal phase-critical scalar balance*.

118 Stability of the formal fixed point

The exact profile identity from Part 53 is

$$\frac{d\log F_N}{d\log s} = \beta - \frac{H_N(s)}{1 - b_N(s)^{-1}}. \tag{1512}$$

Introduce terminal logarithmic time

$$\tau := -\log(s/s_e), \qquad \tau \to +\infty \quad (s \downarrow 0). \tag{1513}$$

Then, using (1509),

$$\frac{d\log F_N}{d\tau} = -\beta + g_{*,N} + \frac{\beta - g_{*,N}}{F_N} + \mathcal{E}_N(\tau), \tag{1514}$$

where $\mathcal{E}_N$ contains the convergence error in $g_{N,\alpha}$, the factor $(1 - b_N^{-1})^{-1} - 1$, the fractional positive-power remainder, and E_N^{in}.

Let $Y_N = \log F_N$. The autonomous part is

$$\mathscr{V}_N(Y) = -(\beta - g_{*,N}) + (\beta - g_{*,N})e^{-Y}. \tag{1515}$$

It has the unique zero $Y = 0$, and

$$\boxed{\mathscr{V}_N'(0) = -(\beta - g_{*,N}) < 0.} \tag{1516}$$

Hence the amplitude fixed point is linearly asymptotically stable in terminal logarithmic time. More quantitatively, for $|Y| \le Y_0$,

$$Y\mathscr{V}_N(Y) \le -c_N Y^2, \qquad c_N > 0. \tag{1517}$$

If

$$|\mathcal{E}_N(\tau)| \le C_N e^{-\eta\tau} \tag{1518}$$

for some $\eta > 0$, the variation-of-constants/Gronwall estimate gives

$$\boxed{|Y_N(\tau)| \le C_N e^{-\min\{c_N,\eta\}\tau},} \tag{1519}$$

provided the trajectory enters a sufficiently small fixed neighborhood of the equilibrium. Equivalently,

$$\boxed{F_N(s) = 1 + O_N(s^{\eta_*}), \qquad \eta_* := \min\{c_N, \eta\} > 0.} \tag{1520}$$

Thus the formal phase-critical amplitude is not merely algebraically matched; it is a stable scalar fixed point if the normalized errors possess positive powers.

119 Positive-power reconstruction error at the phase-critical exponent

The reconstruction estimates of Part 56 were obtained for general β:

$$|E_N(s)| \le C_N \left(\sigma^{q_\beta} + \sigma^{m_\beta}\right), \qquad q_\beta = \frac{\beta+1}{6-\beta}, \qquad m_\beta > q_\beta. \tag{1521}$$

The corresponding characteristic exponent is

$$p_\beta = \frac{12-2\beta}{7}. \tag{1522}$$

At $\beta = 2\alpha - 1$,

$$p_{\rm ph} = \frac{14-4\alpha}{7} = \frac{2(7-2\alpha)}{7}, \tag{1523}$$

and

$$q_{\rm ph} = \frac{2\alpha}{7-2\alpha}. \tag{1524}$$

Their product simplifies exactly to

$$\boxed{p_{\rm ph} q_{\rm ph} = \frac{4\alpha}{7}.} \tag{1525}$$

Consequently, on a two-sided phase-critical characteristic bootstrap $\sigma \asymp s^{p_{\rm ph}}$,

$$\boxed{E_N^{\rm rec/forc}(s) = O_N(s^{4\alpha/7}).} \tag{1526}$$

This is a genuine positive power. In particular it is logarithmically integrable:

$$\int_0^{s_e} |E_N^{\rm rec/forc}(s)| \frac{ds}{s} < \infty. \tag{1527}$$

The Lambert sensitivity error also has the positive power

$$\boxed{g_{N,\alpha}(s) - g_{*,N} = O_N(s^{2^{-k_N}}).} \tag{1528}$$

Finally, $b_N^{-1} = O(s^{2\alpha-1})$ on the candidate bootstrap. Thus, apart from the localized fractional remainder already isolated in Part 57, all explicitly audited terms entering $\mathcal{E}_N$ have positive powers.

120 Conditional persistence theorem for the rebuilt inner equation

Assume that the complete localized phase fractional expansion at the candidate exponent satisfies

$$\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu h_\alpha^{\rm ph} s^{-2\alpha} \left(1 + O_N(s^{\eta_F})\right), \qquad \eta_F > 0, \tag{1529}$$

and that the two-sided characteristic reconstruction used in (1526) holds. Then, with

$$\eta = \min\left\{2^{-k_N},\ \frac{4\alpha}{7},\ 2\alpha - 1,\ \eta_F\right\} > 0, \tag{1530}$$

we have

$$|\mathcal{E}_N(\tau)| \le C_N e^{-\eta\tau}. \tag{1531}$$

The stability estimate (1519) therefore closes the local bootstrap and yields

$$\boxed{b_N(s) = B_{*,N} s^{-(2\alpha-1)}\left(1 + O_N(s^{\eta_*})\right).} \tag{1532}$$

This is the desired persistence theorem for the *rebuilt scalar inner equation*, conditional only on the positive-power localized fractional remainder (1529) and entry into the stable neighborhood.

121 Compatibility with the exact Lambert invariant

There is, however, a second exact relation already present in the active finite-N construction. Part 57 used

$$\Delta_{N,k_N} = K_N q_N e^{-q_N}, \qquad q_N = \frac{1}{b_N}, \tag{1533}$$

together with

$$\Delta_{N,k_N}(s) = D_N(s/w_N)^{p_N}\left(1 + O_N(s^{p_N})\right), \qquad p_N = 2^{-k_N}, \tag{1534}$$

to obtain the exact inner consequence

$$\boxed{b_N(s) = B_N^{\mathrm{in}} s^{-p_N}\left(1 + O_N(s^{p_N})\right).} \tag{1535}$$

If (1533)–(1535) are imposed on the *same* b_N all the way to $s = 0$, then (1532) can be compatible with them only if

$$p_N = 2\alpha - 1. \tag{1536}$$

But for $k_N \ge 1$ and $\alpha \ge 1$,

$$p_N = 2^{-k_N} \le \frac{1}{2}, \qquad 2\alpha - 1 \ge 1. \tag{1537}$$

Hence

$$\boxed{p_N \ne 2\alpha - 1.} \tag{1538}$$

This is an exponent-level incompatibility; no choice of positive amplitude can remove it.

The same obstruction is visible directly in the normalized scalar equation. If the exact Lambert law (1535) is retained, then $H_N \to p_N$, whereas the positive phase fractional term in (1496) behaves like

$$\nu h_{N,\alpha}^{\mathrm{in}} \frac{s^{1-2\alpha}}{b_N(s)} \asymp s^{1+p_N-2\alpha}. \tag{1539}$$

Since

$$1 + p_N - 2\alpha < 0, \tag{1540}$$

this term diverges positively as $s \downarrow 0$ while the left side tends to the finite number p_N. The reconstruction error (1526), being a positive power on the corresponding bootstrap, cannot cancel that negative-power divergence. Therefore the raw Lambert invariant profile cannot solve the full scalar PDE through the terminal endpoint when the positive phase fractional term is retained.

Theorem 53 (Inner amplitude and compatibility audit)**.** *Fix $N < \infty$, $k_N \geq 1$, and $1 \leq \alpha < 1 + \delta_{\mathrm{ph}}$. For the rebuilt phase-critical scalar balance, the exponent $\beta = 2\alpha - 1$ has the unique positive formal amplitude (1509). This amplitude is linearly asymptotically stable in terminal logarithmic time, and the already audited reconstruction/forcing and Lambert-sensitivity errors decay with positive powers, including the explicit reconstruction exponent $4\alpha/7$. If the remaining localized fractional remainder has a positive-power expansion, then the rebuilt scalar equation has the conditional persistent asymptotic (1532). However, the exact finite-N Lambert invariant used in Part 57 simultaneously enforces the distinct exponent $p_N = 2^{-k_N}$. For $k_N \geq 1$ and $\alpha \geq 1$ these exponents cannot agree. Consequently the current construction cannot simultaneously retain the exact Lambert invariant (1533) and the phase-critical persistent solution of the full scalar PDE all the way to $s = 0$. A finite-time scalar blow-up theorem does not follow until this compatibility conflict is resolved by identifying which relation belongs to the actual full PDE rather than to a reduced characteristic model.*

122 Finite physical time: conditional calculation and present status

If the phase-critical persistent law (1532) were established for the actual scalar PDE after resolving the compatibility issue, then the characteristic identity

$$\dot{s} = 1 - b_N \tag{1541}$$

would give

$$T_N - t = \int_0^{s(t)} \frac{d\xi}{b_N(\xi) - 1} = \frac{s(t)^{2\alpha}}{2\alpha B_{*,N}} \left(1 + O_N(s^{\eta_{**}})\right), \tag{1542}$$

for some $\eta_{**} > 0$. Therefore

$$\boxed{b_N(t) \asymp (T_N - t)^{-(2\alpha-1)/(2\alpha)}.} \tag{1543}$$

This finite-time computation is rigorous *conditional on* the persistent phase-critical law, but the compatibility theorem above shows that the current exact Lambert invariant prevents us from promoting that conditional statement to the actual finite-N construction as presently written.

123 Updated proof ledger

phase-critical exponent:	$\beta_{\mathrm{ph}} = 2\alpha - 1$,
correct coefficient at that exponent:	$h_\alpha^{\mathrm{ph}} = -2K_\alpha^+(-(2\alpha-1)/2)$,
anchor sign:	$h_1^{\mathrm{ph}} = 3/2 > 0$,
formal amplitude:	$B_{*,N} = \nu h_\alpha^{\mathrm{ph}}/(2\alpha - 1 - 2^{-k_N}) > 0$,
linear stability:	proved for the normalized rebuilt inner equation,
Lambert sensitivity error:	$O_N(s^{2^{-k_N}})$,
reconstruction/forcing error:	$O_N(s^{4\alpha/7})$,
localized fractional positive-power remainder:	still required for unconditional persistence,
exact Lambert invariant exponent:	$p_N = 2^{-k_N}$,
compatibility with $2\alpha - 1$:	impossible for $k_N \geq 1$, $\alpha \geq 1$,
finite physical-time rate:	conditional rate $(T_N - t)^{-(2\alpha-1)/(2\alpha)}$,
finite-time scalar blow-up theorem:	not established; exact-model compatibility must be resolved first.

(1544)

The amplitude calculation therefore closes algebraically and dynamically, but it also exposes a sharper obstruction than a missing remainder estimate: the raw finite-N Lambert invariant and the full phase-fractional scalar balance select different terminal exponents. The next rigorous task is to trace the derivation of (1533) and determine whether it is an exact invariant of the full reconstructed scalar PDE or only of the reduced Lambert characteristic subsystem. Only after that distinction is settled can one consistently choose the terminal equation whose persistence is to be proved.

Part VII
Provenance Audit of the Lambert Characteristic Invariant

124 The equation from which the invariant was actually derived

The compatibility issue isolated in Part 58 can now be decided directly by tracing the derivation of

$$\Delta_{N,k_N} = K_N q_N e^{-q_N}, \qquad q_N = \frac{1}{b_N}. \tag{1545}$$

We did not derive (1545) from the full reconstructed scalar equation. It first replaces that equation by the reduced transport model

$$b_t + bb_z = -b^2 \partial_z \log \Delta_{N,k}. \tag{1546}$$

With the affine phase

$$\theta = t - z, \tag{1547}$$

a characteristic $Z(t)$ of (1546) satisfies

$$\dot{Z} = B, \qquad B(t) = b(t, Z(t)), \qquad \dot{\theta} = 1 - B. \tag{1548}$$

Along the same reduced characteristic,

$$\dot{B} = B^2 \partial_\theta \log \Delta_{N,k}. \tag{1549}$$

Introducing $q = 1/B$ therefore gives

$$\dot{q} = -\partial_\theta \log \Delta_{N,k}, \qquad \dot{\theta} = \frac{q-1}{q}. \tag{1550}$$

Dividing the two equations yields

$$\frac{dq}{d\theta} = -\frac{q}{q-1} \partial_\theta \log \Delta_{N,k}, \tag{1551}$$

and hence

$$\left(1 - \frac{1}{q}\right) dq = -d \log \Delta_{N,k}. \tag{1552}$$

Integration gives

$$q - \log q + \log \Delta_{N,k} = C, \tag{1553}$$

which exponentiates to

$$\boxed{\Delta_{N,k} = K q e^{-q}.} \tag{1554}$$

Thus (1554) is indeed exact for (1546), but its proof uses the reduced identity (1549) at the decisive step.

125 Comparison with the actual reconstructed scalar equation

The scalar equation audited in Parts 51–58 is instead

$$\boxed{\dot{b}_N = \Gamma_N b_N^2 + \mathfrak{F}_{N,\alpha} + 2 C_N G_N - \mathcal{R}_N^{\mathrm{rec}},} \qquad \Gamma_N = \partial_\theta \log \Delta_{N,k_N} = \frac{g_{N,\alpha}(s)}{s}. \tag{1555}$$

For clarity define the complete non-Lambert contribution

$$\boxed{\mathscr{P}_{N,\alpha} := \mathfrak{F}_{N,\alpha} + 2 C_N G_N - \mathcal{R}_N^{\mathrm{rec}}.} \tag{1556}$$

Then (1555) is

$$\dot{b}_N = b_N^2 \partial_\theta \log \Delta_{N,k_N} + \mathscr{P}_{N,\alpha}. \tag{1557}$$

Consequently, for $q_N = 1/b_N$,

$$\boxed{\dot{q}_N = -\partial_\theta \log \Delta_{N,k_N} - \frac{\mathscr{P}_{N,\alpha}}{b_N^2}.} \tag{1558}$$

The second term in (1558) is absent from the reduced derivation. Therefore one cannot pass from the full scalar PDE to (1550) unless $\mathscr{P}_{N,\alpha} \equiv 0$ along the characteristic.

126 Exact defect formula for the would-be invariant

Define

$$\mathscr{I}_N(t) := q_N - \log q_N + \log \Delta_{N,k_N}(\theta(t)). \tag{1559}$$

Using

$$\dot{\theta} = 1 - b_N = 1 - \frac{1}{q_N} \tag{1560}$$

and differentiating (1559) gives

$$\begin{aligned} \dot{\mathscr{I}}_N &= \left(1 - \frac{1}{q_N}\right) \dot{q}_N + (\partial_\theta \log \Delta_{N,k_N})\dot{\theta} \\ &= \left(1 - \frac{1}{q_N}\right) (\dot{q}_N + \partial_\theta \log \Delta_{N,k_N}) \,. \end{aligned} \tag{1561}$$

Substitution of (1558) therefore gives the exact identity

$$\boxed{\dot{\mathscr{I}}_N = \frac{b_N - 1}{b_N^2}\, \mathscr{P}_{N,\alpha}.} \tag{1562}$$

In particular,

$$\boxed{\dot{\mathscr{I}}_N = 0 \quad \Longleftrightarrow \quad (b_N - 1)\mathscr{P}_{N,\alpha} = 0.} \tag{1563}$$

On the terminal branch $b_N > 1$, constancy of the Lambert invariant is thus equivalent to the vanishing of the entire fractional/reconstruction/forcing correction.

127 The correct full-PDE replacement: a running Lambert constant

Set

$$\mathscr{K}_N(t) := e^{\mathscr{I}_N(t)}. \tag{1564}$$

Then the algebraic relation

$$\boxed{\Delta_{N,k_N}(\theta(t)) = \mathscr{K}_N(t) q_N(t) e^{-q_N(t)}} \tag{1565}$$

is an identity by definition, but $\mathscr{K}_N$ is no longer constant. Indeed (1562) gives

$$\boxed{\frac{\dot{\mathscr{K}}_N}{\mathscr{K}_N} = \frac{b_N - 1}{b_N^2} \left(\mathfrak{F}_{N,\alpha} + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}}\right).} \tag{1566}$$

Hence

$$\boxed{\mathscr{K}_N(t) = \mathscr{K}_N(t_e) \exp\left[\int_{t_e}^{t} \frac{b_N(\tau) - 1}{b_N(\tau)^2}\, \mathscr{P}_{N,\alpha}(\tau)\, d\tau\right].} \tag{1567}$$

Equation (1567) is the exact full-scalar-PDE replacement for the constant K_N used in the reduced Lambert model.

128 Why the phase-critical fixed point is no longer excluded

Part 58 found an apparent exponent conflict by imposing simultaneously

$$b_N \sim B_N^{\mathrm{in}} s^{-2^{-k_N}} \tag{1568}$$

from constant-K_N Lambert invariance and

$$b_N \sim B_{*,N} s^{-(2\alpha-1)} \tag{1569}$$

from the complete phase-fractional balance. The provenance calculation above shows that (1568) is not an exact consequence of the full scalar PDE: it requires $\mathscr{K}_N$ to be constant, while the full equation forces (1566).

This can be seen quantitatively at the phase-critical profile. Suppose conditionally that

$$b_N(s) = B_{*,N} s^{-\beta}(1 + o(1)), \qquad \beta = 2\alpha - 1, \tag{1570}$$

and that the amplitude balance from Part 58 holds. Since

$$H_N = \frac{s\dot{b}_N}{b_N^2} = g_{N,\alpha} + \frac{s\mathscr{P}_{N,\alpha}}{b_N^2}, \tag{1571}$$

we have at the matched equilibrium

$$\frac{s\mathscr{P}_{N,\alpha}}{b_N^2} = \beta - g_{*,N} + o(1), \qquad g_{*,N} = 2^{-k_N}. \tag{1572}$$

Using $\dot{s} = 1 - b_N$ in (1566),

$$\frac{d\log \mathscr{K}_N}{d\log s} = \frac{s}{1-b_N}\frac{b_N - 1}{b_N^2}\mathscr{P}_{N,\alpha} = -\frac{s\mathscr{P}_{N,\alpha}}{b_N^2}. \tag{1573}$$

Therefore

$$\boxed{\frac{d\log \mathscr{K}_N}{d\log s} = -(\beta - g_{*,N}) + o(1).} \tag{1574}$$

Integration yields

$$\boxed{\mathscr{K}_N(s) = \mathscr{K}_{0,N} s^{-(\beta - g_{*,N})}(1 + o(1)).} \tag{1575}$$

Since the Lambert cascade itself has

$$\Delta_{N,k_N}(s) \asymp s^{g_{*,N}}, \tag{1576}$$

while $q_N \asymp s^{\beta}$, relation (1565) requires precisely

$$\mathscr{K}_N(s) \asymp \frac{s^{g_{*,N}}}{s^{\beta}} = s^{-(\beta - g_{*,N})}, \tag{1577}$$

which agrees exactly with (1575). Thus the running constant absorbs the exponent difference found in Part 58. The two exponents are not incompatible once the full-PDE defect is retained.

129 Stability interpretation

The conclusion above does not by itself prove persistence, but it removes the artificial constant-invariant obstruction. The full scalar equation must be studied with the coupled pair

$$\boxed{\begin{aligned} H_N &= g_{N,\alpha} + \frac{s\mathscr{P}_{N,\alpha}}{b_N^2}, \\ \frac{d\log \mathscr{K}_N}{d\log s} &= -\frac{s\mathscr{P}_{N,\alpha}}{b_N^2}. \end{aligned}} \tag{1578}$$

Adding the two equations gives the exact identity

$$\boxed{H_N + \frac{d\log \mathscr{K}_N}{d\log s} = g_{N,\alpha}.} \tag{1579}$$

At the phase-critical equilibrium $H_N \to \beta$ and $g_{N,\alpha} \to g_{*,N}$, so (1579) again forces the running exponent $-(\beta - g_{*,N})$. There is therefore no need to require $\beta = g_{*,N}$ for the full PDE.

Theorem 54 (Provenance of the Lambert invariant and exact full-PDE defect)**.** *The relation*

$$\Delta_{N,k_N} = K_N q_N e^{-q_N}$$

with constant K_N is an exact first integral of the reduced characteristic model (1546). Its derivation uses $\dot{b}_N = b_N^2 \partial_\theta \log \Delta_{N,k_N}$ and therefore does not remain an exact first integral after the fractional, forcing, and reconstruction terms of the actual scalar equation are restored. For the full scalar PDE, the exact replacement is (1565)–(1567), with

$$\frac{\dot{\mathscr{K}}_N}{\mathscr{K}_N} = \frac{b_N - 1}{b_N^2}\mathscr{P}_{N,\alpha}.$$

Consequently the exponent conflict of Part 58 is not an obstruction to the phase-critical fixed point: at a persistent profile $b_N \asymp s^{-(2\alpha-1)}$, the running factor develops exactly the power needed to reconcile the Lambert cascade exponent 2^{-k_N} with the phase-critical scalar exponent. This removes the constant-invariant compatibility obstruction, but it does not by itself prove the remaining localized fractional positive-power estimate or nonlinear persistence of the full scalar PDE.

130 Corrected proof ledger after the provenance audit

constant Lambert invariant:	exact only for the reduced transport model (1546),
full scalar equation:	contains $\mathscr{P}_{N,\alpha} = \mathfrak{F}_{N,\alpha} + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}}$,
exact invariant defect:	$\dot{\mathscr{I}}_N = (b_N - 1)\mathscr{P}_{N,\alpha}/b_N^2$,
running Lambert factor:	equations (1565)–(1567),
Part 58 exponent conflict:	removed for the full scalar PDE,
phase-critical amplitude:	no longer excluded by constant-K_N compatibility,
running-factor exponent:	$\mathscr{K}_N \asymp s^{-(2\alpha-1-2^{-k_N})}$ conditionally at the fixed point,
localized fractional positive-power remainder:	still to be proved for unconditional persistence,
finite physical-time law:	remains conditional on full-PDE persistence,
finite-time scalar blow-up theorem:	not yet established.

(1580)

The decisive compatibility question is therefore resolved: the constant Lambert invariant belongs to the reduced characteristic subsystem, not to the complete reconstructed b_3 equation. The stable phase-critical amplitude derived in Part 58 is consequently admissible with respect to this particular obstruction. The remaining scalar task is now genuinely dynamical: prove the localized fractional positive-power expansion and use it to close the stable persistence bootstrap for the full scalar equation, with the running factor $\mathscr{K}_N$ rather than a constant K_N.

Part VIII
Localized Fractional Positive-Power Persistence Audit

131 The remaining estimate after the running-invariant correction

The provenance audit changes the role of the Lambert relation but does not change the local phase-fractional homogeneity. We therefore return to the phase-critical exponent

$$\beta := 2\alpha - 1, \qquad b_N(s) = B_{*,N} s^{-\beta} F_N(s), \qquad B_{*,N} = \frac{\nu h_\alpha^{\mathrm{ph}}}{\beta - g_{*,N}}, \tag{1581}$$

where

$$g_{*,N} = 2^{-k_N}, \qquad h_\alpha^{\mathrm{ph}} = -2K_\alpha^+\left(-\frac{\beta}{2}\right) > 0 \tag{1582}$$

on the continuation interval already fixed in Part 58. The exact normalized profile equation is

$$\frac{d\log F_N}{d\log s} = \beta - \frac{H_N(s)}{1 - b_N(s)^{-1}}, \qquad H_N = g_{N,\alpha} + \frac{s\mathscr{P}_{N,\alpha}}{b_N^2}. \tag{1583}$$

All terms in the right side except the localized fractional block have already been assigned positive powers in Part 58. The question is whether one can justify

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{b_N} = \nu h_\alpha^{\mathrm{ph}} s^{-2\alpha}\left(1 + O_N(s^{\eta_F})\right), \qquad \eta_F > 0,} \tag{1584}$$

for the *full reconstructed fields*, without returning to the false constant-K_N invariant.

132 A proof localization is not a modification of the construction

Fix a transverse compact set

$$0 < c_\chi \le |\chi(x)| \le C_\chi \tag{1585}$$

and choose smooth proof cutoffs ψ_s, ψ_x which equal one on a smaller terminal cylinder. They are used only to decompose the operator; they do not alter b_N, u_x, u_y, or the PDE. Write

$$u_j = u_j^{\mathrm{loc}} + u_j^{\mathrm{far}}, \qquad u_j^{\mathrm{loc}} = \psi_s\psi_x u_j, \qquad j = x, y. \tag{1586}$$

On the support of the evaluation cutoff, the Schwartz kernel of $(-\Delta)^\alpha$ applied to u_j^{far} is smooth because the two supports are separated. Consequently, for every fixed finite N and every finite number of derivatives,

$$\boxed{|\partial_s^m(-\Delta)^\alpha u_j^{\mathrm{far}}| \le C_{N,m},} \tag{1587}$$

provided the finite-N reconstructed field is smooth away from the terminal phase point. Relative to the singular local scale $s^{-\beta/2-2\alpha}$, (1587) is a positive-power error of order at least $s^{\beta/2+2\alpha}$.

Thus the nonlocality of the torus operator is not itself an obstruction in the terminal chart. The only delicate part is the singular local operator acting on the conormal phase profile.

133 Fourier reduction of the localized full operator

Expand a localized reconstructed component in transverse Fourier modes,

$$u_j^{\mathrm{loc}}(x, s) = \sum_{m\in\mathbb{Z}} a_{j,m}(s)e^{imx}. \tag{1588}$$

For each m, the spatial fractional multiplier is

$$\mathcal{L}_{\alpha,m} := (m^2 - \partial_{ss})^\alpha. \tag{1589}$$

The singular phase part is $\mathcal{L}_{\alpha,0} = (-\partial_{ss})^\alpha$. For fixed m the high phase-frequency symbol has the expansion

$$(\xi^2 + m^2)^\alpha = |\xi|^{2\alpha}\left(1 + \alpha\frac{m^2}{\xi^2} + O_\alpha\left(\frac{m^4}{\xi^4}\right)\right). \tag{1590}$$

After localization this is the standard conormal pseudodifferential expansion

$$\boxed{\mathcal{L}_{\alpha,m} = (-\partial_{ss})^\alpha + \alpha m^2(-\partial_{ss})^{\alpha-1} + \mathcal{R}^{(4)}_{\alpha,m},} \tag{1591}$$

where, on a phase power s^λ and its conormal perturbations, the second term is lower than the first by s^2, while $\mathcal{R}^{(4)}_{\alpha,m}$ is lower by s^4. Summation over m is legitimate for the proof-localized finite-N fields because their smooth transverse dependence gives rapid Fourier decay.

In particular, transverse geometry does not change the leading Mellin coefficient. It enters the localized fractional block only through positive even powers beginning with s^2.

134 Conormal Mellin lemma for a positive-power perturbation

Let

$$\lambda = -\frac{\beta}{2} = -\frac{2\alpha - 1}{2}. \tag{1592}$$

Suppose on the terminal cylinder

$$a(s) = a_0 s^\lambda(1 + r(s)), \qquad \max_{0\le j\le M} |E^j r(s)| \le C_N s^\eta, \qquad E = s\partial_s, \tag{1593}$$

for some $\eta > 0$ and sufficiently large fixed M. Shrink η, if necessary, so that the Mellin line segment from λ to $\lambda + \eta$ avoids the poles of the meromorphic Mellin symbol. Then

$$\boxed{(-\partial_{ss})^\alpha a(s) = a_0 K^+_\alpha(\lambda) s^{\lambda-2\alpha}\left(1 + O_N(s^\eta)\right).} \tag{1594}$$

The same estimate holds after a fixed finite number of Euler derivatives.

Proof. Write $r(s) = s^\eta\rho(s)$ after decreasing η slightly if necessary, with $E^j\rho$ bounded. Mellin conjugation gives

$$(-\partial_{ss})^\alpha\big(s^\lambda r(s)\big) = s^{\lambda+\eta-2\alpha}\mathcal{M}_{\lambda+\eta}\rho(\log s). \tag{1595}$$

On a closed pole-free Mellin strip, the multiplier and the finitely many Euler commutators are bounded on the weighted conormal space used in the continuation argument. Hence the right side of (1595) is $O_N(s^{\lambda+\eta-2\alpha})$. The leading term is the exact Mellin identity

$$(-\partial_{ss})^\alpha s^\lambda = K^+_\alpha(\lambda) s^{\lambda-2\alpha}. \tag{1596}$$

Combining (1595)–(1596) proves (1594). The proof uses a positive-width Mellin strip, not a pointwise $o(1)$ estimate; this is why conormal control is the correct persistence norm. □ □

135 Application to the reconstructed velocity pair

On (1585), the square-root reconstruction is

$$u_x = \left(-\frac{1}{2\chi}\right)^{1/2} b_N^{1/2}, \qquad u_y = (-2\chi)^{1/2} b_N^{1/2}. \tag{1597}$$

Normalize the phase-critical amplitude by

$$F_N = 1 + v_N. \tag{1598}$$

If the terminal tube is measured in the conormal norm

$$\boxed{\|v_N\|_{\mathcal{Z}_\eta} := \max_{0\le j\le M} \sup_{0<s\le s_e} s^{-\eta} |E^j v_N(s)|,} \tag{1599}$$

then smooth composition of $(1+v)^{1/2}$ gives

$$u_x = a_x(x) s^{-\beta/2} \left(1 + \frac{1}{2} v_N + O(v_N^2)\right), \qquad u_y = a_y(x) s^{-\beta/2} \left(1 + \frac{1}{2} v_N + O(v_N^2)\right), \tag{1600}$$

with $a_x a_y = B_{*,N}$. Equations (1587), (1591), and (1594) therefore give

$$(-\Delta)^\alpha u_j = a_j K_\alpha^+(-\beta/2) s^{-\beta/2-2\alpha} \left[1 + O_N(s^{\eta_F})\right], \tag{1601}$$

where one may take

$$\boxed{\eta_F < \min\{\eta, 2, \beta/2 + 2\alpha\}.} \tag{1602}$$

The exact product form of the complete fractional block is

$$\mathfrak{F}_{N,\alpha} = -\nu \left[u_y (-\Delta)^\alpha u_x + u_x (-\Delta)^\alpha u_y\right]. \tag{1603}$$

Substituting (1600)–(1601), using $u_x u_y = b_N$, yields

$$\boxed{\frac{\mathfrak{F}_{N,\alpha}}{b_N} = -2\nu K_\alpha^+ \left(-\frac{\beta}{2}\right) s^{-2\alpha} \left[1 + O_N(s^{\eta_F})\right] = \nu h_\alpha^{\mathrm{ph}} s^{-2\alpha} \left[1 + O_N(s^{\eta_F})\right].} \tag{1604}$$

Thus the formerly missing positive-power fractional estimate is a direct consequence of a positive-power *conormal* tube for the reconstructed fields. No constant Lambert invariant is used.

136 The bootstrap closes in the conormal tube

The pointwise stability calculation of Part 58 can now be upgraded to the same conormal norm. Put

$$\tau = -\log(s/s_e), \qquad Y_N = \log F_N. \tag{1605}$$

The normalized equation has the form

$$\partial_\tau Y_N = \mathscr{V}_N(Y_N) + \mathcal{E}_N(\tau), \qquad \mathscr{V}_N'(0) = -(\beta - g_{*,N}) < 0. \tag{1606}$$

The already audited terms give

$$g_{N,\alpha} - g_{*,N} = O_N(s^{2^{-k_N}}), \qquad E_N^{\mathrm{rec/forc}} = O_N(s^{4\alpha/7}), \qquad b_N^{-1} = O_N(s^{\beta}), \tag{1607}$$

and (1604) supplies the missing fractional power. Choose

$$\boxed{0 < \eta_0 < \min\left\{\beta - g_{*,N},\ 2^{-k_N},\ \frac{4\alpha}{7},\ \beta,\ \eta_F\right\}.} \tag{1608}$$

Differentiating the localized equation by E^j, $0 \le j \le M$, and using the Mellin bounds in (1594) gives

$$\frac{d}{d\tau}\|Y_N\|_{\mathcal{Z}_{\eta_0}} \le -c_N\|Y_N\|_{\mathcal{Z}_{\eta_0}} + C_N\|Y_N\|_{\mathcal{Z}_{\eta_0}}^2 + C_N s_e^{\eta_1}, \tag{1609}$$

for some $c_N > 0$ and $\eta_1 > \eta_0$, as long as the tube is sufficiently small. Hence, after choosing the entry scale and tube radius so that the quadratic and forcing terms are dominated by the linear gap,

$$\boxed{\|Y_N\|_{\mathcal{Z}_{\eta_0}} \le C_N, \qquad |E^j Y_N(s)| \le C_N s^{\eta_0}, \quad 0 \le j \le M.} \tag{1610}$$

In particular,

$$\boxed{F_N(s) = 1 + O_N(s^{\eta_0}), \qquad b_N(s) = B_{*,N} s^{-(2\alpha-1)}\left(1 + O_N(s^{\eta_0})\right).} \tag{1611}$$

This is the desired local scalar persistence estimate once the full localized PDE trajectory has entered the conormal tube.

137 Compatibility with the running Lambert factor

The exact full-PDE relation from Part 59 is

$$\Delta_{N,k_N} = \mathscr{K}_N q_N e^{-q_N}, \qquad \frac{d\log \mathscr{K}_N}{d\log s} = -\frac{s\mathscr{P}_{N,\alpha}}{b_N^2}. \tag{1612}$$

Using (1604), (1607), and (1611),

$$\frac{s\mathscr{P}_{N,\alpha}}{b_N^2} = \beta - g_{*,N} + O_N(s^{\eta_0}), \tag{1613}$$

after reducing η_0 if necessary. Therefore

$$\boxed{\mathscr{K}_N(s) = \mathscr{K}_{0,N} s^{-(\beta - g_{*,N})}\left(1 + O_N(s^{\eta_0})\right).} \tag{1614}$$

The running invariant is therefore not merely exponent-compatible with the stable profile; it inherits the same positive-power persistence remainder.

138 Finite physical time at fixed N

From (1611) and $\dot{s} = 1 - b_N$,

$$T_{*,N} - t = \int_0^{s(t)} \frac{d\rho}{b_N(\rho) - 1} = \frac{s(t)^{\beta+1}}{(\beta+1)B_{*,N}} \left(1 + O_N(s(t)^{\eta_t})\right), \tag{1615}$$

where

$$\eta_t = \min\{\eta_0, \beta\} > 0. \tag{1616}$$

Since $\beta + 1 = 2\alpha$,

$$\boxed{b_N(t) = B_{*,N}^{1/(2\alpha)} (2\alpha)^{-(2\alpha-1)/(2\alpha)} (T_{*,N} - t)^{-(2\alpha-1)/(2\alpha)} \left(1 + o_N(1)\right).} \tag{1617}$$

This is a fixed-N scalar characteristic conclusion inside the localized conormal persistence tube.

139 What has actually been closed and the remaining PDE-level entry issue

The positive-power estimate (1604) closes the *operator* obstruction isolated in Parts 57–59: once the reconstructed fields are in a positive-power conormal neighborhood of the phase-critical profile, the full localized fractional operator maps that neighborhood into itself with the required positive-power remainder. Equations (1609)–(1611) then close the local scalar stability bootstrap.

There is, however, an important distinction between this local persistence statement and an unconditional construction theorem. The characteristic ODE alone does not prove that an arbitrary solution of the full spatial scalar PDE enters the conormal tube (1599). A complete PDE theorem still requires an entry/propagation statement controlling the localized spatial Euler derivatives of the reconstructed fields up to the entry section. That is a different issue from the fractional positive-power calculation.

Theorem 55 (Localized fractional persistence at the phase-critical exponent)**.** *Fix $N < \infty$, $k_N \geq 1$, and $1 \leq \alpha < 1 + \delta_{\mathrm{ph}}$. Work on a transverse compact set satisfying (1585), and suppose a finite-N smooth solution of the complete reconstructed scalar PDE enters a sufficiently small localized conormal tube (1599) about the phase-critical profile, with the previously audited continuity/reconstruction estimates valid there. Then the complete reconstructed fractional block satisfies the positive-power expansion (1604). The normalized amplitude fixed point $B_{*,N}$ is asymptotically stable in that tube, and for some $\eta_0 > 0$ the solution obeys (1611). The exact running Lambert factor obeys (1614), so no constant-invariant contradiction occurs. Along the corresponding trapped characteristic, the terminal phase point is reached in finite physical time with scalar rate (1617).*

This theorem is a local persistence theorem for the complete scalar characteristic dynamics. It does not establish that general finite-N initial data enter the conormal tube,and does not remove the separate $\chi \to 0$ fixed-space reconstruction issue.

140 Updated proof ledger after the persistence audit

constant Lambert invariant:	not used; replaced by the exact running factor,
phase-critical coefficient:	$h_\alpha^{\mathrm{ph}} = -2K_\alpha^+(-(2\alpha-1)/2) > 0$,
nonlocal far field:	smooth after proof localization, hence positive-power relative to the terminal singular scale,
transverse fractional correction:	lower phase order, beginning at relative order s^2,
conormal Mellin perturbation:	preserves a positive power by (1594),
localized complete fractional remainder:	closed in the conormal tube by (1604),
reconstruction/forcing remainder:	$O_N(s^{4\alpha/7})$,
Lambert sensitivity remainder:	$O_N(s^{2^{-k_N}})$,
local scalar amplitude persistence:	closed after conormal entry, equations (1609)–(1611),
running Lambert factor:	positive-power law (1614),
finite physical-time scalar rate:	(1617) along the persistent characteristic,
remaining scalar PDE issue:	prove entry/propagation of the required localized conormal tube from the actual finite-N data.

(1618)

The decisive gain is that the formerly missing fractional positive-power remainder is no longer an independent asymptotic assumption once the terminal profile is formulated in the correct conormal norm. The next non-circular question is now the *entry theorem*: derive the localized conormal tube (1599), including its Euler derivative bounds, directly from the actual finite-N scalar PDE and reconstruction before the terminal bootstrap begins.

Part IX
Finite-N Conormal Entry and Propagation Audit

141 Purpose: separate propagation from entry

The persistence theorem of Part 60 begins at a section $s = s_e$ on which the normalized phase-critical amplitude

$$b_N(s) = B_{*,N} s^{-\beta} F_N(s), \qquad \beta = 2\alpha - 1, \qquad F_N = 1 + v_N, \tag{1619}$$

already lies in the localized conormal tube

$$\|v_N\|_{\mathcal{Z}_\eta} = \max_{0\le j\le M} \sup_{0<s\le s_e} s^{-\eta}|E^j v_N(s)| < \infty, \qquad E = s\partial_s. \tag{1620}$$

There are two logically different questions:

$$\boxed{\text{propagation: tube at one entry section} \Longrightarrow \text{tube for all smaller } s,} \tag{1621}$$

and

$$\boxed{\text{entry: prescribed data at } t = 0 \Longrightarrow \text{tube at some } s = s_e.} \tag{1622}$$

Part 60 already contains the zeroth-order mechanism for (1621). The present audit upgrades it to Euler derivatives and then tests whether the information currently specified in the construction is sufficient to prove (1622) without inserting the terminal profile into the data.

142 Normalized complete finite-N equation

Let

$$Y_N = \log F_N, \qquad \tau = -\log(s/s_e). \tag{1623}$$

After the amplitude choice

$$B_{*,N} = \frac{\nu h_\alpha^{\mathrm{ph}}}{\beta - g_{*,N}}, \qquad g_{*,N} = 2^{-k_N}, \tag{1624}$$

the exact normalized scalar equation on the localized branch can be written in the form

$$\partial_\tau Y_N = \mathscr{V}_N(Y_N) + \mathscr{E}_N(\tau, Y_N), \tag{1625}$$

where

$$\mathscr{V}_N(Y) = (\beta - g_{*,N})(e^{-Y} - 1), \tag{1626}$$

and all differences between the complete finite-N PDE and the autonomous phase-critical balance are placed in $\mathscr{E}_N$. By (1604), (1607), and the reconstruction estimates,

$$|\mathscr{E}_N(\tau, Y)| \le C_N e^{-\eta_0 \tau} \tag{1627}$$

whenever $|Y|$ and the required localized conormal derivatives remain in a fixed sufficiently small tube. Here

$$\eta_0 < \min\left\{2^{-k_N}, \frac{4\alpha}{7}, 2\alpha - 1, \eta_F\right\}. \tag{1628}$$

The important point is that (1625) is not used to assume entry. It is used only after a section on which the local coordinate chart and reconstruction bounds are valid has been reached.

143 Strict inward pointing at the two amplitude boundaries

Choose $0 < \delta_Y < 1$. Since

$$\mathscr{V}_N'(Y) = -(\beta - g_{*,N})e^{-Y} < 0, \tag{1629}$$

there is a number

$$c_{N,Y} := (\beta - g_{*,N})e^{-\delta_Y} > 0 \tag{1630}$$

such that

$$Y\mathscr{V}_N(Y) \le -c_{N,Y}Y^2, \qquad |Y| \le \delta_Y. \tag{1631}$$

Choose τ_e so large that

$$C_N e^{-\eta_0\tau_e} \le \frac{1}{4}c_{N,Y}\delta_Y. \tag{1632}$$

Then on $Y = \delta_Y$,

$$\partial_\tau Y \le -c_{N,Y}\delta_Y + \frac{1}{4}c_{N,Y}\delta_Y < 0, \tag{1633}$$

whereas on $Y = -\delta_Y$,

$$\partial_\tau Y \ge c_{N,Y}\delta_Y - \frac{1}{4}c_{N,Y}\delta_Y > 0. \tag{1634}$$

Thus the scalar amplitude interval

$$\boxed{|Y_N| \le \delta_Y} \tag{1635}$$

is forward invariant in terminal logarithmic time after the entry section. This proves amplitude propagation without assuming convergence of F_N.

144 Euler derivative equations

Set

$$Z_{j,N} := \partial_\tau^j Y_N = (-1)^j E^j Y_N, \qquad 1 \le j \le M. \tag{1636}$$

Differentiating (1625) j times gives

$$\partial_\tau Z_{j,N} = \mathscr{V}_N'(Y_N)Z_{j,N} + \mathscr{Q}_{j,N}(Z_{1,N}, \dots, Z_{j-1,N}) + \partial_\tau^j \mathscr{E}_N, \tag{1637}$$

where $\mathscr{Q}_{1,N} = 0$ and, for $j \ge 2$, every monomial in $\mathscr{Q}_{j,N}$ contains at least two lower-order $Z_{\ell,N}$ factors. On (1635),

$$\mathscr{V}_N'(Y_N) \le -c_{N,Y}. \tag{1638}$$

The localized fractional conormal calculus of Part 60 supplies, after shrinking the tube if necessary,

$$|\partial_\tau^j \mathscr{E}_N| \le C_{N,j}e^{-\eta_0\tau} + \varepsilon_{N,\mathrm{tube}} \sum_{\ell=1}^{j} |Z_{\ell,N}|, \qquad 0 \le j \le M, \tag{1639}$$

with $\varepsilon_{N,\mathrm{tube}}$ as small as desired by the tube radius. This is the derivative form of the already proved conormal mapping property; it introduces no new fractional asymptotic hypothesis.

Define

$$\mathcal{E}_M(\tau) := \sum_{j=0}^{M} a_j |Z_{j,N}(\tau)|^2, \qquad Z_{0,N} = Y_N, \tag{1640}$$

with positive triangular weights a_j. Choosing the weights successively and then the tube radius sufficiently small gives

$$\frac{d}{d\tau}\mathcal{E}_M + c_{N,M}\mathcal{E}_M \le C_{N,M} e^{-2\eta_0 \tau}, \qquad c_{N,M} > 0. \tag{1641}$$

Hence

$$\boxed{\mathcal{E}_M(\tau) \le e^{-c_{N,M}(\tau-\tau_e)}\mathcal{E}_M(\tau_e) + C_{N,M} e^{-2\eta_* \tau},} \tag{1642}$$

where

$$\eta_* = \frac{1}{2}\min\{c_{N,M}, 2\eta_0\} > 0. \tag{1643}$$

Consequently

$$\boxed{|E^j v_N(s)| \le C_{N,j} s^{\eta_*}, \qquad 0 \le j \le M,} \tag{1644}$$

provided the finite-N solution reaches one sufficiently small entry section with the conormal energy below the invariant threshold.

Theorem 56 (Finite-N conormal propagation)**.** *Fix $N < \infty$, $k_N \ge 1$, and $1 \le \alpha < 1 + \delta_{\mathrm{ph}}$. Assume the localized reconstruction chart is valid on $0 < s \le s_e$ and the complete finite-N solution reaches $s = s_e$ with $\mathcal{E}_M(\tau_e)$ below the strict invariant threshold determined by (1632), (1638), and (1639). Then the conormal tube is propagated for every smaller $s > 0$, the estimates (1642)–(1644) hold, and the persistence conclusions (1611), (1614), and (1617) follow. In particular, propagation of the full finite-N conormal tube is a consequence of the complete scalar PDE once entry is known.*

145 What the prescribed scalar data do and do not determine

The earlier finite-k numerical construction prescribes the smooth scalar profile

$$\boxed{b_3(z, 0) = 3 + \sin z,} \tag{1645}$$

or, in the phase coordinate $\theta = t - z$ at $t = 0$,

$$B(\theta, 0) = 3 - \sin\theta. \tag{1646}$$

This is legitimate initial information for the reduced scalar transport problem. It gives immediately

$$2 \le b_3(z, 0) \le 4, \qquad \|\partial_z^m b_3(\cdot, 0)\|_{L^\infty} \le 1 \quad (m \ge 1). \tag{1647}$$

Thus the prescribed scalar datum is smooth, positive, and has finite conormal derivatives on every region separated from the terminal phase point.

For the *complete reconstructed scalar PDE*, however, entry into (1620) is not a consequence of (1645) alone. The full equation contains

$$\mathscr{P}_{N,\alpha} = \mathfrak{F}_{N,\alpha} + 2C_N G_N - \mathcal{R}_N^{\mathrm{rec}}, \tag{1648}$$

and the reconstruction uses the spatial fields $A_N = u_y$ and $C_N = u_z$. Therefore the map from the initial scalar profile to the entry section also depends on the initial/reconstructed values and spatial derivatives of these coefficients, on the chosen real Lambert branch region, and on the forcing and pressure-compatible reconstruction. A scalar pointwise datum does not by itself specify those quantities.

This gives a precise non-circular obstruction. To claim

$$b_3(z,0) = 3 + \sin z \quad \Longrightarrow \quad \mathcal{E}_M(\tau_e) < \varepsilon_{\mathrm{tube}} \tag{1649}$$

one must first prove a preterminal estimate for the *complete* PDE, not for the reduced Lambert transport equation.

146 A sufficient preterminal entry functional

Let $t_e < T_{*,N}$ be the first time at which the distinguished characteristic reaches a chosen phase section $s = s_e$. Define the exact entry mismatch

$$\mathfrak{D}_{N,M}(t_e) := \sum_{j=0}^{M} \left| E^j \log\left(\frac{s_e^{\beta} b_N(X_N(t_e), t_e)}{B_{*,N}} \right) \right|^2, \tag{1650}$$

where $X_N(t)$ denotes the complete reconstructed characteristic. The propagation theorem shows that it is sufficient to establish

$$\boxed{\mathfrak{D}_{N,M}(t_e) \le \varepsilon_{N,M}^{\mathrm{ent}}} \tag{1651}$$

for a computable threshold depending only on the finite-N coefficient bounds in the localized chart.

Before t_e all coefficients are smooth at fixed N. Commuting the complete scalar PDE with spatial derivatives up to order m and using the fractional dissipative term gives the standard finite-N differential inequality

$$\frac{d}{dt}\|b_N(t)\|_{H^m}^2 + \nu \|b_N(t)\|_{H^{m+\alpha}}^2 \le \mathcal{C}_{N,m}(t)\big(1 + \|b_N(t)\|_{H^m}^2\big), \tag{1652}$$

where $\mathcal{C}_{N,m}$ is an explicit polynomial in the smooth reconstruction coefficients and their derivatives on $[0, t_e]$. Hence

$$\|b_N(t_e)\|_{H^m}^2 \le \left(\|b_N(0)\|_{H^m}^2 + \int_0^{t_e} \mathcal{C}_{N,m}(r)\, dr \right) \exp\left(\int_0^{t_e} \mathcal{C}_{N,m}(r)\, dr \right). \tag{1653}$$

For m large enough, Sobolev embedding controls the pointwise derivatives needed in (1650). Thus smooth finite-N evolution gives *finiteness* of the entry functional. It does not give its *smallness* relative to the phase-critical profile.

The missing quantitative statement is therefore exactly

$$\boxed{\text{preterminal attraction:} \qquad \mathfrak{D}_{N,M}(t_e) < \varepsilon^{\mathrm{ent}}_{N,M}.} \tag{1654}$$

No estimate derived so far in the manuscript forces (1654) from (1645). Energy regularity such as (1653) controls size and smoothness, whereas entry requires closeness to a specific amplitude and its Euler jet.

147 Why propagation does not imply entry

This distinction can be seen already in the autonomous normalized equation. The equilibrium $Y = 0$ is asymptotically stable in increasing terminal time, but the local Lyapunov estimate (1631) only applies after $|Y| \le \delta_Y$. It does not imply that a trajectory beginning outside that neighborhood ever crosses its boundary. In the complete PDE the error term is not known to be small on the entire preterminal interval, so extending the local Lyapunov function backward to $t = 0$ would be circular.

In particular, the smooth positivity (1647) cannot be converted into $F_N \approx 1$ at s_e without comparing the actual solution with the phase-critical orbit. Such a comparison requires either a global trapping region for the normalized PDE or a direct finite-N estimate of the entry map.

148 Two rigorous routes to close the entry theorem

The first route is an invariant-region argument. One seeks two explicit preterminal barriers $B_N^-(s)$ and $B_N^+(s)$ satisfying

$$B_N^-(s) \le b_N \le B_N^+(s), \tag{1655}$$

with strict inward signs for the *complete* scalar PDE and

$$\frac{B_N^{\pm}(s)}{B_{*,N} s^{-\beta}} \longrightarrow 1 \quad \text{as the barriers enter the terminal chart.} \tag{1656}$$

If the prescribed datum lies between the barriers at $t = 0$, comparison gives (1651) at a finite section. The present manuscript has not yet constructed such complete-PDE barriers.

The second route is a quantitative entry-map estimate. Let $\mathscr{S}_N(t)$ denote the finite-N solution operator for the complete scalar PDE coupled to its reconstruction. One must estimate

$$\mathscr{E}_N^{\mathrm{map}} : b_{N,0} \longmapsto \left(E^j \log \frac{s_e^{\beta} b_N(X_N(t_e), t_e)}{B_{*,N}} \right)_{j=0}^{M} \tag{1657}$$

and prove

$$\|\mathscr{E}_N^{\mathrm{map}}(3 + \sin z)\|_{\ell^2} < (\varepsilon^{\mathrm{ent}}_{N,M})^{1/2}. \tag{1658}$$

At fixed N this is a finite quantitative problem, but it depends on the actual complete reconstruction coefficients. A calculation using only the reduced equation

$$b_t + b b_z = (\partial_\theta \log \Delta_{N,k}) b^2 \tag{1659}$$

cannot establish (1658), because Part 59 proved that the omitted fractional, forcing, and reconstruction terms change the exact Lambert invariant.

149 Entry-propagation theorem with the exact remaining hypothesis

Theorem 57 (Complete finite-N entry-propagation criterion)**.** *Fix $N < \infty$, $k_N \geq 1$, $1 \leq \alpha < 1 + \delta_{\mathrm{ph}}$, and sufficiently large derivative index M. Let a smooth solution of the complete reconstructed scalar PDE exist from the prescribed initial section to a terminal chart $s = s_e$, with the real branch and transverse reconstruction remaining regular on the preterminal interval. If the exact entry map satisfies (1651), then the conormal tube is forward invariant in terminal logarithmic time, (1644) holds through the endpoint, and the scalar persistence and finite-time characteristic laws (1611), (1614), and (1617) follow.*

For the historical scalar datum $b_3(z, 0) = 3 + \sin z$, the currently audited manuscript proves smooth finite-N preterminal control of the type (1653) under bounded reconstruction coefficients, but it does not yet prove the small entry mismatch (1651). Therefore entry from that prescribed scalar datum is not established merely by the local persistence theorem.

150 Updated proof ledger after the entry audit

localized fractional positive-power remainder:	closed in Part 60,
zeroth-order amplitude propagation:	strict invariant interval (1633)–(1635),
Euler-derivative propagation:	conormal energy estimate (1641)–(1644),
finite-N smooth preterminal regularity:	follows conditionally from bounded reconstruction coefficients by (1652)–(1653),
prescribed scalar datum:	$b_3(z, 0) = 3 + \sin z$ in the earlier reduced finite-k construction,
entry functional:	exact mismatch $\mathfrak{D}_{N,M}(t_e)$ in (1650),
propagation after entry:	proved if $\mathfrak{D}_{N,M}(t_e)$ is below the invariant threshold,
entry from scalar datum alone:	not proved; scalar data do not specify the full reconstructed coefficient jet,
non-circular remaining estimate:	prove (1651) by complete-PDE barriers or the entry map (1657)–(1658),
finite-time scalar law:	follows after verified entry and propagation.

(1660)

The entry audit therefore closes one half of the problem completely: *propagation* of the conormal tube, including finitely many Euler derivatives, follows from the complete finite-

N scalar PDE and the localized fractional estimate already proved. It also prevents a circular claim about the other half. The prescribed scalar profile supplies smooth positive data, but we still need a quantitative complete-PDE entry estimate proving that the actual reconstructed trajectory reaches the small stable tube. Equation (1651), rather than a qualitative appeal to smoothness or to the reduced Lambert invariant, is the exact remaining finite-N scalar condition.

Part X
Quantitative Complete-PDE Entry from the Prescribed Finite-N Data

151 Purpose of the entry calculation

Part 61 reduced the remaining scalar issue to the quantitative inequality

$$\mathfrak{D}_{N,M}(t_e) \le \varepsilon^{\mathrm{ent}}_{N,M}. \tag{1661}$$

The point of the present part is to push this condition backward toward the prescribed initial section without using the reduced Lambert invariant. We work at fixed finite N throughout. All constants in this part are therefore allowed to depend on N, M, and the compact preterminal chart, but not on the terminal variable after the entry section has been selected.

Let

$$\beta = 2\alpha - 1, \qquad \bar{b}_N(s) = B_{*,N} s^{-\beta}, \qquad Y_N = \log \frac{b_N}{\bar{b}_N}. \tag{1662}$$

Along the complete reconstructed characteristic introduce terminal logarithmic time

$$\tau = -\log s, \qquad \tau_e = -\log s_e. \tag{1663}$$

The entry functional (1650) is equivalent, up to fixed finite-N norm constants, to the squared conormal jet of Y_N at τ_e.

152 Exact normalized equation before the local tube

Write the complete characteristic equation in the form

$$\dot{b}_N = \mathcal{Q}_N[b_N] + \mathcal{R}^{\mathrm{full}}_N, \tag{1664}$$

where $\mathcal{Q}_N$ denotes the phase-critical principal part used in Parts 58–60 and $\mathcal{R}^{\mathrm{full}}_N$ is the exact difference between the complete reconstructed PDE and that principal part. No term is dropped in this definition. Dividing by b_N, subtracting the logarithmic derivative of $\bar{b}_N$, and changing from t to τ gives the exact identity

$$\partial_\tau Y_N = -\Lambda_N(\tau, Y_N) Y_N + \mathscr{R}_N(\tau), \tag{1665}$$

where, for $Y \neq 0$,

$$\Lambda_N(\tau, Y) := -\frac{\mathscr{P}_N(\tau, Y) - \mathscr{P}_N(\tau, 0)}{Y}, \qquad \mathscr{R}_N(\tau) := \mathscr{P}_N(\tau, 0) + \mathscr{E}_N^{\mathrm{full}}(\tau). \tag{1666}$$

Here $\mathscr{P}_N$ is the exact normalized principal vector field and $\mathscr{E}_N^{\mathrm{full}}$ is the normalized complete-PDE remainder. At $Y = 0$ we define

$$\Lambda_N(\tau, 0) = -\partial_Y \mathscr{P}_N(\tau, 0). \tag{1667}$$

Thus (1665) is an algebraic rewriting of the complete equation, not a linearization assumption.

153 Preterminal coercive corridor

The local stability calculation proves positivity of the limiting damping coefficient at $Y = 0$. To obtain entry one needs this sign on a corridor larger than the final conormal tube. Fix $R_N > 0$ and suppose that on

$$\mathcal{C}_N = \{(\tau, Y) : \tau_0 \leq \tau \leq \tau_e,\ |Y| \leq R_N\} \tag{1668}$$

the exact normalized principal field satisfies

$$\boxed{\Lambda_N(\tau, Y) \geq \lambda_N^{\mathrm{cor}} > 0.} \tag{1669}$$

This is a finite-N sign condition on a compact set and is strictly stronger than the local derivative sign used in Part 60. It can be checked directly from the actual coefficient functions; it is not implied by local stability alone.

Under (1669), multiplying (1665) by $\operatorname{sgn} Y_N$ yields

$$\frac{d}{d\tau}|Y_N| \leq -\lambda_N^{\mathrm{cor}}|Y_N| + |\mathscr{R}_N|. \tag{1670}$$

Consequently

$$|Y_N(\tau)| \leq e^{-\lambda_N^{\mathrm{cor}}(\tau-\tau_0)}|Y_N(\tau_0)| + \int_{\tau_0}^{\tau} e^{-\lambda_N^{\mathrm{cor}}(\tau-r)}|\mathscr{R}_N(r)|\,dr. \tag{1671}$$

This is the first non-circular entry estimate: the right side depends only on the actual normalized initial mismatch and the complete-PDE residual measured along the actual characteristic.

154 Euler-jet entry estimate

Set

$$Z_{N,j} = E^j Y_N, \qquad 0 \leq j \leq M, \qquad \mathbf{Z}_N = (Z_{N,0}, \ldots, Z_{N,M}). \tag{1672}$$

Commuting the exact normalized equation with E^j gives

$$\partial_\tau \mathbf{Z}_N = -\mathbf{A}_N(\tau)\mathbf{Z}_N + \mathbf{N}_N(\tau, \mathbf{Z}_N) + \mathbf{R}_{N,M}(\tau), \tag{1673}$$

where $\mathbf{A}_N$ contains the diagonal corridor damping and the bounded finite-N commutators. Assume the symmetric part obeys the compact-corridor bound

$$\frac{1}{2}\big(\mathbf{A}_N + \mathbf{A}_N^*\big) \geq \lambda_{N,M}^{\mathrm{jet}} I, \qquad \lambda_{N,M}^{\mathrm{jet}} > 0, \tag{1674}$$

and that on $\|\mathbf{Z}\| \leq R_{N,M}$,

$$\|\mathbf{N}_N(\tau, \mathbf{Z})\| \leq L_{N,M}^{\mathrm{nl}} \|\mathbf{Z}\|^2. \tag{1675}$$

If

$$L_{N,M}^{\mathrm{nl}} R_{N,M} \leq \frac{1}{4}\lambda_{N,M}^{\mathrm{jet}}, \tag{1676}$$

then the energy $\mathcal{Y}_{N,M} = \|\mathbf{Z}_N\|_{\ell^2}$ satisfies

$$\frac{d}{d\tau}\mathcal{Y}_{N,M} \leq -\frac{3}{4}\lambda_{N,M}^{\mathrm{jet}}\mathcal{Y}_{N,M} + \|\mathbf{R}_{N,M}(\tau)\|. \tag{1677}$$

Hence

$$\boxed{\mathcal{Y}_{N,M}(\tau_e) \leq e^{-\frac{3}{4}\lambda_{N,M}^{\mathrm{jet}}(\tau_e - \tau_0)}\mathcal{Y}_{N,M}(\tau_0) + \int_{\tau_0}^{\tau_e} e^{-\frac{3}{4}\lambda_{N,M}^{\mathrm{jet}}(\tau_e - r)}\|\mathbf{R}_{N,M}(r)\|\, dr.} \tag{1678}$$

The bootstrap condition $\mathcal{Y}_{N,M} \leq R_{N,M}$ is propagated provided the same right side, with τ_e replaced by every $\tau \in [\tau_0, \tau_e]$, is strictly smaller than $R_{N,M}$.

155 A computable sufficient condition for quantitative entry

Define the complete preterminal residual budget

$$\mathfrak{B}_{N,M}(\tau_0, \tau_e) := \int_{\tau_0}^{\tau_e} e^{-\frac{3}{4}\lambda_{N,M}^{\mathrm{jet}}(\tau_e - r)}\|\mathbf{R}_{N,M}(r)\|\, dr \tag{1679}$$

and the damped initial mismatch

$$\mathfrak{I}_{N,M}(\tau_0, \tau_e) := e^{-\frac{3}{4}\lambda_{N,M}^{\mathrm{jet}}(\tau_e - \tau_0)}\mathcal{Y}_{N,M}(\tau_0). \tag{1680}$$

Then (1678) gives the explicit implication

$$\boxed{\mathfrak{I}_{N,M} + \mathfrak{B}_{N,M} \leq (\varepsilon_{N,M}^{\mathrm{ent}})^{1/2} \quad \Longrightarrow \quad \mathfrak{D}_{N,M}(t_e) \leq \varepsilon_{N,M}^{\mathrm{ent}}.} \tag{1681}$$

Thus the formerly qualitative entry problem has been reduced to a finite-N coercivity check and a single weighted residual integral for the *complete* reconstructed PDE.

A useful stronger condition is obtained if, on the corridor,

$$\|\mathbf{R}_{N,M}(\tau)\| \le C^{\mathrm{pre}}_{N,M} e^{-\eta^{\mathrm{pre}}_{N,M}\tau}, \qquad \eta^{\mathrm{pre}}_{N,M} > 0. \tag{1682}$$

Then

$$\mathfrak{B}_{N,M} \le C^{\mathrm{pre}}_{N,M} \frac{e^{-\eta^{\mathrm{pre}}_{N,M}\tau_e} + e^{-\frac{3}{4}\lambda^{\mathrm{jet}}_{N,M}(\tau_e-\tau_0)} e^{-\eta^{\mathrm{pre}}_{N,M}\tau_0}}{|\frac{3}{4}\lambda^{\mathrm{jet}}_{N,M} - \eta^{\mathrm{pre}}_{N,M}|}, \tag{1683}$$

with the usual $(\tau_e-\tau_0)e^{-\eta\tau_e}$ replacement in the resonant case. Therefore both the initial mismatch and the complete residual are forced below the entry threshold by taking a sufficiently deep but finite entry section, provided the solution remains inside the coercive corridor.

156 What the prescribed datum contributes

For the historical scalar datum

$$b_3(z,0) = 3 + \sin z, \tag{1684}$$

positivity gives

$$2 \le b_3(z,0) \le 4. \tag{1685}$$

At fixed N this supplies a finite zeroth-order initial mismatch wherever the reference profile and reconstruction chart are regular. Its derivatives are also uniformly finite at the initial section. Hence

$$\mathcal{Y}_{N,M}(\tau_0) < \infty. \tag{1686}$$

Equation (1680) then shows that the initial contribution can be made arbitrarily small by a sufficiently long coercive terminal-time interval. The datum therefore causes no obstruction to entry *once* the complete trajectory has been proved to remain in the corridor (1668).

This observation does not dispose of the reconstruction coefficients. The quantity $\mathbf{R}_{N,M}$ contains the actual fractional, forcing, continuity, pressure/reconstruction, and transverse terms. Their boundedness alone is insufficient: (1682), or an integrable substitute strong enough for (1681), must be verified from the actual reconstruction.

157 Corridor persistence from an outer barrier

The remaining bootstrap can itself be expressed without assuming entry. At the two faces $Y = \pm R_N$, (1665) gives

$$\partial_\tau Y_N\big|_{Y=R_N} \le -\lambda^{\mathrm{cor}}_N R_N + |\mathscr{R}_N|, \qquad \partial_\tau Y_N\big|_{Y=-R_N} \ge \lambda^{\mathrm{cor}}_N R_N - |\mathscr{R}_N|. \tag{1687}$$

Therefore the corridor is forward invariant whenever

$$\boxed{\sup_{\tau_0\le\tau\le\tau_e} |\mathscr{R}_N(\tau)| < \lambda^{\mathrm{cor}}_N R_N.} \tag{1688}$$

If the initial normalized state lies in the corridor, (1688) prevents exit; then (1678) contracts the complete Euler jet and (1681) proves quantitative entry. This is precisely the desired outer-to-inner trapping mechanism.

158 Finite-N quantitative entry theorem

Theorem 58 (Outer-corridor criterion for complete-PDE conormal entry)**.** *Fix $N < \infty$, $k_N \geq 1$, $1 \leq \alpha < 1 + \delta_{\mathrm{ph}}$ and M large enough for the reconstruction estimates. Suppose that on a finite preterminal interval $[\tau_0, \tau_e]$ the complete reconstructed solution is smooth and remains on the real branch and away from the independent reconstruction zero set $\chi = 0$. Suppose further that:*

1. *the exact normalized principal field satisfies the corridor coercivity (1669) and the Euler-jet coercivity (1674);*
2. *the nonlinear commutator bound (1675)–(1676) holds on the chosen outer jet ball;*
3. *the complete residual satisfies the inward boundary condition (1688) and the weighted budget inequality*
$$\mathfrak{I}_{N,M} + \mathfrak{B}_{N,M} \leq (\varepsilon^{\mathrm{ent}}_{N,M})^{1/2}; \tag{1689}$$
4. *the normalized initial state belongs to the outer corridor and jet ball.*

Then the actual complete finite-N trajectory cannot leave the outer corridor, reaches the entry section with
$$\boxed{\mathfrak{D}_{N,M}(t_e) \leq \varepsilon^{\mathrm{ent}}_{N,M},} \tag{1690}$$
and thereafter Part 61 propagates the positive-power conormal tube through the terminal endpoint. Consequently the conditional scalar persistence and finite-physical-time characteristic laws of Part 60 apply.

Proof. Condition (1688) gives strict inward pointing at both zeroth-order corridor faces by (1687), so the first-exit argument excludes loss of the outer amplitude bound. Conditions (1674)–(1676), together with the corresponding first-exit argument for the Euler jet, justify (1677) on the whole interval. Integration gives (1678), and (1689) gives (1690). The hypotheses of the entry-propagation theorem in Part 61 are therefore met at τ_e; its conormal propagation estimate then continues the solution inside the local tube. No reduced characteristic invariant is used. □ □

159 Status for the actual reconstruction

The theorem above is stronger than the entry criterion (1651): it identifies the exact finite collection of estimates that would prove (1651) directly from the complete evolution. It also shows exactly what has *not* yet been obtained from the formulas presently audited. We do not yet have a global preterminal estimate proving either
$$\Lambda_N \geq \lambda_N^{\mathrm{cor}} > 0 \quad \text{on an outer corridor} \tag{1691}$$
or
$$\|\mathbf{R}_{N,M}(\tau)\| \leq C^{\mathrm{pre}}_{N,M} e^{-\eta^{\mathrm{pre}}_{N,M}\tau} \quad \text{before local conormal entry.} \tag{1692}$$

The positive-power estimates of Part 60 establish the analogue of (1692) *inside* the local conormal tube; using them to prove entry into that same tube would be circular. Therefore (1691)–(1692) must be proved on a strictly larger preterminal region from the raw reconstruction formulas.

In particular, the calculation here does not convert the historical datum (1684) into an unconditional scalar blow-up theorem. It proves instead that quantitative entry follows from two checkable complete-PDE estimates: outer-corridor coercivity and a weighted residual budget. These are now the only preterminal estimates separating the prescribed data from the already proved local propagation mechanism.

160 Updated proof ledger after the quantitative-entry reduction

localized fractional persistence:	closed in Part 60,
conormal propagation after entry:	closed in Part 61,
exact normalized preterminal equation:	rewritten as the damping–residual identity (1665),
zeroth-order outer trapping:	follows from corridor coercivity and the inward residual bound (1688),
Euler-jet contraction before entry:	follows from (1674)–(1678),
quantitative entry:	follows from the explicit budget criterion (1681),
prescribed datum $3 + \sin z$:	finite positive initial mismatch; its damped contribution decays by (1680),
remaining complete-PDE estimate I:	prove outer-corridor coercivity (1691) from the actual reconstructed coefficients,
remaining complete-PDE estimate II:	prove the preterminal residual decay/integrability (1692) without assuming local conormal closeness,
finite-time scalar law:	follows once (1691)–(1692) close and entry is obtained.

(1693)

The quantitative entry problem has therefore been reduced from an abstract smallness requirement at t_e to an explicit outer-to-inner contraction argument. The next decisive calculation is no longer to assume the terminal profile, but to evaluate Λ_N and $\mathbf{R}_{N,M}$ directly from the raw finite-N reconstruction on a larger preterminal corridor and verify (1691)–(1692).

Part XI
Raw Outer-Corridor Coercivity and Preterminal Residual Audit

161 Purpose and non-circular starting point

The preceding quantitative-entry reduction left two estimates:

$$\Lambda_N \geq \lambda_N^{\mathrm{cor}} > 0, \tag{1694}$$

and

$$\|\mathbf{R}_{N,M}(\tau)\| \leq C_{N,M}^{\mathrm{pre}} e^{-\eta_{N,M}^{\mathrm{pre}}\tau}. \tag{1695}$$

The first estimate can in fact be proved directly from the exact phase-critical principal vector field, without assuming conormal entry. The second estimate requires a separate audit because the complete fractional operator depends on the shape of the normalized profile, not only on its amplitude.

Fix finite N and write

$$\beta = 2\alpha - 1, \qquad g_{*,N} = 2^{-k_N}, \qquad a_N := \beta - g_{*,N}. \tag{1696}$$

For $k_N \geq 1$ and $\alpha \geq 1$,

$$a_N \geq \frac{1}{2} > 0. \tag{1697}$$

The matched phase-critical amplitude is

$$B_{*,N} = \frac{\nu h_\alpha^{\mathrm{ph}}}{a_N} > 0. \tag{1698}$$

Introduce

$$b_N = B_{*,N} s^{-\beta} F_N, \qquad Y_N = \log F_N. \tag{1699}$$

No smallness of Y_N is assumed in the present section.

162 Exact principal vector field on an arbitrary finite corridor

The phase-critical matching calculation of Part 58 gives the normalized principal equation

$$\partial_\tau \log F_N = -a_N + \frac{a_N}{F_N} \tag{1700}$$

before the complete remainder is restored. Since $F_N = e^{Y_N}$, the exact principal vector field is therefore

$$\boxed{\mathscr{P}_N(Y) = a_N(e^{-Y} - 1).} \tag{1701}$$

In particular,

$$\mathscr{P}_N(0) = 0, \qquad \mathscr{P}'_N(0) = -a_N < 0. \tag{1702}$$

Substitution into the definition (1666) gives, for $Y \neq 0$,

$$\boxed{\Lambda_N(Y) = a_N \frac{1 - e^{-Y}}{Y}.} \tag{1703}$$

The continuous value at the origin is

$$\Lambda_N(0) = a_N. \tag{1704}$$

163 Global sign of the corridor damping

Define

$$\phi(Y) = \frac{1 - e^{-Y}}{Y}, \qquad \phi(0) = 1. \tag{1705}$$

For every real $Y \neq 0$, the numerator $1 - e^{-Y}$ has the same sign as Y. Hence

$$\phi(Y) > 0 \qquad (Y \in \mathbb{R}). \tag{1706}$$

Moreover, on a symmetric finite corridor $|Y| \leq R_N$,

$$\min_{|Y| \leq R_N} \phi(Y) = \frac{1 - e^{-R_N}}{R_N}. \tag{1707}$$

Consequently

$$\boxed{\Lambda_N(Y) \geq \lambda_N^{\mathrm{cor}} := a_N \frac{1 - e^{-R_N}}{R_N} > 0, \qquad |Y| \leq R_N.} \tag{1708}$$

Thus the zeroth-order outer-corridor coercivity left open in (1691) is not an additional hypothesis: it follows algebraically from the exact matched phase-critical principal field. Notice that no Taylor expansion in Y and no local conormal smallness have been used.

At the two corridor faces one therefore has the exact principal inward margin

$$-\Lambda_N(R_N)R_N = -a_N(1 - e^{-R_N}) < 0, \tag{1709}$$

and

$$-\Lambda_N(-R_N)(-R_N) = a_N(e^{R_N} - 1) > 0. \tag{1710}$$

The negative face has the stronger restoring force. The smaller inward margin is

$$\boxed{m_N^{\mathrm{cor}} = a_N(1 - e^{-R_N}) > 0.} \tag{1711}$$

Therefore the corridor is forward invariant whenever the complete normalized remainder obeys

$$\sup_{\tau_0 \leq \tau \leq \tau_e} |\mathscr{R}_N(\tau)| < m_N^{\mathrm{cor}}. \tag{1712}$$

164 What can be proved for the raw algebraic residuals on the corridor

The corridor bound $|Y_N| \le R_N$ is equivalent to the two-sided amplitude comparison

$$e^{-R_N} B_{*,N} s^{-\beta} \le b_N(s) \le e^{R_N} B_{*,N} s^{-\beta}. \tag{1713}$$

This comparison is sufficient for all purely algebraic power-counting estimates that do not require decay of conormal derivatives. In particular, the exact finite-N Lambert sensitivity gives

$$g_{N,\alpha}(s) - g_{*,N} = O_N(s^{g_{*,N}}), \tag{1714}$$

and the reconstruction/forcing audit at $\beta = 2\alpha - 1$ gives, uniformly on any fixed amplitude corridor,

$$\boxed{E_N^{\mathrm{rec/forc}}(s) = O_{N,R_N}(s^{4\alpha/7}).} \tag{1715}$$

The geometric characteristic relation similarly gives the two-sided power comparison

$$c_{N,R_N} s^{p_{\mathrm{ph}}} \le \sigma_N(s) \le C_{N,R_N} s^{p_{\mathrm{ph}}}, \qquad p_{\mathrm{ph}} = \frac{2(7-2\alpha)}{7} > 0, \tag{1716}$$

as long as the reconstruction chart remains on the same real branch and away from the independent zero set where that chart degenerates.

Consequently the non-fractional part of the normalized preterminal residual obeys

$$\boxed{|\mathscr{R}_N^{\mathrm{alg}}(\tau)| \le C_{N,R_N} e^{-\eta_N^{\mathrm{alg}}\tau}, \qquad \eta_N^{\mathrm{alg}} = \min\left\{2^{-k_N}, \frac{4\alpha}{7}, p_{\mathrm{ph}}\right\} > 0.} \tag{1717}$$

The same estimate holds for finitely many Euler derivatives provided the raw coefficient maps are differentiated only a finite number of times on the compact reconstruction chart.

165 The complete fractional residual and the remaining gap

The fractional term is qualitatively different. Let

$$F_N = e^{Y_N}. \tag{1718}$$

After subtraction of the matched phase-critical monomial, the localized fractional remainder has the schematic exact form

$$\mathscr{R}_N^{\mathrm{frac}} = \mathcal{T}_{N,\alpha}[F_N] - \mathcal{T}_{N,\alpha}[1] + \mathscr{R}_N^{\mathrm{far}} + \mathscr{R}_N^{\mathrm{geo}}, \tag{1719}$$

where $\mathcal{T}_{N,\alpha}$ is the normalized finite-N fractional operator in the reconstruction chart. Parts 60–61 proved

$$\mathscr{R}_N^{\mathrm{frac}} = O_N(s^{\eta_F}) \tag{1720}$$

inside the conormal tube because there

$$|E^j Y_N| \le C_N s^{\eta_*}. \tag{1721}$$

On the larger amplitude corridor, however, (1713) gives only

$$e^{-R_N} \le F_N \le e^{R_N}, \tag{1722}$$

and does not force $F_N - 1$, EF_N, or higher Euler derivatives to decay. For a nonlocal operator this distinction is essential: bounded amplitude does not imply

$$\mathcal{T}_{N,\alpha}[F_N] - \mathcal{T}_{N,\alpha}[1] \longrightarrow 0. \tag{1723}$$

Thus the positive-power fractional estimate cannot be deduced from zeroth-order corridor trapping alone.

More quantitatively, the localized operator estimate gives on a compact finite-N chart

$$\boxed{\|\mathbf{R}^{\mathrm{frac}}_{N,M}(\tau)\| \le C_{N,M,R_N} \left(\sum_{j=0}^{M_*} \|E^j Y_N(\tau)\| + e^{-\eta_N^{\mathrm{ker}}\tau} \right),} \tag{1724}$$

for a finite derivative index M_* determined by the fractional order and the reconstruction product estimates. The second term contains the genuine kernel/geometric positive powers. The first term is the profile-shape defect. Equation (1724) identifies exactly why (1692) cannot be proved from the raw amplitude corridor without an additional preterminal shape estimate.

166 A non-circular replacement for the missing residual hypothesis

The correct preterminal target is therefore weaker than assuming the final conormal asymptotic, but stronger than amplitude trapping. It is enough to prove a finite-jet decay estimate

$$\boxed{\sum_{j=1}^{M_*} \|E^j Y_N(\tau)\| \le C^{\mathrm{sh}}_{N,M_*} e^{-\eta_N^{\mathrm{sh}}\tau}, \qquad \eta_N^{\mathrm{sh}} > 0,} \tag{1725}$$

together with decay of the zeroth-order deviation generated by the coercive ODE. Combining (1717), (1724), and (1725) yields

$$\boxed{\|\mathbf{R}_{N,M}(\tau)\| \le C^{\mathrm{pre}}_{N,M} e^{-\eta^{\mathrm{pre}}_{N,M}\tau},} \tag{1726}$$

where one may take

$$\eta^{\mathrm{pre}}_{N,M} = \min\{\eta_N^{\mathrm{alg}}, \eta_N^{\mathrm{ker}}, \eta_N^{\mathrm{sh}}\} > 0. \tag{1727}$$

Unlike the final conormal-tube hypothesis, (1725) asks only for decay of a finite number of shape derivatives on the larger outer corridor. It does not prescribe the terminal amplitude or its limiting constant.

167 Differentiated corridor equation and the shape-entry mechanism

Applying $E = s\partial_s = -\partial_\tau$ to the exact normalized equation gives for $Z_{1,N} = EY_N$

$$\partial_\tau Z_{1,N} = -\Lambda_N(Y_N)Z_{1,N} + \mathcal{C}_{1,N}(Y_N, Z_{1,N}) + E\mathscr{R}_N, \tag{1728}$$

where $\mathcal{C}_{1,N}$ contains the derivative of the exact damping coefficient. Since $|Y_N| \le R_N$, all coefficients are bounded, and the strict positivity (1708) provides a zeroth-order restoring margin. Iterating gives a triangular finite-jet system

$$\partial_\tau \mathbf{Z}_N^{\mathrm{sh}} = -\mathbf{A}_N^{\mathrm{sh}}(\tau)\mathbf{Z}_N^{\mathrm{sh}} + \mathbf{N}_N^{\mathrm{sh}}(\mathbf{Z}_N^{\mathrm{sh}}) + \mathbf{G}_N^{\mathrm{raw}}(\tau), \tag{1729}$$

with

$$\mathbf{Z}_N^{\mathrm{sh}} = (EY_N, E^2Y_N, \dots, E^{M_*}Y_N). \tag{1730}$$

The algebraic forcing part of $\mathbf{G}_N^{\mathrm{raw}}$ decays by (1717). The only component not already controlled by raw coefficient powers is the fractional shape operator acting on $\mathbf{Z}_N^{\mathrm{sh}}$ itself.

Therefore a direct non-circular closure would follow from the finite-N outer-corridor accretivity estimate

$$\boxed{\left\langle \mathbf{Z}, \mathbf{A}_N^{\mathrm{sh}}\mathbf{Z} - \mathbf{L}_{N,\alpha}^{\mathrm{frac,sh}}\mathbf{Z} \right\rangle \ge \lambda_{N,M_*}^{\mathrm{sh}} \|\mathbf{Z}\|^2, \qquad \lambda_{N,M_*}^{\mathrm{sh}} > 0,} \tag{1731}$$

on $|Y| \le R_N$. Under (1731), the same first-exit/variation-of-constants argument as in Part 62 gives (1725), then (1726), and finally quantitative entry. Importantly, (1731) is an operator sign estimate on a compact preterminal corridor; it does not assume the conclusion (1725).

168 What has and has not been closed

The present audit therefore resolves one of the two estimates left in (1691)–(1692) and reduces the other to an operator-level statement:

$$\boxed{\Lambda_N \ge (2\alpha - 1 - 2^{-k_N})\frac{1 - e^{-R_N}}{R_N} > 0.} \tag{1732}$$

This is the requested outer-corridor coercivity, proved directly from the matched principal field.

For the complete residual, the raw reconstruction already supplies positive powers for the Lambert, reconstruction, forcing, geometric, and separated kernel pieces. What remains is not another scalar power count but the fractional shape accretivity (1731), or an equivalent finite-jet estimate. Without such an estimate, claiming (1695) from amplitude trapping would be circular because the missing decay is precisely the decay of the profile shape toward the conormal state.

169 Updated proof ledger after the raw-corridor audit

localized fractional persistence:	closed in Part 60,
conormal propagation after entry:	closed in Part 61,
quantitative-entry reduction:	closed in Part 62,
outer-corridor coercivity:	closed explicitly by (1703)–(1708),
non-fractional preterminal residual:	positive-power decay closed by (1714)–(1717),
complete fractional residual on amplitude corridor:	reduced to the finite-jet shape defect (1724),
preterminal residual decay:	follows from the non-circular shape estimate (1725),
remaining operator estimate:	prove outer-corridor fractional shape accretivity (1731),
quantitative entry from prescribed data:	follows once (1731) closes, via Parts 62–63,
finite-time scalar persistence law:	follows after entry and propagation.

(1733)

The decisive remaining preterminal calculation is therefore no longer the zeroth-order damping sign. That sign is globally positive on every finite amplitude corridor. The remaining question is whether the localized fractional shape operator is accretive relative to this damping on the larger preterminal corridor, as quantified by (1731). Establishing that estimate would produce the exponential residual bound (1726), quantitative conormal entry, and hence the already proved terminal scalar persistence mechanism.

Part XII
Outer-Corridor Fractional Shape Operator Audit

170 Purpose and a necessary correction to the formulation of (1731)

Part 63 reduced quantitative entry to the sign of the localized fractional shape operator. The purpose of the present audit is to compute that operator at the phase-critical profile and to determine what coercive estimate is actually compatible with its differential order. The calculation shows that (1731), if interpreted as a purely zeroth-order Euclidean estimate on the finite jet, is not the natural statement. The fractional shape term contains a positive derivative quadratic form and must be retained in the energy rather than estimated as an arbitrary bounded perturbation.

Work on the proof-localized transverse chart

$$0 < c_\chi \le |\chi(x)| \le C_\chi, \tag{1734}$$

and write

$$\beta = 2\alpha - 1, \qquad a_\beta(s) = s^{-\beta/2}, \qquad b = B_{*,N} s^{-\beta} e^{Y}. \tag{1735}$$

Up to the already audited positive-power geometric and transverse remainders, the two reconstructed factors have the same phase amplitude

$$u_x = c_x(x)\, a_\beta(s) e^{Y/2}, \qquad u_y = c_y(x)\, a_\beta(s) e^{Y/2}, \qquad c_x c_y = B_{*,N}. \tag{1736}$$

Thus the principal localized fractional block is governed by the conjugated operator

$$\boxed{\mathscr{T}_{\alpha,\beta} Z := a_\beta^{-1}(-\partial_{ss})^\alpha (a_\beta Z) - Z\, a_\beta^{-1}(-\partial_{ss})^\alpha a_\beta.} \tag{1737}$$

The subtraction in (1737) is forced by differentiating the quotient $((-\partial_{ss})^\alpha a)/a$; in particular constants lie in the kernel,

$$\mathscr{T}_{\alpha,\beta} 1 = 0. \tag{1738}$$

This is the correct linearized shape operator before the finite-jet commutators and positive-power reconstruction errors are restored.

171 Exact anchor computation at $\alpha = 1$

At $\alpha = 1$ one has $\beta = 1$, $a_\beta = s^{-1/2}$ and $(-\partial_{ss})^1 = -\partial_{ss}$. More generally, for $a = s^\lambda$, $\lambda = -\beta/2$,

$$a^{-1}\{-\partial_{ss}(aZ)\} - Z a^{-1}\{-\partial_{ss} a\} = -Z_{ss} - 2\frac{a_s}{a} Z_s. \tag{1739}$$

Since $E = s\partial_s$,

$$Z_s = s^{-1} E Z, \qquad Z_{ss} = s^{-2}(E^2 - E) Z, \tag{1740}$$

and therefore

$$\boxed{\mathscr{T}_{1,\beta} Z = s^{-2}\{-E^2 + (1+\beta)E\} Z.} \tag{1741}$$

At the phase-critical anchor $\beta = 1$ this becomes

$$\boxed{\mathscr{T}_{1,1} Z = s^{-2}(-E^2 + 2E) Z.} \tag{1742}$$

The normalized scalar equation multiplies the fractional block by $s/b \sim s^{1+\beta}/B_{*,N}$. At $\alpha = 1$, $\beta = 1$, the factor s^2 exactly cancels the s^{-2} in (1742). Hence the principal linearized fractional shape contribution in terminal logarithmic time is a constant-coefficient second-order operator.

Put

$$\tau = -\log(s/s_e), \qquad E = -\partial_\tau. \tag{1743}$$

Then

$$-E^2 + 2E = -\partial_\tau^2 - 2\partial_\tau. \tag{1744}$$

The first derivative is skew with respect to the unweighted whole-line quadratic form (or contributes only boundary/weight terms on a localized interval), while the second derivative gives the positive form

$$\big\langle Z, -\partial_\tau^2 Z\big\rangle = \|\partial_\tau Z\|_2^2. \tag{1745}$$

Thus the anchor principal fractional shape operator is not an uncontrolled zeroth-order defect. Its highest-order part is accretive.

172 Mellin symbol for general α near one

The same statement can be expressed without differentiating an integer-order operator. The exact Mellin formula already established in the manuscript is

$$(-\partial_{ss})^\alpha s^\lambda = K_\alpha^+(\lambda) s^{\lambda-2\alpha}, \qquad K_\alpha^+(\lambda) = \frac{\Gamma(\lambda+1)\Gamma(2\alpha-\lambda)}{\pi}\sin\pi(\alpha-\lambda). \tag{1746}$$

For a logarithmic Fourier mode $Z = e^{i\xi\log s}$, (1737) has symbol

$$\boxed{t_{\alpha,\beta}(\xi) = K_\alpha^+\left(-\frac{\beta}{2}+i\xi\right) - K_\alpha^+\left(-\frac{\beta}{2}\right).} \tag{1747}$$

The normalized phase-critical equation removes the common factor $s^{-2\alpha}$ because $1+\beta = 2\alpha$. Consequently (1747) is precisely the terminal logarithmic shape symbol.

At $(\alpha,\beta) = (1,1)$, using $K_1^+(\lambda) = -\lambda(\lambda-1)$,

$$t_{1,1}(\xi) = \xi^2 - 2i\xi. \tag{1748}$$

Therefore

$$\boxed{\Re t_{1,1}(\xi) = \xi^2 \geq 0.} \tag{1749}$$

The zero at $\xi = 0$ is the amplitude mode and is already controlled by the strict scalar corridor damping $\Lambda_N(Y)$. Nonzero shape frequencies receive a positive derivative contribution.

The general-α Gamma-symbol perturbation estimates proved earlier imply, for $1 \leq \alpha < 1+\delta$ and $\beta = 2\alpha - 1$,

$$|t_{\alpha,\beta}(\xi) - t_{1,1}(\xi)| \leq C\,|\alpha-1|\,\langle\xi\rangle^{2\alpha}(1+\log\langle\xi\rangle), \tag{1750}$$

on the common weighted Mellin domain. This proves continuity of the shape operator, but by itself it is not a global positive lower bound at every frequency: the logarithmic loss in (1750) cannot simply be absorbed by the anchor ξ^2 uniformly as $|\xi| \to \infty$. The correct high-frequency statement must use the principal symbol directly.

173 High-frequency ellipticity and low-frequency compactness

Stirling asymptotics for (1746) give the order-2α principal behavior

$$\Re t_{\alpha,\beta}(\xi) \geq c_\alpha|\xi|^{2\alpha} - C_\alpha, \qquad |\xi| \geq \Xi_\alpha, \tag{1751}$$

for α in a sufficiently small interval above one, after fixing the same branch and weighted Mellin line used in Part 60. Hence the only possible loss of accretivity occurs on a compact frequency interval.

For every fixed finite N and finite amplitude radius R_N, the localized reconstruction coefficients range over a compact coefficient set. The low-frequency shape symbol and its finite-jet commutators therefore have a finite lower bound. Define the actual corridor spectral margin

$$\boxed{\mu_{N,R_N,\alpha} := \inf_{|Y|\le R_N} \inf_{\substack{\xi\in\mathbb{R} \\ \text{shape mode}}} \Re\, \ell^{\mathrm{sh}}_{N,\alpha}(Y,\xi),} \tag{1752}$$

where the amplitude mode is excluded because it is handled by Λ_N. The anchor computation gives a positive derivative form, but the information currently present in the manuscript does not determine the sign of the full compact-frequency number (1752) for an arbitrary large R_N. In particular, compactness proves that $\mu_{N,R_N,\alpha} > -\infty$; it does not prove $\mu_{N,R_N,\alpha} \ge 0$.

This is the key outcome of the audit. The global statement (1731) cannot be promoted to a theorem merely from amplitude boundedness and the anchor sign. One must either compute the compact-frequency margin (1752), or choose a corridor on which perturbation from the already certified conormal operator is quantitatively smaller than its spectral gap.

174 A rigorous coercive replacement for (1731)

The natural energy statement retains the derivative norm. Let $\mathfrak{Z} = (Z_1, \dots, Z_{M_*})$ and let $\|\mathfrak{Z}\|_{\mathcal{H}^\alpha_N}$ denote the finite-jet weighted Mellin norm containing the order-α derivative of every shape component. The localized pseudodifferential Garding estimate has the form

$$\boxed{\Re\langle \mathfrak{Z}, \mathbf{L}^{\mathrm{frac,sh}}_{N,\alpha}\mathfrak{Z}\rangle \ge c_{N,\alpha}\|\mathfrak{Z}\|^2_{\mathcal{H}^\alpha_N} - C_{N,R_N}\|\mathfrak{Z}\|^2_2.} \tag{1753}$$

Combining this with the already proved scalar corridor margin

$$\lambda^{\mathrm{cor}}_N = (2\alpha - 1 - 2^{-k_N})\frac{1-e^{-R_N}}{R_N} \tag{1754}$$

shows that the desired preterminal shape contraction follows whenever the compact-frequency loss obeys

$$\boxed{C^{\mathrm{eff}}_{N,R_N} < \lambda^{\mathrm{cor}}_N.} \tag{1755}$$

Equivalently, in the spectral notation (1752), it is enough that

$$\boxed{\lambda^{\mathrm{cor}}_N + \mu_{N,R_N,\alpha} > 0.} \tag{1756}$$

Under (1756), one obtains

$$\frac{1}{2}\frac{d}{d\tau}\|\mathbf{Z}^{\mathrm{sh}}_N\|^2_2 + \lambda^{\mathrm{sh}}_{N,M_*}\|\mathbf{Z}^{\mathrm{sh}}_N\|^2_2 + c_{N,\alpha}\|\mathbf{Z}^{\mathrm{sh}}_N\|^2_{\mathcal{H}^\alpha_N} \le C_N e^{-2\eta^{\mathrm{alg}}_N\tau}, \tag{1757}$$

for some

$$\lambda^{\mathrm{sh}}_{N,M_*} > 0. \tag{1758}$$

Integration gives

$$\boxed{\|\mathbf{Z}_N^{\mathrm{sh}}(\tau)\|_2 \le C_N e^{-\eta_N^{\mathrm{sh}}(\tau-\tau_0)}\|\mathbf{Z}_N^{\mathrm{sh}}(\tau_0)\|_2 + C_N e^{-\eta_N^{\mathrm{sh}}\tau}.} \tag{1759}$$

Equations (1724) and (1759) then imply the complete preterminal residual bound

$$\boxed{\|\mathbf{R}_{N,M}(\tau)\| \le C_{N,M}^{\mathrm{pre}} e^{-\eta_{N,M}^{\mathrm{pre}}\tau}.} \tag{1760}$$

Thus (1756), rather than the schematic zeroth-order version of (1731), is the precise finite-N spectral condition needed for quantitative entry.

175 What is proved at the anchor and what remains for the full corridor

At $\alpha = 1$ the principal shape symbol is explicit:

$$\Re t_{1,1}(\xi) = \xi^2. \tag{1761}$$

Hence the phase fractional operator has the favorable high-frequency sign and no negative principal spectrum. The amplitude mode is separately damped by $\lambda_N^{\mathrm{cor}} > 0$. This proves that there is no principal-symbol obstruction to outer-corridor shape coercivity.

What is not supplied by the present raw reconstruction is a certified lower bound for the compact-frequency margin (1752) over an arbitrary entire amplitude corridor $|Y| \le R_N$. The finite-N coefficients are explicit, so this has now become a finite-dimensional/compact-frequency certification problem rather than an uncontrolled nonlocal remainder problem. A rigorous closure can proceed by either

$$\text{(i) analytic interval bounds for } \ell_{N,\alpha}^{\mathrm{sh}}(Y,\xi), \qquad \text{or} \qquad \text{(ii) a certified Galerkin/interval enclosure on } |\xi| \le \Xi \tag{1762}$$

with the high-frequency tail already controlled by (1751).

Theorem 59 (Outer-corridor fractional shape audit)**.** *Fix $N < \infty$ and the proof-localized transverse chart (1734). At the phase-critical anchor $\alpha = 1$, the principal linearized localized fractional shape operator has Mellin symbol $t_{1,1}(\xi) = \xi^2 - 2i\xi$ and therefore nonnegative real part ξ^2. For α sufficiently close to one, its high-frequency part remains elliptic and accretive in the weighted Mellin energy, up to a bounded compact-frequency loss. Consequently the preterminal finite-jet estimate and residual decay follow from the explicit spectral margin condition (1756).*

The present paper's data presently prove the anchor sign and high-frequency accretivity, but do not by themselves certify the sign of the full compact-frequency margin uniformly on an arbitrary prescribed outer corridor. Therefore (1731) is not declared closed on the entire corridor without the additional finite-N compact-frequency certification (1762).

176 Updated proof ledger after the shape-operator audit

outer-corridor scalar damping:	closed by (1732),
linearized fractional shape operator:	computed explicitly by (1737)–(1748),
anchor fractional shape sign:	$\Re t_{1,1}(\xi) = \xi^2 \geq 0$,
high-frequency fractional shape sector:	accretive/elliptic by (1751),
compact-frequency corridor sector:	reduced to the explicit margin $\mu_{N,R_N,\alpha}$ in (1752),
correct coercive condition:	$\lambda_N^{\mathrm{cor}} + \mu_{N,R_N,\alpha} > 0$,
preterminal shape decay:	follows from (1756) by (1757)–(1759),
complete preterminal residual:	follows as (1760) once the compact-frequency margin is certified,
quantitative entry:	then follows from Parts 62–63,
terminal scalar persistence:	already proved after verified entry.

(1763)

The remaining calculation is now finite and sharply localized: certify the compact-frequency number (1752) for the actual finite-N coefficient family. The nonlocal high-frequency sign is favorable; what remains is a bounded spectral enclosure, not a new terminal asymptotic assumption.

Part XIII
Certified Compact–Frequency Enclosure on a Finite Outer Corridor

177 Purpose: replace the unresolved number $\mu_{N,R_N,\alpha}$ by a quantitative enclosure

Part 64 reduced the remaining preterminal issue to the compact-frequency margin

$$\mu_{N,R_N,\alpha} = \inf_{|Y|\leq R_N} \inf_{|\xi|\leq \Xi_\alpha} \Re\, \ell_{N,\alpha}^{\mathrm{sh}}(Y,\xi), \tag{1764}$$

with the amplitude mode treated separately by the positive scalar damping. The point of the present calculation is that one does not need to prove $\mu_{N,R_N,\alpha} \geq 0$. It is enough to prove that its possible negative part is strictly smaller than the already available damping margin λ_N^{cor}.

Write the complete localized symbol on the bounded frequency block as

$$\ell_{N,\alpha}^{\mathrm{sh}}(Y,\xi) = t_{1,1}(\xi) + \mathfrak{p}_{N,\alpha}(Y,\xi), \qquad |\xi| \leq \Xi, \tag{1765}$$

where

$$t_{1,1}(\xi) = \xi^2 - 2i\xi, \qquad \Re t_{1,1}(\xi) = \xi^2. \tag{1766}$$

All finite-N proof-localized coefficients are smooth on the compact transverse chart and on a finite amplitude corridor. Hence, for every fixed N and fixed finite cutoff Ξ, the symbol is C^1 in (Y, α) on a neighborhood of $(0, 1)$ away from the already excluded Mellin poles. Define

$$L^Y_{N,\Xi} := \sup_{\substack{|Y|\le R_0,\ |\xi|\le\Xi \\ 1\le\alpha\le 1+\delta_0}} \left|\partial_Y \Re \ell^{\mathrm{sh}}_{N,\alpha}(Y,\xi)\right|, \tag{1767}$$

and

$$L^\alpha_{N,\Xi} := \sup_{\substack{|Y|\le R_0,\ |\xi|\le\Xi \\ 1\le\alpha\le 1+\delta_0}} \left|\partial_\alpha \Re \ell^{\mathrm{sh}}_{N,\alpha}(Y,\xi)\right|. \tag{1768}$$

These constants are finite for each fixed N, R_0, Ξ, and sufficiently small δ_0.

178 Anchor normalization and the finite-N coefficient defect

The exact anchor calculation of Part 64 gives the principal phase symbol $t_{1,1}$. The proof-localized finite-N reconstruction may also contain a bounded anchor coefficient defect. Isolate it by

$$\varepsilon^{\mathrm{anc}}_{N,\Xi} := \sup_{|\xi|\le\Xi} \left|\Re \ell^{\mathrm{sh}}_{N,1}(0,\xi) - \xi^2\right|. \tag{1769}$$

This quantity is zero if the localized coefficient normalization at $Y = 0$ is exactly the phase-critical anchor, and otherwise it is an explicit finite-N number determined by the raw reconstruction. The mean-value theorem now implies, uniformly for $|Y| \le R \le R_0$, $|\xi| \le \Xi$, and $1 \le \alpha \le 1 + \delta \le 1 + \delta_0$,

$$\boxed{\Re \ell^{\mathrm{sh}}_{N,\alpha}(Y,\xi) \ge \xi^2 - \varepsilon^{\mathrm{anc}}_{N,\Xi} - L^Y_{N,\Xi} R - L^\alpha_{N,\Xi}(\alpha - 1).} \tag{1770}$$

Consequently

$$\boxed{\mu_{N,R,\alpha} \ge -\varepsilon^{\mathrm{anc}}_{N,\Xi} - L^Y_{N,\Xi} R - L^\alpha_{N,\Xi}(\alpha - 1).} \tag{1771}$$

This is the desired analytic compact-frequency enclosure. It does not use smallness of the shape derivatives and therefore does not assume conormal entry.

179 Comparison with the scalar damping margin

Recall

$$\lambda^{\mathrm{cor}}_N(R,\alpha) = \left(2\alpha - 1 - 2^{-k_N}\right) \frac{1 - e^{-R}}{R}. \tag{1772}$$

For $0 < R \le R_0$ and $\alpha \ge 1$,

$$\lambda^{\mathrm{cor}}_N(R,\alpha) \ge \left(1 - 2^{-k_N}\right) \frac{1 - e^{-R_0}}{R_0} =: \lambda_{N,0} > 0. \tag{1773}$$

Combining (1771) and (1773) gives the explicit sufficient inequality

$$\boxed{\varepsilon_{N,\Xi}^{\mathrm{anc}} + L_{N,\Xi}^{Y} R + L_{N,\Xi}^{\alpha}(\alpha - 1) < \lambda_{N,0}.} \tag{1774}$$

Whenever (1774) holds,

$$\boxed{\lambda_N^{\mathrm{cor}} + \mu_{N,R,\alpha} \geq \lambda_{N,0} - \varepsilon_{N,\Xi}^{\mathrm{anc}} - L_{N,\Xi}^{Y} R - L_{N,\Xi}^{\alpha}(\alpha - 1) > 0.} \tag{1775}$$

Thus the compact-frequency obstruction is perturbatively closed on every corridor satisfying the explicit budget (1774).

180 A nonzero certified corridor exists when the anchor defect is below the damping gap

Assume the checkable finite-N anchor condition

$$\boxed{\varepsilon_{N,\Xi}^{\mathrm{anc}} < \frac{1}{4}\lambda_{N,0}.} \tag{1776}$$

Choose

$$R_N^{\mathrm{spec}} := \min\left\{ R_0, \frac{\lambda_{N,0}}{4(1 + L_{N,\Xi}^{Y})} \right\}, \tag{1777}$$

and

$$\delta_N^{\mathrm{spec}} := \min\left\{ \delta_0, \frac{\lambda_{N,0}}{4(1 + L_{N,\Xi}^{\alpha})} \right\}. \tag{1778}$$

Then for

$$|Y| \leq R_N^{\mathrm{spec}}, \qquad 1 \leq \alpha < 1 + \delta_N^{\mathrm{spec}}, \tag{1779}$$

one has

$$\varepsilon_{N,\Xi}^{\mathrm{anc}} + L_{N,\Xi}^{Y} R_N^{\mathrm{spec}} + L_{N,\Xi}^{\alpha}(\alpha - 1) < \frac{3}{4}\lambda_{N,0}, \tag{1780}$$

and hence the uniform margin

$$\boxed{\lambda_N^{\mathrm{cor}} + \mu_{N,R_N^{\mathrm{spec}},\alpha} \geq \frac{1}{4}\lambda_{N,0} > 0.} \tag{1781}$$

This proves existence of a genuine nonzero finite-N spectral corridor. It is stronger than the small conormal tube because no decay of $EY, E^2Y, \ldots$ is assumed in deriving (1770). It is, however, not a certification of an arbitrarily large preassigned R_N.

181 The zero-frequency issue is absorbed by the amplitude equation

The compact interval contains $\xi = 0$, where the anchor shape symbol vanishes. This causes no loss because the corresponding constant logarithmic mode is precisely the amplitude direction. Let Π_0 denote the amplitude projection and $\Pi_{\mathrm{sh}} = I - \Pi_0$. The normalized linearized evolution has the block form

$$\partial_\tau \begin{pmatrix} \Pi_0 Z \\ \Pi_{\mathrm{sh}} Z \end{pmatrix} = - \begin{pmatrix} \Lambda_N(Y) & \mathcal{B}_{0\mathrm{sh}} \\ \mathcal{B}_{\mathrm{sh}0} & \mathcal{A}_{\mathrm{sh}} + \mathcal{L}_{\mathrm{frac}} \end{pmatrix} \begin{pmatrix} \Pi_0 Z \\ \Pi_{\mathrm{sh}} Z \end{pmatrix} + \mathbf{G}_N^{\mathrm{raw}}. \tag{1782}$$

The off-diagonal blocks are smooth finite-N coefficient perturbations and are included in the constants in (1767)–(1769). By Young's inequality they consume only part of the positive margin in (1781). Shrinking the spectral budget from $\lambda_{N,0}/4$ to $\lambda_{N,0}/8$ if necessary yields

$$\boxed{\Re\langle Z, (\mathbf{A}_N^{\mathrm{sh}} - \mathbf{L}_{N,\alpha}^{\mathrm{frac,sh}}) Z\rangle \geq \frac{\lambda_{N,0}}{8} \|Z\|_2^2 + c_{N,\alpha} \|\Pi_{\mathrm{sh}} Z\|_{\mathcal{H}_N^\alpha}^2.} \tag{1783}$$

Thus frequencies arbitrarily close to zero are controlled by the combination of amplitude damping and the nonnegative fractional derivative form; no artificial frequency gap is required.

182 High-frequency tail and compact block fit together

Choose $\Xi = \Xi_{N,\alpha}$ so that the elliptic estimate (1751) gives

$$\Re \ell_{N,\alpha}^{\mathrm{sh}}(Y, \xi) \geq c_{N,\alpha} |\xi|^{2\alpha} - C_{N,R_0} \qquad (|\xi| \geq \Xi). \tag{1784}$$

Increasing Ξ if necessary absorbs the bounded lower-order coefficient loss and gives

$$\boxed{\Re \ell_{N,\alpha}^{\mathrm{sh}}(Y, \xi) \geq \frac{c_{N,\alpha}}{2} |\xi|^{2\alpha} > 0, \qquad |\xi| \geq \Xi.} \tag{1785}$$

On $|\xi| \leq \Xi$, (1781)–(1783) supply the missing bounded-frequency coercivity. Therefore, subject only to the explicit anchor-defect check (1776), the full frequency axis is controlled on the certified corridor (1779).

183 Consequences for preterminal entry

Insert (1783) into the finite-jet energy equation of Part 63. The nonlinear triangular terms obey, on a sufficiently small but fixed jet radius,

$$|\langle \mathbf{Z}, \mathbf{N}_N^{\mathrm{sh}}(\mathbf{Z}) \rangle| \leq \frac{\lambda_{N,0}}{16} \|\mathbf{Z}\|_2^2. \tag{1786}$$

The raw algebraic forcing already satisfies the positive-power estimate (1717). Hence

$$\frac{1}{2} \frac{d}{d\tau} \|\mathbf{Z}_N^{\mathrm{sh}}\|_2^2 + \frac{\lambda_{N,0}}{16} \|\mathbf{Z}_N^{\mathrm{sh}}\|_2^2 + c_{N,\alpha} \|\Pi_{\mathrm{sh}} \mathbf{Z}_N^{\mathrm{sh}}\|_{\mathcal{H}_N^\alpha}^2 \leq C_N e^{-2\eta_N^{\mathrm{alg}} \tau}. \tag{1787}$$

Therefore

$$\boxed{\|\mathbf{Z}_N^{\mathrm{sh}}(\tau)\|_2 \le C_N e^{-\eta_N^{\mathrm{spec}}(\tau-\tau_0)}\|\mathbf{Z}_N^{\mathrm{sh}}(\tau_0)\|_2 + C_N e^{-\eta_N^{\mathrm{spec}}\tau},} \tag{1788}$$

where

$$\eta_N^{\mathrm{spec}} := \min\left\{\frac{\lambda_{N,0}}{16}, \eta_N^{\mathrm{alg}}\right\} > 0. \tag{1789}$$

Using (1724),

$$\boxed{\|\mathbf{R}_{N,M}(\tau)\| \le C_{N,M}^{\mathrm{pre}} e^{-\eta_{N,M}^{\mathrm{pre}}\tau}, \qquad \eta_{N,M}^{\mathrm{pre}} > 0.} \tag{1790}$$

Thus the preterminal residual estimate follows on the certified spectral corridor. Parts 62–63 then convert this decay into quantitative conormal entry provided the actual trajectory reaches this corridor without leaving the raw reconstruction chart.

184 What remains for the prescribed data

The spectral problem has therefore changed status. It is no longer necessary to determine an unrestricted global infimum over an arbitrary corridor. For each fixed N it is enough to verify the finite anchor number $\varepsilon_{N,\Xi}^{\mathrm{anc}}$ and the explicit derivative bounds $L_{N,\Xi}^{Y}$ and $L_{N,\Xi}^{\alpha}$. Once (1776) is certified, a nonzero corridor and continuation interval are produced explicitly by (1777)–(1779).

There remains one logically separate entry-map question: does the complete trajectory generated by the prescribed reconstructed data reach $|Y| \le R_N^{\mathrm{spec}}$ with its finite jet inside the radius required in (1786)? The present spectral enclosure proves contraction *after that larger corridor is reached*; it does not manufacture that first crossing from scalar amplitude data alone. This distinction prevents a circular entry claim.

Theorem 60 (Perturbative compact-frequency spectral enclosure)**.** *Fix $N < \infty$ and a proof-localized transverse chart. Let $\Xi < \infty$ be chosen so that the high-frequency elliptic estimate (1785) holds. Suppose the finite-N anchor defect (1769) satisfies (1776). Then there exist explicit positive numbers R_N^{spec} and δ_N^{spec} given by (1777)–(1778) such that, for $|Y| \le R_N^{\mathrm{spec}}$ and $1 \le \alpha < 1 + \delta_N^{\mathrm{spec}}$, the combined amplitude–shape operator has a strictly positive coercive margin. In particular (1783), the finite-jet decay (1788), and the complete preterminal residual estimate (1790) hold after entry into that corridor.*

If the anchor normalization is exact, then $\varepsilon_{N,\Xi}^{\mathrm{anc}} = 0$ and the anchor-defect hypothesis is automatic. The theorem certifies a nonzero finite-N outer corridor; it does not assert coercivity on an arbitrarily large prescribed corridor without evaluating the actual coefficient bounds there.

185 Updated proof ledger after the compact-frequency enclosure

anchor fractional shape sign:	closed, $\Re t_{1,1}(\xi) = \xi^2$,
high-frequency sector:	closed by ellipticity (1785),
compact-frequency sector:	analytic enclosure (1770)–(1771),
finite-N anchor defect:	explicit check $\varepsilon^{\mathrm{anc}}_{N,\Xi} < \lambda_{N,0}/4$ still required unless exact normalization gives zero,
certified spectral corridor:	exists by (1777)–(1781) once the anchor check holds,
combined amplitude–shape coercivity:	(1783),
preterminal finite-jet decay:	(1788),
complete preterminal residual:	(1790),
quantitative conormal entry after reaching the spectral corridor:	follows from Parts 62–63,
first crossing of the spectral corridor from the prescribed full reconstructed data:	remains to be proved by the complete entry map,
terminal scalar persistence after verified entry:	already closed.

(1791)

The bounded-frequency problem is therefore no longer an open sign question in an infinite-dimensional frequency tail. It has been reduced to explicit finite-N coefficient bounds on a compact block. Under the anchor-defect check, these bounds produce a nonzero coercive outer corridor and close the preterminal residual estimate there. The next non-circular issue is the first-crossing/entry-map estimate from the prescribed complete reconstructed data into this certified spectral corridor.

Part XIV
First-Crossing Audit from the Prescribed Complete Finite-N Data

The preceding compact-frequency audit proves contraction after the complete trajectory reaches the certified spectral corridor

$$|Y_N| \le R_N^{\mathrm{spec}}.$$

The remaining question is whether this first crossing follows from the prescribed finite-N data without assuming the conclusion. In this part we separate the scalar restoring mechanism

from the finite-jet shape mechanism and derive the exact quantitative criterion for first crossing. The result is deliberately stated at fixed finite N.

186 Global restoring sign of the scalar principal field

Recall

$$a_N := 2\alpha - 1 - 2^{-k_N} > 0, \qquad \mathscr{P}_N(Y) = a_N(e^{-Y} - 1). \tag{1792}$$

For $Y \neq 0$ define

$$\Lambda_N(Y) := -\frac{\mathscr{P}_N(Y)}{Y} = a_N \frac{1 - e^{-Y}}{Y}. \tag{1793}$$

The elementary function $(1 - e^{-Y})/Y$ is strictly positive for every finite real Y. Hence for every finite bridge radius $R > 0$,

$$\boxed{\lambda_N^{\mathrm{br}}(R) := a_N \min_{|Y| \le R} \frac{1 - e^{-Y}}{Y} = a_N \frac{1 - e^{-R}}{R} > 0.} \tag{1794}$$

Thus the scalar principal equation has an inward restoring sign on every bounded amplitude corridor. No smallness of Y is needed for this fact.

Let the exact normalized complete equation be

$$\partial_\tau Y_N = -\Lambda_N(Y_N) Y_N + \mathscr{R}_N^{\mathrm{cmp}}. \tag{1795}$$

If $|Y_N| \le R$ and

$$|\mathscr{R}_N^{\mathrm{cmp}}(\tau)| \le \rho_N(\tau), \tag{1796}$$

then

$$\frac{d}{d\tau}|Y_N| \le -\lambda_N^{\mathrm{br}}(R)|Y_N| + \rho_N(\tau). \tag{1797}$$

Consequently

$$\boxed{|Y_N(\tau)| \le e^{-\lambda_N^{\mathrm{br}}(R)(\tau - \tau_0)}|Y_N(\tau_0)| + \int_{\tau_0}^{\tau} e^{-\lambda_N^{\mathrm{br}}(R)(\tau - r)} \rho_N(r)\, dr.} \tag{1798}$$

This is the exact scalar first-crossing inequality.

187 A quantitative first-crossing time

Assume for the moment that on a bridge interval

$$\rho_N(\tau) \le C_N^{\mathrm{br}} e^{-\eta_N^{\mathrm{br}} \tau}, \qquad \eta_N^{\mathrm{br}} > 0. \tag{1799}$$

For $\lambda_N^{\mathrm{br}} \neq \eta_N^{\mathrm{br}}$, (1798) gives

$$|Y_N(\tau)| \le e^{-\lambda_N^{\mathrm{br}}(\tau - \tau_0)}|Y_N(\tau_0)| + \frac{C_N^{\mathrm{br}}}{|\lambda_N^{\mathrm{br}} - \eta_N^{\mathrm{br}}|} \left(e^{-\eta_N^{\mathrm{br}} \tau} + e^{-\lambda_N^{\mathrm{br}}(\tau - \tau_0)} e^{-\eta_N^{\mathrm{br}} \tau_0} \right). \tag{1800}$$

The resonant case has the usual factor $(\tau-\tau_0)e^{-\lambda_N^{\mathrm{br}}\tau}$ and also tends to zero. Therefore, provided the bridge estimate (1799) is valid until crossing, there is a finite τ_{cr} such that

$$\boxed{|Y_N(\tau_{\mathrm{cr}})|\le R_N^{\mathrm{spec}}.} \tag{1801}$$

For example, any τ_{cr} for which the two terms on the right of (1800) are each at most $R_N^{\mathrm{spec}}/2$ is sufficient. This gives an explicit finite first-crossing time in terms of the initial mismatch and the bridge constants.

188 Finite shape jet on the bridge

The amplitude crossing alone is not enough. Let

$$\mathbf{Z}_N^{\mathrm{sh}}=(EY_N,E^2Y_N,\dots,E^{M_*}Y_N). \tag{1802}$$

On a finite bridge corridor $|Y|\le R$ write the exact differentiated equation as

$$\partial_\tau\mathbf{Z}_N^{\mathrm{sh}}=-\mathbf{A}_N^{\mathrm{br}}(Y,\tau)\mathbf{Z}_N^{\mathrm{sh}}+\mathbf{N}_N^{\mathrm{br}}(Y,\mathbf{Z}_N^{\mathrm{sh}})+\mathbf{G}_N^{\mathrm{br}}(\tau). \tag{1803}$$

Define the bridge spectral margin

$$\boxed{\mu_{N,R,\alpha}^{\mathrm{br}}:=\inf_{|Y|\le R}\inf_{\xi\in\mathbb{R}}\Re\ell_{N,\alpha}^{\mathrm{sh}}(Y,\xi).} \tag{1804}$$

The same amplitude–shape decomposition used in Part 64 shows that a sufficient bridge coercivity condition is

$$\boxed{\lambda_N^{\mathrm{br}}(R)+\mu_{N,R,\alpha}^{\mathrm{br}}>0.} \tag{1805}$$

If the margin in (1805) is $2\kappa_N^{\mathrm{br}}>0$, the nonlinear jet radius is chosen so that its quadratic loss is at most $\kappa_N^{\mathrm{br}}/2$. Then

$$\frac{d}{d\tau}\|\mathbf{Z}_N^{\mathrm{sh}}\|_2\le-\kappa_N^{\mathrm{br}}\|\mathbf{Z}_N^{\mathrm{sh}}\|_2+\|\mathbf{G}_N^{\mathrm{br}}(\tau)\|_2. \tag{1806}$$

If

$$\|\mathbf{G}_N^{\mathrm{br}}(\tau)\|_2\le C_{N,M_*}^{\mathrm{br}}e^{-\eta_{N,M_*}^{\mathrm{br}}\tau}, \tag{1807}$$

then

$$\boxed{\|\mathbf{Z}_N^{\mathrm{sh}}(\tau)\|_2\le e^{-\kappa_N^{\mathrm{br}}(\tau-\tau_0)}\|\mathbf{Z}_N^{\mathrm{sh}}(\tau_0)\|_2+C_Ne^{-\min\{\kappa_N^{\mathrm{br}},\eta_{N,M_*}^{\mathrm{br}}\}\tau}.} \tag{1808}$$

Hence the finite shape jet also reaches any prescribed positive contraction radius in finite terminal logarithmic time, provided (1805)–(1807) hold on the bridge.

189 What the compact-frequency calculation actually proves on a bridge

Part 64 gives, on $|\xi| \le \Xi$,

$$\Re \ell^{\mathrm{sh}}_{N,\alpha}(Y,\xi) \ge -\varepsilon^{\mathrm{anc}}_{N,\Xi} - L^{Y}_{N,\Xi}|Y| - L^{\alpha}_{N,\Xi}(\alpha-1). \tag{1809}$$

Therefore for any finite R,

$$\boxed{\mu^{\mathrm{br}}_{N,R,\alpha} \ge -\varepsilon^{\mathrm{anc}}_{N,\Xi} - L^{Y}_{N,\Xi}R - L^{\alpha}_{N,\Xi}(\alpha-1).} \tag{1810}$$

Combining (1794) and (1810) yields the fully explicit sufficient bridge condition

$$\boxed{a_N \frac{1-e^{-R}}{R} > \varepsilon^{\mathrm{anc}}_{N,\Xi} + L^{Y}_{N,\Xi}R + L^{\alpha}_{N,\Xi}(\alpha-1).} \tag{1811}$$

When (1811) holds, the entire frequency axis is coercive on the bridge after the high-frequency elliptic tail is added exactly as in (1784)–(1785). Thus the first-crossing problem is finite and quantitative: it is not an unknown sign at arbitrarily high frequency.

190 Insertion of the prescribed scalar datum

For

$$b_3(z,0) = 3 + \sin z \tag{1812}$$

we have

$$2 \le b_3(z,0) \le 4, \qquad \|\partial_z^m b_3(\cdot,0)\|_{L^\infty} \le 1 \quad (m \ge 1). \tag{1813}$$

Let s_0 denote the phase value of the chosen initial section. The exact initial normalized amplitude is

$$Y_N(\tau_0, z) = \log \frac{(3+\sin z)s_0^{\beta}}{B_{*,N}}, \tag{1814}$$

and hence

$$\boxed{|Y_N(\tau_0,z)| \le R^{\mathrm{data}}_{N,0} := \max\left\{ \left| \log \frac{2s_0^{\beta}}{B_{*,N}} \right|, \left| \log \frac{4s_0^{\beta}}{B_{*,N}} \right| \right\}.} \tag{1815}$$

Every finite Euler jet of (1814) is finite because the denominator $3+\sin z$ is bounded below by 2. Thus the prescribed scalar datum gives an explicit finite bridge radius and a finite initial shape-jet norm.

However, the phrase "prescribed complete reconstructed data" must include the initial fields A_N, C_N, their required spatial derivatives, the forcing, and the pressure-compatible reconstruction entering the complete residual. The scalar datum (1812) alone does not determine the constants $\varepsilon^{\mathrm{anc}}_{N,\Xi}$, $L^{Y}_{N,\Xi}$, $L^{\alpha}_{N,\Xi}$, C^{br}_{N}, or C^{br}_{N,M_*}. Therefore (1811), (1799), and (1807) cannot be numerically or symbolically certified from (1812) alone.

191 First-crossing theorem for the complete reconstructed data

Theorem 61 (Finite-N first crossing into the certified spectral corridor)**.** *Fix $N < \infty$ and let the complete reconstructed initial data determine a smooth solution on a real-branch chart. Let $R > R_N^{\mathrm{spec}}$ contain the initial normalized amplitude and finite shape jet. Assume the raw reconstruction coefficients satisfy the explicit bridge spectral inequality (1811), the complete scalar residual estimate (1799), and the differentiated residual estimate (1807) until the first crossing. Then there exists a finite terminal logarithmic time τ_{cr} such that*

$$|Y_N(\tau_{\mathrm{cr}})| \le R_N^{\mathrm{spec}} \tag{1816}$$

and, after increasing τ_{cr} by a finite amount if necessary,

$$\|\mathbf{Z}_N^{\mathrm{sh}}(\tau_{\mathrm{cr}})\|_2 \le r_{N,M_*}^{\mathrm{contr}}, \tag{1817}$$

where $r_{N,M_*}^{\mathrm{contr}}$ is the nonlinear contraction radius in (1786). The compact-frequency spectral estimate of Part 64 then gives the preterminal residual decay (1790), Parts 62–63 give quantitative conormal entry, and Part 61 propagates the local conormal tube.

Proof. The bridge inequality (1811) and the high-frequency elliptic estimate imply (1805). The scalar comparison (1798) together with (1799) tends to zero as $\tau \to \infty$, so it crosses R_N^{spec} in finite τ. The shape estimate (1808) likewise tends to zero and therefore crosses the finite contraction radius. Strict inward pointing on the bridge boundary follows from (1797) and (1806), so the estimates remain valid up to the first crossing. After crossing, Part 64 supplies the smaller-corridor coercivity and the previous entry/propagation results apply. □ □

192 Can the requested first crossing be proved from the data presently written in the paper?

The preceding theorem isolates all quantities that must be evaluated. The answer from the currently written data is more precise than a generic "entry remains open." The scalar principal field is globally restoring on every finite amplitude corridor, the historical datum supplies an explicit finite amplitude radius (1815), and the high-frequency shape sector is coercive. What is not supplied by the scalar datum is the complete reconstruction coefficient jet needed to verify (1811), (1799), and (1807) on the bridge from $R_{N,0}^{\mathrm{data}}$ down to R_N^{spec}.

Accordingly, a proof that the *actual* trajectory reaches the spectral corridor cannot be completed from $b_3(z,0) = 3 + \sin z$ alone without inserting those complete reconstructed initial fields and their finite-N coefficient bounds. Assuming them would be exactly the circular step this audit is meant to avoid. Once those raw data are inserted, (1811), (1800), and (1808) are a finite checklist and give an explicit first-crossing time rather than a new asymptotic hypothesis.

193 Updated proof ledger after the first-crossing audit

scalar restoring sign on every bounded corridor:	closed by (1793)–(1794),
prescribed scalar initial amplitude radius:	explicit in (1815),
finite initial scalar Euler jet:	closed from (1813)–(1814),
high-frequency shape sector:	closed in Part 64,
compact-frequency bridge enclosure:	reduced to explicit inequality (1811),
complete bridge scalar residual:	requires raw reconstructed coefficient bounds in (1799),
complete differentiated bridge residual:	requires raw reconstructed coefficient bounds in (1807),
first crossing once bridge bounds hold:	proved by (1798), (1800), and (1808),
entry and propagation after crossing:	already closed by Parts 61–64,
first crossing from the scalar datum alone:	not proved because it does not specify the complete reconstructed coefficient jet,
finite-time scalar law after verified complete-data crossing:	follows from the established persistence theorem.

(1818)

The first-crossing problem has therefore been reduced to a finite raw-data verification, not to a new terminal asymptotic argument. The next legitimate step is to substitute the manuscript's actual prescribed A_N, C_N, forcing, and pressure-compatible initial reconstruction into (1811), (1799), and (1807). Without those data, claiming unconditional first crossing would go beyond what the current manuscript specifies.

Part 66. Raw-Data Substitution Audit for the First-Crossing Conditions

194 Purpose of the raw-data substitution

Part 65 reduced unconditional first crossing to the finite verification of (1811), (1799), and (1807). We now perform the promised substitution using the actual reconstruction formulas stated than introducing new free coefficient fields.

The first point is structural. The revised source prescribes

$$\boxed{Q_N = \Delta_{N,k(N)}, \qquad A_N = u_y = V_N - D_N(V_N), \qquad C_N = u_z = V_N + D_N(V_N),} \tag{1819}$$

with

$$\boxed{D_N(V) = \frac{-(2V + R_N) + \sqrt{R_N^2 + 8R_N V}}{2}, \qquad \frac{C_N^2}{A_N} = R_N.} \tag{1820}$$

The reconstructed first velocity component is

$$\boxed{u_x = \frac{b_N}{A_N}, \qquad u_x u_y = b_N.} \tag{1821}$$

Thus the raw reconstruction is algebraic in the pair (V_N, R_N) once the scalar b_N is supplied.

The corrected forcing appearing in the same source is

$$\boxed{G(x, y, z) = A_0 + (\sin x \sin y \sin z)^2, \qquad F_c = 2C_N G.} \tag{1822}$$

In particular, if $A_0 > 0$ and $C_N \geq c_- > 0$, then

$$\boxed{F_c \geq 2c_- A_0 > 0.} \tag{1823}$$

This sign is favorable for entry, but by itself does not control the complete reconstruction error.

195 The actual raw scalar equation

The revised source writes the corrected amplitude equation as

$$\boxed{\mathcal{D}_N b_N = \frac{Q_N'}{Q_N} b_N^2 + 2C_N G - \mathcal{E}_N,} \tag{1824}$$

where, in the displayed two-correction reconstruction,

$$\boxed{\mathcal{E}_N = b_N \frac{C_{N,t} + C_{N,z}}{C_N} + b_N A_N C_{N,x} + \frac{b_N^2}{A_N} C_{N,y} + 2C_N \nabla b_N \cdot \nabla C_N.} \tag{1825}$$

This formula is the quantity that must be inserted into the bridge residual. It is not legitimate to replace it by a different reconstruction remainder before proving that the two expressions agree.

For the prescribed scalar datum

$$\boxed{b_N(z, 0) = 3 + \sin z, \qquad 2 \leq b_N(z, 0) \leq 4,} \tag{1826}$$

all scalar derivatives are bounded at $t = 0$. Hence any failure of the raw bridge estimate at entry must come from the reconstructed coefficient fields or from the nonlocal fractional block, not from a singular scalar datum.

196 Direct comparison with the reconstruction used in Parts 51–65

The first-crossing machinery in the present continuation was developed after introducing the square-root reconstruction

$$\boxed{A_N^{\mathrm{sq}} = \sqrt{-2\chi b_N}, \qquad u_x^{\mathrm{sq}} = \sqrt{-\frac{b_N}{2\chi}}, \qquad C_N^{\mathrm{sq}} = C_N(x,s),} \tag{1827}$$

with the corresponding remainder

$$\boxed{\mathcal{R}_N^{\mathrm{sq}} = b_N \frac{C_{N,t} + C_{N,z}}{C_N} + 2A_N b_N C_{N,x} + 2\frac{b_N^2}{A_N} C_{N,y}.} \tag{1828}$$

By contrast, the raw two-correction source gives (1825). Their difference is therefore exactly

$$\boxed{\mathcal{E}_N - \mathcal{R}_N^{\mathrm{sq}} = -A_N b_N C_{N,x} - \frac{b_N^2}{A_N} C_{N,y} + 2C_N \nabla b_N \cdot \nabla C_N.} \tag{1829}$$

There is no identity in the currently cited raw reconstruction that makes the right-hand side of (1829) vanish identically. Thus the substitution audit reveals a genuine reconstruction mismatch: equations (1799) and (1807) were proved for a remainder architecture that is not automatically identical to the two-correction architecture (1819)–(1825).

This is not a small numerical defect. It changes the finite-jet residual that enters the first-crossing estimate. Consequently, the raw data cannot simply be inserted into the old constants C_N^{br} and C_{N,M_*}^{br} without first re-deriving those constants for (1825).

197 What can nevertheless be substituted exactly

The algebraic reconstruction gives several exact bridge identities. From $C_N^2/A_N = R_N$,

$$\boxed{A_N = \frac{C_N^2}{R_N}, \qquad u_x = \frac{b_N R_N}{C_N^2}.} \tag{1830}$$

Hence, on a raw bridge on which

$$0 < c_C \le C_N \le C_C, \qquad 0 < c_R \le R_N \le C_R, \tag{1831}$$

we obtain the explicit bounds

$$\boxed{\frac{c_C^2}{C_R} \le A_N \le \frac{C_C^2}{c_R}, \qquad \frac{c_R}{C_C^2} b_N \le u_x \le \frac{C_R}{c_C^2} b_N.} \tag{1832}$$

Thus positivity and nondegeneracy of the raw reconstructed velocity reduce to finite bounds on C_N and R_N.

Moreover, (1825) gives the pointwise estimate

$$\begin{aligned}|\mathcal{E}_N| \le{}& \frac{b_N}{c_C}|C_{N,t}+C_{N,z}| + b_N\frac{C_C^2}{c_R}|C_{N,x}| \\ &+ \frac{C_R}{c_C^2}b_N^2|C_{N,y}| + 2C_C|\nabla b_N|\,|\nabla C_N|.\end{aligned} \tag{1833}$$

Therefore a sufficient raw scalar bridge bound is

$$\boxed{\rho_N^{\mathrm{raw}}(\tau) \le \rho_N^{\mathrm{frac}}(\tau) + \frac{s}{b_N^2}|\mathcal{E}_N| + \frac{2sC_C}{b_N^2}\|G\|_\infty,} \tag{1834}$$

where ρ_N^{frac} denotes the normalized complete fractional defect. Equation (1834), not the older schematic residual, is the direct raw-data replacement for the quantity in (1799).

198 Raw differentiated residual

Let $E = s\partial_s$ and let $\mathfrak{J}_M$ denote the collection of Euler jets through order M. Differentiating (1825) gives, by Leibniz' rule,

$$\boxed{\|\mathfrak{J}_M\mathcal{E}_N\| \le C_{N,M}^{\mathrm{alg}}\mathcal{P}_M\left(\|\mathfrak{J}_{M+1}b_N\|, \|\mathfrak{J}_{M+1}C_N\|, \|A_N\| + \|A_N^{-1}\|\right),} \tag{1835}$$

where $\mathcal{P}_M$ is an explicit finite polynomial with nonnegative coefficients. Using (1830), every occurrence of A_N and A_N^{-1} can be replaced by C_N and R_N. Hence the differentiated bridge estimate (1807) is reduced to the finite raw condition

$$\boxed{\begin{aligned}&\sup_{\tau_0\le\tau\le\tau_e} e^{\eta\tau}\Bigg[\|\mathbf{R}_{N,M}^{\mathrm{frac}}(\tau)\| \\ &\qquad + \frac{s}{b_N^2}C_{N,M}^{\mathrm{alg}}\mathcal{P}_M\left(\|\mathfrak{J}_{M+1}b_N\|, \|\mathfrak{J}_{M+1}C_N\|, \|\mathfrak{J}_{M+1}R_N\|\right)\Bigg] < \infty.\end{aligned}} \tag{1836}$$

This is a genuine finite-N checklist. But the source formulas displayed so far do not supply the required uniform bridge bounds for all derivatives in (1836); they define the fields but do not prove the needed decay on the whole first-crossing interval.

199 Audit of the compact-frequency inequality (1811)

The compact-frequency condition from Part 65 is

$$\boxed{\lambda_N^{\mathrm{br}}(R) > \varepsilon_{N,\Xi}^{\mathrm{anc}} + L_{N,\Xi}^{Y}R + L_{N,\Xi}^{\alpha}(\alpha-1).} \tag{1837}$$

The raw reconstruction (1819)–(1820) determines the coefficients entering the left and right sides once V_N, R_N and their finite jets are known. However, the source identities

$$\frac{C_N^2}{A_N} = R_N, \qquad D_N(V) = \frac{-(2V+R_N)+\sqrt{R_N^2+8R_NV}}{2} \tag{1838}$$

do not by themselves provide a numerical upper bound for $\varepsilon^{\mathrm{anc}}_{N,\Xi}$, $L^{Y}_{N,\Xi}$, or $L^{\alpha}_{N,\Xi}$ on the bridge. In particular, those constants contain localized fractional-symbol derivatives and therefore require bounds on the actual coefficient jet, not merely positivity of A_N, C_N.

Accordingly the exact substitution yields

$$\boxed{\text{(1811) is reduced to raw finite coefficient bounds, but is not yet numerically certified by the displayed reconstruction identities alone.}} \tag{1839}$$

200 Pressure compatibility cannot be silently supplied

The raw source material located in this audit specifies the velocity reconstruction, continuity relation, and corrected forcing. It does not, in the same displayed data block, prescribe a complete initial pressure field whose gradient has already been verified against all three momentum equations for the present fractional system. Therefore pressure compatibility cannot be inserted as an unproved coefficient bound in (1836) or (1837).

The correct logical requirement is

$$\boxed{\nabla P_N(0) = \mathbb{Q}[-f_N(0) - u_N(0)\cdot\nabla u_N(0) - \nu(-\Delta)^{\alpha} u_N(0)]\,,} \tag{1840}$$

where $\mathbb{Q} = I - \mathbb{P}$ is the gradient part of the Helmholtz–Leray decomposition, together with the corresponding periodic solvability and regularity conditions. Until this is evaluated for the same raw A_N, C_N used in (1819), pressure-compatible entry remains a separate finite check.

201 Result of the actual substitution audit

The requested substitution therefore has a definite outcome. The raw source supplies

$$\boxed{A_N = V_N - D_N(V_N), \quad C_N = V_N + D_N(V_N), \quad \frac{C_N^2}{A_N} = R_N, \quad G = A_0 + (\sin x \sin y \sin z)^2,} \tag{1841}$$

and the complete raw reconstruction error (1825). These formulas are enough to turn (1799) and (1807) into the explicit finite estimates (1834) and (1836). They also show that the forcing has the favorable sign (1823).

But the substitution simultaneously exposes the mismatch (1829) between the raw two-correction reconstruction and the square-root remainder used in the later first-crossing derivation. Therefore one cannot honestly conclude from the current source that (1799), (1807), and (1811) already hold for the same complete reconstructed solution.

$$\boxed{\begin{array}{l}\text{The first-crossing theorem remains conditional, but the condition is now strictly}\\ \text{finite and reconstruction-specific. To close it one must either (i) prove that the}\\ \text{two-correction reconstruction (1819)–(1825) is equivalent to the square-root recon-}\\ \text{struction used in Parts 51–65 on the bridge, or (ii) rederive the bridge fractional-}\\ \text{shape operator and its spectral constants directly for (1819)–(1825). In either route}\\ \text{one must then verify the raw jet bounds in (1836) and the pressure compatibility}\\ \text{condition (1840).}\end{array}} \tag{1842}$$

This is a stronger conclusion than merely saying that unspecified data are missing: the actual source data have now been inserted, and they identify the precise reconstruction-level incompatibility that must be resolved before an unconditional first crossing can be claimed.

202 Updated proof ledger after raw-data substitution

raw A_N, C_N reconstruction:	identified explicitly in (1819)–(1820),
raw forcing:	identified and favorable by (1822)–(1823),
raw scalar reconstruction error:	identified exactly in (1825),
scalar initial datum:	regular and bounded by (1826),
raw scalar bridge residual:	reduced to explicit estimate (1834),
raw differentiated residual:	reduced to finite jet estimate (1836),
compact-frequency bridge condition:	reduced to finite raw coefficient constants in (1837),
reconstruction equivalence with Parts 51–65:	not established; mismatch is (1829),
pressure-compatible initial field:	still requires the finite check (1840),
unconditional first crossing:	not yet proved,
entry/persistence after a verified crossing:	remains available from Parts 61–65.

(1843)

The next mathematically necessary calculation is therefore no longer to estimate an unspecified bridge constant. It is to reconcile the two reconstruction architectures, or to rebuild the bridge spectral estimate for the raw two-correction fields (1819)–(1825). Only after that reconciliation can the finite raw-data inequalities be evaluated without mixing two distinct velocity reconstructions.

Part 67. Direct Bridge Spectral Rederivation for the Raw Two-Correction Reconstruction

203 Why the two architectures need not be identified pointwise

Part 66 found the exact difference (1829) between the algebraic remainder of the raw two-correction reconstruction and the remainder used in the square-root continuation. That difference prevents substitution of one remainder for the other. It does *not*, however, imply that the fractional shape mechanism must be rebuilt from zero. The correct route is to derive

the shape operator directly from the exact product identity

$$\boxed{b_N = u_{x,N} u_{y,N}, \qquad u_{y,N} = A_N, \qquad u_{x,N} = \frac{b_N}{A_N}.} \tag{1844}$$

This identity is common to the raw construction and is sufficient to fix the principal fractional symbol independently of how the factor A_N is split between Q_N, S_N, D_N.

Let

$$L_\alpha := (-\Delta)^\alpha. \tag{1845}$$

The exact viscous product block is

$$\boxed{\mathfrak{V}_{N,\alpha} := A_N L_\alpha \left(\frac{b_N}{A_N} \right) + \frac{b_N}{A_N} L_\alpha A_N.} \tag{1846}$$

This is the no-double-counting form already implied by the exact product momentum equation. Thus the raw bridge analysis can be carried out directly on (1846), without replacing A_N by $\sqrt{-2\chi b_N}$.

204 Universal principal symbol forced by the product constraint

Fix a smooth finite-N bridge state and write a logarithmic scalar perturbation

$$b_N^{(\varepsilon)} = b_N e^{\varepsilon Z}. \tag{1847}$$

The continuity reconstruction may make A_N depend on the perturbation. Put

$$\Theta_N Z := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \log A_N^{(\varepsilon)}, \tag{1848}$$

where Θ_N denotes the linearized raw reconstruction map. Since $u_x = b/A$, the exact product identity gives

$$\left. \frac{d}{d\varepsilon} \right|_0 \log u_{y,N}^{(\varepsilon)} = \Theta_N Z, \qquad \left. \frac{d}{d\varepsilon} \right|_0 \log u_{x,N}^{(\varepsilon)} = (I - \Theta_N) Z. \tag{1849}$$

Consequently the order-2α pieces in the Frechet derivative of (1846) are

$$\begin{aligned} \delta \mathfrak{V}_{N,\alpha}^{\mathrm{prin}}[Z] &= u_{y,N} u_{x,N} L_\alpha ((I - \Theta_N) Z) + u_{x,N} u_{y,N} L_\alpha (\Theta_N Z) \\ &= b_N L_\alpha Z + b_N [L_\alpha, \Theta_N] Z. \end{aligned} \tag{1850}$$

If the reconstruction map Θ_N is order zero on the bridge, as is the case for the algebraic D_N part and for a bounded solution operator for the first-order continuity correction, then the commutator in (1850) has order at most $2\alpha - 1$. Therefore

$$\boxed{\sigma_{2\alpha} \left(b_N^{-1} \delta \mathfrak{V}_{N,\alpha} \right) (X, \xi) = |\xi|^{2\alpha}.} \tag{1851}$$

The coefficient is exactly one. This is the reconstruction-independent principal-symbol identity that reconciles the two architectures at the level needed for the bridge spectral argument.

The point is algebraic: the two logarithmic shares in (1849) add to one. A change of reconstruction redistributes the perturbation between u_x and u_y, but cannot change the top-order symbol of the product block.

205 Exact frozen-A_N formula and the anchor check

It is useful first to freeze the raw coefficient A_N. Put

$$g_N := \frac{b_N}{A_N} = u_{x,N}. \tag{1852}$$

After removal of the scalar amplitude mode, the corresponding exact shape operator is

$$\boxed{\mathscr{T}^{\mathrm{raw}}_{N,\alpha} Z = g_N^{-1} L_\alpha (g_N Z) - Z g_N^{-1} L_\alpha g_N.} \tag{1853}$$

In particular

$$\mathscr{T}^{\mathrm{raw}}_{N,\alpha} 1 = 0. \tag{1854}$$

Thus (1853) is the raw analogue of (1737), with the actual factor $g_N = b_N/A_N$ in place of the square-root power a_β.

At $\alpha = 1$, $L_1 = -\partial_{ss}$, so (1853) is exactly

$$\boxed{\mathscr{T}^{\mathrm{raw}}_{N,1} Z = -Z_{ss} - 2\frac{g_{N,s}}{g_N} Z_s.} \tag{1855}$$

The zeroth-order terms cancel identically. In logarithmic coordinates $E = s\partial_s$, this becomes

$$\mathscr{T}^{\mathrm{raw}}_{N,1} Z = s^{-2} \left[-E^2 + (1 - 2E \log g_N)\, E \right] Z. \tag{1856}$$

Hence the frozen logarithmic symbol is

$$\boxed{t^{\mathrm{raw}}_{N,1}(X,\xi) = \xi^2 + i\xi \left(1 - 2E \log g_N(X)\right), \qquad \Re t^{\mathrm{raw}}_{N,1}(X,\xi) = \xi^2.} \tag{1857}$$

This is stronger than merely retaining the square-root anchor by analogy: for the actual raw factor g_N, the real part at the integer anchor is still exactly ξ^2 before the continuity-response and other lower-order commutators are restored.

206 Restoring the continuity response

The raw source defines

$$A_N = Q_N + S_N - D_N, \qquad C_N = Q_N + S_N + D_N, \tag{1858}$$

and the continuity correction satisfies a first-order equation containing $\partial_x(b_N/A_N)$. Therefore Θ_N in (1848) is not identically zero. The correct conclusion is not to discard it, but to isolate its commutator contribution.

Write the complete normalized linearized raw shape operator as

$$\boxed{\mathbf{L}^{\mathrm{raw,sh}}_{N,\alpha} = \mathscr{T}^{\mathrm{raw}}_{N,\alpha} + \mathbf{K}^{\mathrm{cont}}_{N,\alpha} + \mathbf{K}^{\mathrm{alg}}_{N,\alpha} + \mathbf{K}^{\mathrm{tr}}_{N,\alpha}.} \tag{1859}$$

Here $\mathbf{K}^{\mathrm{cont}}$ contains $[L_\alpha, \Theta_N]$ and the linearized continuity response, $\mathbf{K}^{\mathrm{alg}}$ contains derivatives of $D_N(V_N)$ and the algebraic coefficients, and $\mathbf{K}^{\mathrm{tr}}$ contains the transport/reconstruction

terms from (1825). On a finite bridge on which A_N, C_N, R_N and their required jets are bounded and A_N, C_N, R_N stay away from zero, all three are lower-order finite-N operators relative to the order-2α term in (1851).

Accordingly there are finite raw constants

$$K^{\mathrm{cont}}_{N,M}, \qquad K^{\mathrm{alg}}_{N,M}, \qquad K^{\mathrm{tr}}_{N,M}, \tag{1860}$$

determined only by the same finite coefficient jets already appearing in (1836), such that the localized energy estimate has the form

$$\boxed{\Re\langle Z, \mathbf{L}^{\mathrm{raw,sh}}_{N,\alpha} Z\rangle \geq c^{\mathrm{raw}}_{N,\alpha}\|Z\|^2_{H^\alpha} - C^{\mathrm{raw}}_{N,R}\|Z\|^2_2,} \tag{1861}$$

where

$$C^{\mathrm{raw}}_{N,R} \leq C\left(K^{\mathrm{cont}}_{N,M} + K^{\mathrm{alg}}_{N,M} + K^{\mathrm{tr}}_{N,M}\right). \tag{1862}$$

No square-root coefficient occurs in (1861)–(1862).

207 Raw compact-frequency enclosure

Fix $\Xi < \infty$. Define the actual raw anchor defect

$$\boxed{\varepsilon^{\mathrm{raw}}_{N,\Xi} := \sup_{|\xi|\leq\Xi}\left|\Re\ell^{\mathrm{raw,sh}}_{N,1}(0,\xi) - \xi^2\right|.} \tag{1863}$$

Define also

$$\boxed{L^{Y,\mathrm{raw}}_{N,\Xi} := \sup_{\mathcal{B}_{N,R_0,\delta_0}} |\partial_Y \Re\ell^{\mathrm{raw,sh}}_{N,\alpha}(Y,\xi)|, \qquad L^{\alpha,\mathrm{raw}}_{N,\Xi} := \sup_{\mathcal{B}_{N,R_0,\delta_0}} |\partial_\alpha \Re\ell^{\mathrm{raw,sh}}_{N,\alpha}(Y,\xi)|.} \tag{1864}$$

The bridge set $\mathcal{B}_{N,R_0,\delta_0}$ is the compact set $|Y| \leq R_0$, $|\xi| \leq \Xi$, $1 \leq \alpha \leq 1+\delta_0$, together with the finite raw coefficient-jet bounds (1831), (1836). Smoothness of the finite-N real-branch reconstruction makes the numbers in (1863)–(1864) finite whenever those raw jet bounds hold.

The same mean-value argument as in Part 64 now gives, without using the square-root reconstruction,

$$\boxed{\Re\ell^{\mathrm{raw,sh}}_{N,\alpha}(Y,\xi) \geq \xi^2 - \varepsilon^{\mathrm{raw}}_{N,\Xi} - L^{Y,\mathrm{raw}}_{N,\Xi} R - L^{\alpha,\mathrm{raw}}_{N,\Xi}(\alpha-1), \qquad |\xi| \leq \Xi.} \tag{1865}$$

At high frequency, (1851) gives

$$\boxed{\Re\ell^{\mathrm{raw,sh}}_{N,\alpha}(Y,\xi) \geq c_{N,\alpha}|\xi|^{2\alpha} - C^{\mathrm{raw}}_{N,R}, \qquad |\xi| \geq \Xi_N,} \tag{1866}$$

so Ξ_N can be increased until the right side is strictly positive. Therefore the entire bridge spectral problem for the raw reconstruction is again reduced to a compact-frequency finite-N check.

208 Replacement of equation (1811)

The scalar restoring margin is unchanged because it comes from the normalized b_N equation:

$$\lambda_N^{\mathrm{br}}(R) = a_N \frac{1-e^{-R}}{R}, \qquad a_N = 2\alpha - 1 - 2^{-k_N} > 0. \tag{1867}$$

Combining (1865) with (1867) gives the correct raw two-correction replacement for (1811):

$$\boxed{a_N \frac{1-e^{-R}}{R} > \varepsilon_{N,\Xi}^{\mathrm{raw}} + L_{N,\Xi}^{Y,\mathrm{raw}} R + L_{N,\Xi}^{\alpha,\mathrm{raw}}(\alpha - 1).} \tag{1868}$$

Thus the architecture mismatch in (1829) does *not* destroy the bridge spectral mechanism. It changes the finite compact-frequency constants from the square-root constants to the raw constants (1863)–(1864).

If

$$\varepsilon_{N,\Xi}^{\mathrm{raw}} < \frac{1}{4}\lambda_{N,0}, \qquad \lambda_{N,0} := (1 - 2^{-k_N}) \frac{1-e^{-R_0}}{R_0}, \tag{1869}$$

then one may define

$$\boxed{R_{N,\mathrm{raw}}^{\mathrm{spec}} := \min\left\{ R_0, \frac{\lambda_{N,0}}{4(1 + L_{N,\Xi}^{Y,\mathrm{raw}})} \right\}, \qquad \delta_{N,\mathrm{raw}}^{\mathrm{spec}} := \min\left\{ \delta_0, \frac{\lambda_{N,0}}{4(1 + L_{N,\Xi}^{\alpha,\mathrm{raw}})} \right\}.} \tag{1870}$$

Exactly as before, this yields a positive combined amplitude–shape margin on $|Y| \le R_{N,\mathrm{raw}}^{\mathrm{spec}}$.

209 Replacement of (1799) and (1807) for the same architecture

Part 66 already produced the raw scalar residual (1834) and raw differentiated residual (1836). We now place them in the same operator architecture as (1868). Define

$$\boxed{\rho_N^{\mathrm{raw}} := \rho_N^{\mathrm{frac,raw}} + \frac{s}{b_N^2}|\mathcal{E}_N| + \frac{2s|C_N G|}{b_N^2},} \tag{1871}$$

where $\mathcal{E}_N$ is exactly (1825), and where $\rho_N^{\mathrm{frac,raw}}$ is computed from (1846), not from the square-root factorization. A sufficient replacement for (1799) is

$$\boxed{\rho_N^{\mathrm{raw}}(\tau) \le C_N^{\mathrm{br,raw}} e^{-\eta_N^{\mathrm{br,raw}}\tau}.} \tag{1872}$$

Likewise, if $\mathbf{G}_{N,M}^{\mathrm{raw}}$ denotes the differentiated forcing obtained by applying the bridge Euler jets to (1825), (1846), and the smooth forcing (1822), the exact replacement for (1807) is

$$\boxed{\|\mathbf{G}_{N,M}^{\mathrm{raw}}(\tau)\|_2 \le C_{N,M}^{\mathrm{br,raw}} e^{-\eta_{N,M}^{\mathrm{br,raw}}\tau}.} \tag{1873}$$

Equations (1868), (1872), and (1873) now refer to one and the same raw reconstruction. The logical mixing identified in Part 66 has therefore been removed.

210 What has actually been reconciled

The comparison can now be stated sharply:

> The square-root and raw two-correction remainders are not equal, and no such equality is needed. The bridge spectral calculation can be rederived directly for the raw fields. The exact product constraint forces the normalized order-2α fractional symbol to be $|\xi|^{2\alpha}$; all dependence on the two-correction reconstruction enters through lower-order finite-N commutators and compact-frequency constants. Hence the correct first-crossing check is the raw triple (1868), (1872), (1873), together with pressure compatibility (1840).

(1874)

This resolves the *architectural* obstruction found in Part 66. It does not by itself prove the numerical inequalities (1868), (1872), and (1873), because their constants require the actual finite bridge jets of S_N, hence of A_N, C_N, along the prescribed trajectory. In particular, smoothness at one initial section is not a substitute for exponential bridge decay.

Theorem 62 (Raw two-correction bridge spectral reduction)**.** *Fix $N < \infty$ and a real-branch bridge on which the raw two-correction reconstruction is smooth, A_N, C_N, R_N are bounded above and away from zero, and the linearized continuity solution operator is bounded at order zero on the localized bridge spaces. Then the normalized linearized complete fractional product block associated with $u_y = A_N$, $u_x = b_N/A_N$ has principal symbol $|\xi|^{2\alpha}$. At $\alpha = 1$ its frozen-coefficient amplitude-subtracted shape symbol has exact real part ξ^2. For $1 \le \alpha < 1 + \delta$ the high-frequency sector is elliptic, and the compact-frequency sector satisfies (1865). Therefore the raw first-crossing spectral condition is (1868), with residual conditions (1872)–(1873).*

If these three finite raw inequalities and pressure compatibility (1840) hold until first crossing, the first-crossing theorem of Part 65 applies with all square-root constants replaced by the raw constants above.

Proof. The logarithmic product variations (1849) sum to one, which gives the universal top-order identity (1850)–(1851). Freezing the reconstruction yields (1853); direct differentiation at $\alpha = 1$ gives (1855)–(1857). Restoring the continuity response produces only the commutator and lower-order blocks in (1859) under the stated order-zero boundedness hypothesis. Standard localized elliptic/Garding estimates then give (1861) and the high-frequency estimate (1866). Compactness of the finite bridge coefficient set and the mean-value theorem give (1865). Adding the scalar restoring margin (1867) yields (1868). Finally (1872)–(1873) are exactly the scalar and differentiated raw residual hypotheses required in (1798) and (1808), so the Part 65 crossing argument applies without changing reconstruction. □ □

211 Updated proof ledger after direct raw spectral rederivation

pointwise equality of the two remainders:	false in general; mismatch remains (1829),
need for such equality:	removed by direct raw rederivation,
raw principal fractional symbol:	closed, $\|\xi\|^{2\alpha}$ by (1851),
raw $\alpha = 1$ frozen shape sign:	closed, $\Re \mathfrak{t}^{\mathrm{raw}}_{N,1} = \xi^2$ by (1857),
raw high-frequency sector:	elliptic under the stated finite bridge jet bounds, (1866),
raw compact-frequency sector:	reduced to finite constants (1863)–(1865),
raw replacement of (1811):	equation (1868),
raw replacement of (1799):	equation (1872),
raw replacement of (1807):	equation (1873),
architectural mixing obstruction:	resolved,
finite raw coefficient/decay certification:	still required along the bridge,
pressure compatibility:	still requires (1840),
first crossing:	follows from Part 65 once (1868), (1872), (1873), (1840) are certified.

(1875)

The next calculation is therefore narrower than the obstruction at the end of Part 66. There is no longer a need to identify the raw and square-root velocity reconstructions. One must now evaluate the finite raw continuity response Θ_N, the corresponding compact-frequency constants in (1863)–(1864), and the raw residual decay in (1872)–(1873) for the prescribed Q_N, S_N, D_N trajectory. That is the remaining finite certification needed before the first crossing can be declared unconditional.

Part 68. Finite Certification Audit of the Raw Continuity Response, Compact Spectrum, Residual Decay, and Pressure

212 The raw continuity equation determines the response only after a normalization

The direct raw spectral rederivation reduced the remaining bridge calculation to the continuity response Θ_N. We now return to the actual two-correction identities rather than treating Θ_N as an abstract bounded operator. Recall

$$\boxed{A_N = V_N - D_N(V_N), \qquad C_N = V_N + D_N(V_N), \qquad V_N = Q_N + S_N,} \tag{1876}$$

and

$$\boxed{\partial_x\left(\frac{b_N}{A_N}\right) + A_{N,y} + C_{N,z} = 0.} \tag{1877}$$

Equivalently, the raw source writes

$$\boxed{S_{N,y} + S_{N,z} + D_{N,z} - D_{N,y} = (C_2 + C_3)Q_N' - \partial_x\left(\frac{b_N}{A_N}\right).} \tag{1878}$$

This identity is exact, but it is a first-order equation in the transverse variables. Consequently it does not, by itself, select a unique S_N. A normalization, boundary condition, or zero-mode prescription is required before the map $b_N \mapsto S_N$ and hence the response Θ_N is a well-defined operator.

To make this precise, set

$$d_N(V) := D_N(V), \qquad d_{V,N} := \partial_V D_N(V_N), \tag{1879}$$

and perturb $b_N \mapsto b_N(1 + \varepsilon Z)$ and $S_N \mapsto S_N + \varepsilon\Sigma_N[Z] + O(\varepsilon^2)$. Since Q_N is frozen in the shape variation,

$$\delta V_N = \Sigma_N[Z], \qquad \delta A_N = (1 - d_{V,N})\Sigma_N[Z], \qquad \delta C_N = (1 + d_{V,N})\Sigma_N[Z]. \tag{1880}$$

Linearizing (1877) gives the exact response equation

$$\boxed{\mathcal{B}_N\Sigma_N[Z] = -\partial_x\left(\frac{b_N}{A_N}Z\right),} \tag{1881}$$

where

$$\boxed{\begin{aligned}\mathcal{B}_N\Sigma := -\partial_x\left(\frac{b_N}{A_N^2}(1-d_{V,N})\Sigma\right) + \partial_y((1-d_{V,N})\Sigma) \\ + \partial_z((1+d_{V,N})\Sigma)\,.\end{aligned}} \tag{1882}$$

Thus the actual logarithmic continuity response appearing in Part 67 is

$$\boxed{\Theta_N Z := \frac{\delta A_N}{A_N} = \frac{1-d_{V,N}}{A_N}\,\mathcal{B}_N^{-1}\left[-\partial_x\left(\frac{b_N}{A_N}Z\right)\right],} \tag{1883}$$

provided $\mathcal{B}_N^{-1}$ is defined on the chosen normalized function space. Equation (1883) is the requested explicit reduction of Θ_N to the raw reconstruction.

213 The derivative of the algebraic correction is explicit

For the raw algebraic relation

$$D_N(V) = \frac{-(2V+R_N) + \sqrt{R_N^2 + 8R_N V}}{2}, \tag{1884}$$

one obtains

$$\boxed{d_{V,N}(V) = -1 + \frac{2R_N}{\sqrt{R_N^2 + 8R_N V}}.} \tag{1885}$$

Hence

$$\boxed{1-d_{V,N} = 2 - \frac{2R_N}{\sqrt{R_N^2+8R_N V}}, \qquad 1+d_{V,N} = \frac{2R_N}{\sqrt{R_N^2+8R_N V}}.} \tag{1886}$$

On every finite bridge on which

$$0 < r_- \le R_N \le r_+, \qquad 0 < v_- \le V_N \le v_+, \tag{1887}$$

these coefficients and all finite derivatives required in the M-jet calculation are bounded. In particular, the algebraic part of the response creates no singular coefficient on such a bridge. The nontrivial question is the invertibility and norm of $\mathcal{B}_N^{-1}$ after the missing normalization is fixed.

Let $\mathfrak{X}_{N,M}$ denote the normalized bridge space. If

$$\boxed{\|\mathcal{B}_N^{-1}F\|_{H^{M+1}} \le K_{N,M}^{\mathcal{B}}\|F\|_{H^M},} \tag{1888}$$

then the tame product estimate applied to (1883) gives

$$\boxed{\|\Theta_N Z\|_{H^M} \le K_{N,M}^{\Theta}\|Z\|_{H^M}, \qquad K_{N,M}^{\Theta} \le C_{N,M}K_{N,M}^{\mathcal{B}}.} \tag{1889}$$

This proves the order-zero bound assumed in Part 67 once (1888) is established for the actual normalization. It also identifies exactly which constant must be computed.

214 Compact-frequency constants are now explicit functions of one inverse norm

The continuity contribution in (1859) is composed of the response (1883), its commutator with L_α, and finite multiplication operators. Therefore, on $|\xi| \le \Xi$,

$$\boxed{K^{\mathrm{cont}}_{N,M} \le C^{\mathrm{cont}}_{N,M,\Xi}\big(1 + K^{\mathcal{B}}_{N,M}\big).} \tag{1890}$$

The algebraic and transport constants contain only the already displayed finite jets of Q_N, V_N, D_N, b_N:

$$\boxed{K^{\mathrm{alg}}_{N,M} + K^{\mathrm{tr}}_{N,M} \le \mathcal{P}_{N,M}\left(\|Q_N\|_{C^{M+2}}, \|V_N\|_{C^{M+2}}, \|b_N\|_{C^{M+2}}, r_-^{-1}, v_-^{-1}\right),} \tag{1891}$$

where $\mathcal{P}_{N,M}$ is a finite polynomial with positive coefficients determined by the raw formulas. Consequently the compact-frequency constants in (1863)–(1864) obey

$$\boxed{\varepsilon^{\mathrm{raw}}_{N,\Xi} \le \mathfrak{C}^{(0)}_{N,M,\Xi}\big(1 + K^{\mathcal{B}}_{N,M}\big),} \tag{1892}$$

and

$$\boxed{L^{Y,\mathrm{raw}}_{N,\Xi} + L^{\alpha,\mathrm{raw}}_{N,\Xi} \le \mathfrak{C}^{(1)}_{N,M,\Xi}\big(1 + K^{\mathcal{B}}_{N,M}\big).} \tag{1893}$$

Thus (1868) becomes the finite, directly checkable inequality

$$\boxed{a_N \frac{1 - e^{-R}}{R} > \mathfrak{C}^{(0)}_{N,M,\Xi}(1 + K^{\mathcal{B}}_{N,M}) + R\,\mathfrak{C}^{(1)}_{N,M,\Xi}(1 + K^{\mathcal{B}}_{N,M}) + (\alpha - 1)\mathfrak{C}^{(1)}_{N,M,\Xi}(1 + K^{\mathcal{B}}_{N,M}).} \tag{1894}$$

No unknown square-root coefficient remains in this criterion.

215 What can and cannot be concluded for the raw decay inequalities

The raw reconstruction error is

$$\boxed{\mathcal{E}_N = b_N \frac{C_{N,t} + C_{N,z}}{C_N} + b_N A_N C_{N,x} + \frac{b_N^2}{A_N} C_{N,y} + 2C_N \nabla b_N \cdot \nabla C_N.} \tag{1895}$$

Substituting $A_N = V_N - D_N(V_N)$ and $C_N = V_N + D_N(V_N)$ makes every coefficient in (1895) an explicit smooth function of the finite jets of Q_N, S_N, b_N on a nondegenerate bridge. Applying E^j, $0 \le j \le M$, gives

$$\boxed{\sum_{j=0}^{M} |E^j \mathcal{E}_N| \le \mathcal{P}^{\mathrm{rec}}_{N,M}\left(\mathbf{J}_{M+2} Q_N, \mathbf{J}_{M+2} S_N, \mathbf{J}_{M+2} b_N, r_-^{-1}, v_-^{-1}\right).} \tag{1896}$$

This closes the *finite-jet reduction*, but not the exponential decay claim. To obtain (1872)–(1873) before local conormal entry one still needs a dynamical estimate of the form

$$\boxed{\mathbf{J}_{M+2}(Q_N, S_N, b_N)(\tau) - \mathbf{J}_{M+2}(Q_N, S_N, b_N)_{\mathrm{pr}}(\tau) = O_N(e^{-\eta_N \tau})} \tag{1897}$$

for a principal bridge trajectory. Mere smoothness and boundedness on a finite bridge do not imply (1897). Therefore it would be circular to infer (1872)–(1873) from the local conormal estimates that are only available after entry.

There is, however, a useful exact implication. If the normalized inverse estimate (1888) holds and the raw coefficient jets satisfy

$$\|E^j(Q_N - Q_N^{\mathrm{pr}})\| + \|E^j(S_N - S_N^{\mathrm{pr}})\| + \|E^j(b_N - b_N^{\mathrm{pr}})\| \le C_{N,M} e^{-\eta_N \tau}, \quad 0 \le j \le M+2, \tag{1898}$$

then (1883), (1895), and tame composition estimates yield

$$\boxed{\|\mathbf{R}_{N,M}^{\mathrm{raw}}(\tau)\| \le C_{N,M}^{\mathrm{raw}} e^{-\eta_N^{\mathrm{raw}} \tau}, \qquad \eta_N^{\mathrm{raw}} > 0.} \tag{1899}$$

Thus the decay problem has been reduced to the raw bridge jet estimate (1898), rather than an unspecified fractional remainder.

216 Pressure compatibility at the initial section

The pressure requirement (1840) can be formulated without prescribing pressure by hand. For a smooth periodic divergence-free initial velocity $u_N(0)$ and smooth periodic forcing $f_N(0)$, define

$$\boxed{\nabla P_N(0) = \mathbb{Q}[f_N(0) - u_N(0) \cdot \nabla u_N(0) - \nu(-\Delta)^\alpha u_N(0)]\,,} \tag{1900}$$

with the sign adjusted to the manuscript's convention for moving the pressure term between the two sides of the momentum equation. Since $(-\Delta)^\alpha$ commutes with divergence and $\nabla \cdot u_N(0) = 0$, its contribution has zero gradient projection. Equivalently, after fixing zero mean,

$$\boxed{\Delta P_N(0) = \nabla \cdot f_N(0) - \nabla \cdot \big(u_N(0) \cdot \nabla u_N(0)\big).} \tag{1901}$$

The right-hand side has zero spatial mean on $\mathbb{T}^3$, so (1901) has a unique zero-mean periodic solution. Hence pressure *existence* at a smooth periodic initial section is not an additional obstruction once the complete divergence-free raw velocity and forcing have been fixed.

What remains distinct is momentum compatibility: defining P_N by (1900) removes the gradient part, but the solenoidal projection must still equal the prescribed time derivative. The exact check is

$$\boxed{\partial_t u_N(0) = \mathbb{P}[f_N(0) - u_N(0) \cdot \nabla u_N(0) - \nu(-\Delta)^\alpha u_N(0)]\,.} \tag{1902}$$

Thus (1840) separates into an automatically solvable periodic pressure equation and a nontrivial solenoidal evolution compatibility condition (1902).

217 Finite certification theorem and the remaining normalization datum

Theorem 63 (Raw finite-certification reduction)**.** *Fix $N < \infty$ and a nondegenerate raw two-correction bridge. Choose a normalization for the first-order continuity equation (1878) such that $\mathcal{B}_N$ in (1882) is invertible and satisfies (1888). Then:*

1. *the actual continuity response is given by (1883) and is order zero by (1889);*
2. *the compact-frequency constants are bounded explicitly by (1892)–(1893), so the spectral crossing condition is the finite inequality (1894);*
3. *if the raw bridge jets satisfy (1898), then the complete differentiated residual satisfies the exponential estimate (1899), which supplies (1872)–(1873);*
4. *for smooth periodic divergence-free raw initial data, the zero-mean pressure is uniquely determined by (1901); the remaining momentum compatibility is (1902).*

The theorem is deliberately a reduction rather than an unconditional crossing theorem. The raw manuscript identity (1878) does not specify, in the displayed reconstruction block, the normalization needed to select a unique inverse $\mathcal{B}_N^{-1}$. Consequently a numerical value for $K^{\mathcal{B}}_{N,M}$, and hence for the constants in (1892)–(1894), cannot be inferred uniquely from the currently displayed raw formulas. This is a data-selection issue, not a failure of the principal fractional sign.

218 Updated proof ledger after the finite certification audit

raw continuity response Θ_N:	exact implicit formula (1883),
algebraic derivative $D_{N,V}$:	explicit, (1885)–(1886),
order-zero response bound:	follows from the normalized inverse estimate (1888),
compact-frequency constants:	reduced to $K^{\mathcal{B}}_{N,M}$ and finite raw jets, (1892)–(1893),
raw spectral crossing inequality:	explicit finite criterion (1894),
raw residual finite-jet reduction:	closed by (1895)–(1896),
pre-entry exponential residual decay:	equivalent to the still-needed raw bridge jet decay (1898),
pressure existence on $\mathbb{T}^3$:	closed for smooth periodic divergence-free data by (1900)–(1901),
solenoidal momentum compatibility:	still requires the finite check (1902),
continuity normalization / zero mode:	not specified by (1878) alone; needed to define $\mathcal{B}_N^{-1}$,
unconditional first crossing:	not yet established.

(1903)

The remaining task is therefore sharper than at the end of Part 67. One must specify and audit the actual normalization used to solve the raw continuity equation for S_N, prove the corresponding inverse estimate (1888), and establish the pre-entry raw jet decay (1898). Once these are available, (1894) and (1899) are numerical/finite-N checks and the first-crossing machinery can be invoked without changing reconstruction.

Part 69. Continuity Normalization, Transport Invertibility, and the Pre-Entry Jet-Decay Audit

219 The normalization actually supported by the periodic construction

Part 68 isolated two remaining statements, (1888) and (1898). The first point can now be sharpened by returning to the periodic reconstruction itself. The continuity equation is

$$\mathcal{B}_N \Sigma = F, \qquad F = -\partial_x \left(\frac{b_N}{A_N} Z \right), \tag{1904}$$

and hence

$$\int_{\mathbb{T}^3} F\,dx\,dy\,dz = 0. \tag{1905}$$

Thus the forcing lies automatically in the zero-mean subspace. The same manuscript architecture later inverts a periodic derivative only after fixing a normalization of the kernel variable; the canonical torus completion of the raw continuity prescription is therefore

$$\boxed{\int_{\mathbb{T}^3} \Sigma_N[Z]\,dx\,dy\,dz = 0.} \tag{1906}$$

This is the natural zero-mode normalization. It removes the additive constant whenever constants span the relevant kernel. It must, however, be distinguished from a proof that $\mathcal{B}_N$ is invertible on the entire zero-mean space.

Write

$$p_N := \frac{b_N}{A_N^2}(1 - d_{V,N}), \qquad q_N := 1 - d_{V,N}, \qquad r_N := 1 + d_{V,N}. \tag{1907}$$

Then (1882) is exactly

$$\boxed{\mathcal{B}_N\Sigma = -\partial_x(p_N\Sigma) + \partial_y(q_N\Sigma) + \partial_z(r_N\Sigma).} \tag{1908}$$

Equivalently,

$$\mathcal{B}_N\Sigma = X_N\Sigma + c_N\Sigma, \qquad X_N = -p_N\partial_x + q_N\partial_y + r_N\partial_z, \tag{1909}$$

with

$$c_N = -p_{N,x} + q_{N,y} + r_{N,z}. \tag{1910}$$

Therefore the response problem is a first-order transport equation on the torus, not an elliptic equation.

220 Why the isotropic inverse estimate (1888) is not a consequence of zero mean

The estimate written in (1888),

$$\|\mathcal{B}_N^{-1}F\|_{H^{M+1}} \le K_{N,M}^{\mathcal{B}}\|F\|_{H^M},$$

asserts a full isotropic gain of one derivative. A first-order transport operator does not have such an estimate merely because its zero mode has been removed.

Indeed freeze the coefficients at a bridge point and write

$$\mathcal{B}_N^{\mathrm{fr}} = v_N \cdot \nabla + c_N^{\mathrm{fr}}, \qquad v_N = (-p_N, q_N, r_N). \tag{1911}$$

For the Fourier mode $e^{ik\cdot x}$,

$$\widehat{\mathcal{B}_N^{\mathrm{fr}}}(k) = i\,v_N \cdot k + c_N^{\mathrm{fr}}. \tag{1912}$$

If $c_N^{\mathrm{fr}} = 0$, the normalized inverse multiplier is

$$\frac{1}{i\, v_N \cdot k}, \qquad k \in \mathbb{Z}^3 \setminus \{0\}. \tag{1913}$$

Consequently a bound of the form (1888) would require

$$\boxed{\inf_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|v_N \cdot k|}{\langle k \rangle} > 0.} \tag{1914}$$

In dimension three this is far stronger than nonvanishing of v_N. For a generic irrational direction there are arbitrarily good integer near-resonances, while for a rational direction there are nonzero integer vectors orthogonal to v_N. Hence zero mean alone does not imply (1914), and therefore does not prove (1888).

If the zeroth-order term in (1912) has a uniform real part,

$$|c_N^{\mathrm{fr}}| \geq c_{0,N} > 0, \tag{1915}$$

then it removes the exact kernel but still yields only an order-zero bound at frequencies nearly perpendicular to v_N; it does not supply the factor $\langle k \rangle^{-1}$ required for a full $H^M \to H^{M+1}$ gain. Thus the logical status is

$$\boxed{\text{the zero-mode normalization (1906) is natural, but (1888) is too strong without additional directional structure.}} \tag{1916}$$

221 The correct inverse estimate for the raw continuity response

The continuity response in (1883) contains one derivative on the right-hand side and one transport inverse. What is actually needed for the order-zero map $Z \mapsto \Theta_N Z$ is not an isotropic derivative gain for arbitrary forcing, but a bounded inverse on the specific range

$$\mathfrak{R}_N := \left\{ -\partial_x \left(\frac{b_N}{A_N} Z \right) : Z \in H^M \right\}. \tag{1917}$$

Accordingly replace (1888) by the range-adapted estimate

$$\boxed{\left\| \mathcal{B}_N^{-1} \partial_x \left(\frac{b_N}{A_N} Z \right) \right\|_{H^M} \leq K_{N,M}^{\mathrm{ran}} \|Z\|_{H^M}.} \tag{1918}$$

Then (1883) gives directly

$$\boxed{\|\Theta_N Z\|_{H^M} \leq \left\| \frac{1 - d_{V,N}}{A_N} \right\|_{C^M} K_{N,M}^{\mathrm{ran}} \|Z\|_{H^M}.} \tag{1919}$$

This is exactly the order-zero statement used in the compact-frequency spectral argument. It avoids claiming an elliptic gain that the raw continuity equation does not possess.

For frozen coefficients, (1918) reduces to the finite multiplier condition

$$\boxed{\sup_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|k_x|}{|i\,v_N\cdot k + c_N^{\mathrm{fr}}|} < \infty.} \tag{1920}$$

Thus the true arithmetic/geometric issue is explicit: the numerator is only k_x, not $|k|$. In particular, a sufficient frozen-coefficient condition is the transversality estimate

$$\boxed{|v_N\cdot k| + |c_N^{\mathrm{fr}}| \geq \kappa_N |k_x| \quad \text{for every } k\in\mathbb{Z}^3,} \tag{1921}$$

for some $\kappa_N > 0$. Variable coefficients require the corresponding transport estimate with commutator control. The manuscript has not yet supplied (1921) for the actual raw bridge, so $K_{N,M}^{\mathrm{ran}}$ remains a finite certification constant rather than a proved number.

The compact-frequency estimates (1892)–(1894) remain valid after the systematic replacement

$$\boxed{K_{N,M}^{\mathcal{B}} \quad \rightsquigarrow \quad K_{N,M}^{\mathrm{ran}},} \tag{1922}$$

with the coefficient factor in (1919) absorbed into the finite constants.

222 Audit of the proposed pre-entry raw-jet decay

We next examine (1898). Let

$$U_N := (Q_N, S_N, b_N), \qquad U_N^{\mathrm{pr}} := (Q_N, S_N, b_N)_{\mathrm{pr}}, \tag{1923}$$

and set

$$W_N := U_N - U_N^{\mathrm{pr}}. \tag{1924}$$

The desired estimate is

$$\|W_N(\tau)\|_{\mathcal{J}_{M+2}} \leq C_N e^{-\eta_N \tau}. \tag{1925}$$

Smoothness of the finite-N reconstructed solution and boundedness of its bridge jets do not imply (1925). Exponential decay is a dynamical stability statement. Subtracting the principal bridge equations from the complete raw equations has the schematic exact form

$$\partial_\tau W_N = \mathbb{L}_N(\tau) W_N + \mathbb{N}_N(\tau, W_N) + \mathbb{F}_N^{\mathrm{pre}}(\tau), \tag{1926}$$

where $\mathbb{L}_N$ is the full pre-entry linearization, $\mathbb{N}_N = O(|W_N|^2)$, and $\mathbb{F}_N^{\mathrm{pre}}$ is the defect of the chosen principal bridge trajectory. A proof of (1925) therefore requires a genuine pre-entry stability estimate such as

$$\boxed{\|\mathcal{U}_N(\tau,r)\|_{\mathcal{J}_{M+2}\to\mathcal{J}_{M+2}} \leq M_N e^{-\lambda_N(\tau-r)}, \qquad \tau\geq r,} \tag{1927}$$

for the evolution family generated by $\mathbb{L}_N$, together with

$$\|\mathbb{F}_N^{\mathrm{pre}}(\tau)\|_{\mathcal{J}_{M+2}} \leq C_N^F e^{-\eta_N^F \tau}. \tag{1928}$$

Duhamel then gives

$$\begin{aligned}\|W_N(\tau)\|_{\mathcal{J}_{M+2}} &\le M_N e^{-\lambda_N(\tau-\tau_0)}\|W_N(\tau_0)\|_{\mathcal{J}_{M+2}} \\ &\quad + M_N\int_{\tau_0}^{\tau} e^{-\lambda_N(\tau-r)}\left(C_N^{\mathrm{nl}}\|W_N(r)\|_{\mathcal{J}_{M+2}}^2 + C_N^F e^{-\eta_N^F r}\right)dr.\end{aligned} \tag{1929}$$

For a sufficiently small bootstrap radius this yields

$$\boxed{\|W_N(\tau)\|_{\mathcal{J}_{M+2}} \le \widetilde{C}_N e^{-\eta_N\tau}, \qquad 0<\eta_N<\min\{\lambda_N,\eta_N^F\},} \tag{1930}$$

which is precisely (1898). But (1927) is not contained in the raw reconstruction identities themselves. It is the missing pre-entry stability theorem.

223 No circular use of the local conormal spectral gap

The spectral gap already proved after local entry cannot be substituted for (1927): doing so would assume the trajectory has reached the small shape tube in order to prove that it reaches that tube. The correct division is

$$\boxed{\begin{array}{c}\text{outer bridge: prove (1918)/(1921) and the pre-entry stability estimate (1927),}\\ \Downarrow\\ \text{obtain (1930), hence the residual estimate (1899),}\\ \Downarrow\\ \text{verify the finite crossing inequality (1894),}\\ \Downarrow\\ \text{enter the local spectral/conormal tube and invoke Parts 60–65.}\end{array}} \tag{1931}$$

This keeps the entry proof logically independent of the post-entry contraction theorem.

224 Result of the normalization and jet-decay audit

The requested two-item certification therefore has a definite outcome. The periodic continuity equation canonically suggests the zero-mean normalization (1906), consistent with the manuscript's later treatment of periodic derivative kernels. However, that normalization does not prove the isotropic inverse estimate (1888). The correct estimate needed by the spectral argument is the range-adapted order-zero bound (1918), whose frozen form is the explicit transversality/small-divisor condition (1920)–(1921).

Likewise, (1898) is not a consequence of finite-N smoothness or coefficient boundedness. It follows from the Duhamel estimate (1929) once the genuinely pre-entry evolution estimate (1927) and the principal-defect decay (1928) are proved. Therefore the two remaining

certifications are now

(I) prove the raw range-adapted continuity estimate (1918), equivalently a variable-coefficient version of the finite transversality condition (1921);
(II) prove the outer-bridge evolution-family decay (1927) and defect bound (1928).

(1932)

Only after (I)–(II) are certified can (1894) and (1899) be declared verified for the prescribed complete reconstructed trajectory.

225 Updated proof ledger

periodic forcing zero mode:	closed automatically by (1905),
canonical continuity normalization:	zero mean, (1906),
full isotropic inverse estimate (1888):	not implied by zero mean; generally too strong for a transport operator,
correct response estimate:	range-adapted order-zero bound (1918),
frozen finite certification:	transversality/small-divisor condition (1920)–(1921),
compact-frequency constants:	use $K^{\mathrm{ran}}_{N,M}$ via (1922),
pre-entry raw-jet decay (1898):	follows from (1927)–(1929), not from smoothness alone,
principal-defect decay:	still requires the raw estimate (1928),
post-entry conormal contraction:	remains available and must not be used circularly before crossing,
pressure existence:	remains closed as in Part 68,
solenoidal momentum compatibility:	still requires (1902),
unconditional first crossing:	not yet established.

(1933)

The next calculation is therefore more precise than the pair (1888), (1898): compute the actual transport direction $v_N = (-p_N, q_N, r_N)$ on the prescribed finite bridge and test the range transversality (1921); in parallel, linearize the complete raw bridge dynamics about U^{pr}_N and establish the pre-entry evolution estimate (1927). These are the two noncircular estimates that would turn the finite certification into a first-crossing proof.

Part 70. Actual Transport Direction and Complete Outer-Bridge Linearization Audit

226 The actual raw transport direction is explicit

Part 69 reduced the continuity-response question to the range estimate (1918) and proposed the frozen transversality test (1921). We now substitute the actual two-correction coefficients rather than leaving the vector v_N abstract. Put

$$\mathscr{D}_N := \sqrt{R_N^2 + 8R_N V_N}, \qquad \vartheta_N := \frac{R_N}{\mathscr{D}_N}. \tag{1934}$$

By (1885)–(1886),

$$\boxed{q_N = 1 - d_{V,N} = 2(1-\vartheta_N), \qquad r_N = 1 + d_{V,N} = 2\vartheta_N.} \tag{1935}$$

In particular

$$\boxed{q_N + r_N = 2.} \tag{1936}$$

The coefficient multiplying the x derivative is

$$\boxed{p_N = \frac{b_N}{A_N^2} q_N = 2\frac{b_N}{A_N^2}(1-\vartheta_N).} \tag{1937}$$

Using $A_N = C_N^2/R_N$ from (1830), this may also be written

$$\boxed{p_N = 2\frac{b_N R_N^2}{C_N^4}(1-\vartheta_N).} \tag{1938}$$

Therefore the actual first-order transport direction in the linearized raw continuity operator is

$$\boxed{v_N = (-p_N, q_N, r_N) = 2\left(-\frac{b_N R_N^2}{C_N^4}(1-\vartheta_N), 1-\vartheta_N, \vartheta_N\right).} \tag{1939}$$

This is the requested substitution. On a nondegenerate finite bridge all three coefficients are finite, and $q_N, r_N > 0$ whenever $R_N, V_N > 0$ and $\mathscr{D}_N > R_N$.

227 The unrestricted Fourier transversality test (1921) cannot hold

The substitution (1939) reveals a stronger fact. The sufficient condition (1921), if imposed on the full periodic Fourier lattice, is not merely unverified: it is impossible for any fixed finite nonzero vector v_N and bounded frozen zeroth-order coefficient.

Indeed fix one bridge point and freeze $p_N, q_N, r_N, c_N^{\mathrm{fr}}$. Since $q_N > 0$, for every positive integer n choose

$$m_n := \operatorname{nint}\left(\frac{p_N}{q_N} n\right), \qquad k_n := (n, m_n, 0) \in \mathbb{Z}^3, \tag{1940}$$

where nint denotes a nearest integer. Then

$$|-p_N n + q_N m_n| \le \frac{q_N}{2}. \tag{1941}$$

Consequently

$$\frac{|v_N \cdot k_n| + |c_N^{\mathrm{fr}}|}{|(k_n)_x|} \le \frac{q_N/2 + |c_N^{\mathrm{fr}}|}{n} \longrightarrow 0. \tag{1942}$$

Thus no $\kappa_N > 0$ can satisfy (1921) for all $k \in \mathbb{Z}^3$:

$$\boxed{\inf_{k_x \neq 0} \frac{|v_N \cdot k| + |c_N^{\mathrm{fr}}|}{|k_x|} = 0.} \tag{1943}$$

The same conclusion holds if $q_N = 0$ but $r_N \neq 0$, by using $k_n = (n, 0, \operatorname{nint}(p_N n / r_N))$. Hence the only way an estimate of the form (1921) can survive is after a genuine restriction of the admissible Fourier support, or after replacing the transport inverse by a different anisotropic/characteristic estimate. The full-torus multiplier proof of (1918) is therefore ruled out.

Notice that this does not prove that the actual response Θ_N is unbounded. It proves only that the proposed uniform full-lattice transversality mechanism is unavailable. The range in (1917) has additional structure, and the variable-coefficient equation may be solved along its characteristics with a normalization adapted to that flow. Such a proof must use that structure explicitly.

228 Characteristic form of the actual continuity inverse

Expanding (1882) gives

$$\mathcal{B}_N \Sigma = X_N \Sigma + c_N \Sigma, \qquad X_N = -p_N \partial_x + q_N \partial_y + r_N \partial_z, \tag{1944}$$

with

$$\boxed{c_N = -\partial_x p_N + \partial_y q_N + \partial_z r_N.} \tag{1945}$$

Let $\Phi_N(\ell; X_0)$ be the characteristic flow of X_N,

$$\frac{d}{d\ell} \Phi_N(\ell; X_0) = v_N(\Phi_N(\ell; X_0)), \qquad \Phi_N(0; X_0) = X_0. \tag{1946}$$

Along this flow the response equation (1881) becomes the scalar ODE

$$\frac{d}{d\ell} \Sigma_N(\Phi_N(\ell; X_0)) + c_N(\Phi_N(\ell; X_0)) \Sigma_N(\Phi_N(\ell; X_0)) = F_N(\Phi_N(\ell; X_0)), \tag{1947}$$

where

$$F_N = -\partial_x\left(\frac{b_N}{A_N}Z\right). \tag{1948}$$

Thus, for any section on which one value of Σ_N is prescribed, the exact solution is

$$\boxed{\begin{aligned}\Sigma_N(\Phi_N(\ell;X_0)) &= e^{-C_N(\ell;X_0)}\Sigma_N(X_0)\\ &\quad + \int_0^\ell e^{-[C_N(\ell;X_0)-C_N(r;X_0)]}F_N(\Phi_N(r;X_0))\,dr,\end{aligned}} \tag{1949}$$

where

$$C_N(\ell;X_0) := \int_0^\ell c_N(\Phi_N(r;X_0))\,dr. \tag{1950}$$

Equation (1949) identifies the correct finite-N inverse problem. A bounded order-zero response follows if one proves, on the actual bridge, a global flow-box/return-map estimate controlling the integral in (1949), the chosen normalization, and the differentiated characteristic flow through order M. The zero-mean condition alone does not supply those estimates.

Accordingly the corrected replacement for the failed full-lattice test is the finite characteristic certification

$$\boxed{\sup_{0\le j\le M}\|D^j\mathcal{T}_N\|_{L^\infty(\mathfrak{B}_N)}<\infty, \qquad \sup_{X_0,\ell}\int_0^{|\ell|} e^{-\int_r^{|\ell|}c_N(\Phi_N(a;X_0))\,da}\,dr<\infty,} \tag{1951}$$

where $\mathcal{T}_N$ denotes the relevant bridge return/section map. With a normalization eliminating the homogeneous return mode, (1951) yields a finite constant $K^{\mathrm{char}}_{N,M}$ in the anisotropic estimate

$$\boxed{\|\Theta_N Z\|_{H^M} \le K^{\mathrm{char}}_{N,M}\|Z\|_{H^M}.} \tag{1952}$$

The raw formulas determine every coefficient entering (1951), but the present source does not provide the return-map bounds themselves. Therefore (1952) is now a precise finite characteristic check, not a proved global estimate.

229 Derivation of the complete raw outer-bridge linearization

We next make (1926) explicit at the level supported by the raw equations. The raw scalar equation is (1824),

$$\mathcal{D}_N b_N = \frac{Q_N'}{Q_N}b_N^2 + 2C_N G - \mathcal{E}_N, \tag{1953}$$

with $\mathcal{E}_N$ given by (1825). The reconstruction is constrained by

$$A_N = A(V_N,R_N), \qquad C_N = C(V_N,R_N), \qquad V_N = Q_N + S_N, \tag{1954}$$

and the continuity equation (1878). Let

$$U_N = (Q_N,S_N,b_N), \qquad U_N^{\mathrm{pr}} = (Q_N^{\mathrm{pr}},S_N^{\mathrm{pr}},b_N^{\mathrm{pr}}), \qquad W_N = U_N - U_N^{\mathrm{pr}}. \tag{1955}$$

The algebraic differential is

$$\delta A_N = (1-d_{V,N})(\delta Q_N + \delta S_N) + A_{R,N}\delta R_N, \qquad \delta C_N = (1+d_{V,N})(\delta Q_N + \delta S_N) + C_{R,N}\delta R_N, \tag{1956}$$

where $A_{R,N}, C_{R,N}$ are obtained by differentiating (1820) with respect to R_N. For the shape variation used in Parts 67–69, Q_N and hence R_N are frozen, and (1956) reduces exactly to (1880).

Linearizing continuity gives

$$\boxed{\mathcal{B}_N\,\delta S_N = -\partial_x\left(\frac{\delta b_N}{A_N}\right) + \partial_x\left(\frac{b_N}{A_N^2}\delta A_N\right) + \mathcal{F}_N^{Q,R}[\delta Q_N, \delta R_N],} \tag{1957}$$

where $\mathcal{F}_N^{Q,R}$ is the explicit linear forcing obtained by varying the Q'_N and $D_N(V_N, R_N)$ terms in (1878). Once the characteristic inverse in (1949) is certified, (1957) expresses δS_N as an order-zero nonlocal response to $(\delta Q_N, \delta b_N)$ on the bridge.

For the scalar component, write the complete raw right-hand side of (1953) as

$$\mathscr{H}_N(Q,S,b) := \frac{Q'}{Q}b^2 + 2C(Q,S)G - \mathcal{E}_N(Q,S,b). \tag{1958}$$

Subtracting the principal bridge equation and linearizing gives

$$\boxed{\partial_\tau \delta b_N = L_{bb,N}\delta b_N + L_{bQ,N}\delta Q_N + L_{bS,N}\delta S_N + F_{b,N}^{\mathrm{pr}} + \mathcal{N}_{b,N}(W_N),} \tag{1959}$$

where, before the change from physical time to τ is absorbed into the coefficients,

$$\boxed{D_b\mathscr{H}_N[\delta b] = 2\frac{Q'_N}{Q_N}b_N\delta b - D_b\mathcal{E}_N[\delta b],} \tag{1960}$$

$$\boxed{D_S\mathscr{H}_N[\delta S] = 2G(1+d_{V,N})\delta S - D_S\mathcal{E}_N[\delta S],} \tag{1961}$$

and

$$\boxed{D_Q\mathscr{H}_N[\delta Q] = b_N^2\,D_Q\left(\frac{Q'_N}{Q_N}\right)[\delta Q] + 2G(1+d_{V,N})\delta Q - D_Q\mathcal{E}_N[\delta Q].} \tag{1962}$$

The derivatives of $\mathcal{E}_N$ are finite sums of the raw coefficient jets already displayed in (1825) and (1835). Thus no unspecified reconstruction architecture enters (1959)–(1962).

Eliminating δS_N through (1957) yields the reduced outer-bridge linearized system

$$\boxed{\partial_\tau\begin{pmatrix}\delta Q_N\\ \delta b_N\end{pmatrix} = \mathbb{L}_N^{\mathrm{red}}(\tau)\begin{pmatrix}\delta Q_N\\ \delta b_N\end{pmatrix} + \mathbb{F}_N^{\mathrm{pr}}(\tau) + \mathbb{N}_N^{\mathrm{red}}(W_N),} \tag{1963}$$

where every occurrence of the continuity response is through the actual characteristic inverse of $\mathcal{B}_N$. Equation (1963) is the complete raw outer-bridge linearization available from the displayed reconstruction identities.

230 Why (1927)–(1928) still do not follow from the displayed raw identities

To prove (1927) one needs a negative growth bound for the evolution family of $\mathbb{L}_N^{\mathrm{red}}$. A sufficient energy condition is

$$\boxed{\Re\langle W, \mathbb{L}_N^{\mathrm{red}}(\tau)W\rangle_{\mathcal{J}_{M+2}} \le -\lambda_N \|W\|^2_{\mathcal{J}_{M+2}}, \qquad \lambda_N > 0,} \tag{1964}$$

throughout the outer bridge. The principal fractional block supplies the positive elliptic shape term identified in Part 67 after the sign convention of the evolution equation is taken into account, but the complete matrix also contains the continuity response, transport, algebraic correction, and Q_N-coupling terms in (1957)–(1962). The raw formulas show these terms are finite; they do not give the sign inequality (1964).

There is a second, more basic data issue. In the displayed raw reconstruction, Q_N is prescribed by the Lambert cascade and S_N is fixed at each time by the continuity constraint. No independent evolution equation for S_N is present, and the principal bridge trajectory U_N^{pr} is not specified with enough data to evaluate its complete defect. Consequently

$$\boxed{\mathbb{F}_N^{\mathrm{pr}} = \partial_\tau U_N^{\mathrm{pr}} - \mathscr{G}_N(U_N^{\mathrm{pr}})} \tag{1965}$$

cannot yet be reduced to a numerical function of τ from the displayed formulas alone. Therefore the exponential defect estimate (1928) is not yet proved.

The exact finite certifications needed are now

$$\boxed{\begin{array}{l}\text{(a) certify the characteristic continuity inverse (1951)–(1952);}\\ \text{(b) evaluate the symmetric part of } \mathbb{L}_N^{\mathrm{red}} \text{ and prove (1964);}\\ \text{(c) specify } U_N^{\mathrm{pr}} \text{ completely and verify } \|\mathbb{F}_N^{\mathrm{pr}}(\tau)\| \le C_N e^{-\eta_N \tau}.\end{array}} \tag{1966}$$

If (1966) holds, the standard evolution-family estimate gives (1927), and Duhamel gives (1930), hence (1898)–(1899). But none of these implications should be reversed.

231 Result of the two requested checks

The first requested calculation is now complete at the algebraic level: v_N is exactly (1939). Substitution shows that the full-lattice sufficient condition (1921) fails by the elementary near-resonance construction (1940)–(1943). Therefore the correct continuity certification is not a uniform isotropic Fourier transversality estimate; it must be a range/characteristic estimate such as (1951)–(1952), or an explicitly restricted Fourier-space estimate justified by the construction.

The second requested calculation gives the complete raw linearization in the form (1957)–(1963). It identifies the exact energy inequality (1964) and principal-defect estimate (1965) that would imply (1927)–(1928). The current raw source supplies the coefficients but does not supply the global characteristic inverse bound, the negative symmetric-part estimate, or

a fully specified principal bridge defect. Hence these are finite-N certification problems, not already-proved consequences of smoothness.

actual transport direction v_N:	explicit, (1939),
identity $q_N + r_N = 2$:	exact, (1936),
full-lattice transversality (1921):	false in general; ruled out by (1940)–(1943),
continuity inverse:	reduced to the characteristic formula (1949),
correct finite continuity check:	return-map/flow estimate (1951), giving (1952),
complete raw outer-bridge linearization:	derived in (1957)–(1963),
evolution decay (1927):	follows from the still-needed signed estimate (1964),
defect decay (1928):	requires the fully specified principal bridge and (1965),
pre-entry raw-jet decay (1898):	follows after the preceding two certifications,
raw residual decay (1899):	then follows as in Part 68,
first crossing:	not yet unconditional.

(1967)

The remaining proof task has therefore narrowed again. One should no longer try to verify (1921) on the full Fourier lattice. Instead, compute the actual characteristic return map of (1946) on the prescribed finite bridge and certify (1951); then insert that inverse into (1963), compute the symmetric part of the resulting finite-N linearized operator, and test (1964) together with the principal-defect bound (1965). Those calculations are the noncircular route from the raw reconstruction to (1927)–(1928).

Part 71. Slaved-Continuity Reduction and the Correct Outer-Bridge Stability Criterion

232 The state space in (1963) is larger than the raw reconstruction permits

Part 70 wrote the reduced bridge perturbation as $(\delta Q_N, \delta b_N)$ in (1963). The raw construction, however, prescribes Q_N from the finite Lambert cascade; it is not an independently evolved unknown of the scalar reconstruction. Likewise S_N is selected by continuity once

Q_N and b_N are fixed. Therefore the dynamical perturbation relevant to the first-crossing problem is not a free two-component vector. At fixed N and for the prescribed Lambert coefficient we must impose

$$\boxed{\delta Q_N = 0, \qquad \delta R_N = 0.} \tag{1968}$$

Then (1957) reduces to

$$\boxed{\mathcal{B}_N\,\delta S_N = -\partial_x\left(\frac{\delta b_N}{A_N}\right) + \partial_x\left(\frac{b_N}{A_N^2}(1-d_{V,N})\delta S_N\right).} \tag{1969}$$

Equivalently, after collecting the δS_N terms into the same linearized continuity operator used in (1881)–(1882), there is a slaving operator $\mathfrak{S}_N$ such that

$$\boxed{\delta S_N = \mathfrak{S}_N[\delta b_N].} \tag{1970}$$

Whenever the characteristic inverse certification (1951) holds,

$$\boxed{\|\mathfrak{S}_N h\|_{H^M} \le K^S_{N,M}\|h\|_{H^M}, \qquad K^S_{N,M} < \infty.} \tag{1971}$$

Thus the continuity correction is a slaved order-zero response on the certified bridge. No separate evolution estimate for S_N is required.

233 Exact scalar reduced linearization

Let $\mathscr{H}_N(Q,S,b)$ denote the complete raw scalar right-hand side from (1958), with the physical-to-τ conversion included in the coefficient notation. Since Q_N is prescribed and $S_N = S_N[Q_N, b_N]$, define the exact one-field reduced vector field

$$\boxed{\mathscr{K}_N(b) := \mathscr{H}_N\big(Q_N, S_N[Q_N,b], b\big).} \tag{1972}$$

Its Fréchet derivative is

$$\boxed{D\mathscr{K}_N(b)[h] = D_b\mathscr{H}_N[h] + D_S\mathscr{H}_N\big[\mathfrak{S}_N h\big].} \tag{1973}$$

Using (1960)–(1961),

$$\boxed{\begin{aligned}\mathcal{L}^b_N h = {}& 2\frac{Q_N'}{Q_N}b_N h - D_b\mathcal{E}_N[h] \\ & + 2G(1+d_{V,N})\mathfrak{S}_N h - D_S\mathcal{E}_N[\mathfrak{S}_N h] + \mathcal{L}^{\mathrm{frac}}_{N,\alpha}h,\end{aligned}} \tag{1974}$$

where $\mathcal{L}^{\mathrm{frac}}_{N,\alpha}$ denotes the signed linearized fractional block in the τ evolution. Formula (1974) is the operator whose symmetric part must be tested. In particular, the free Q_N block in (1963) must not be included in the stability norm for the prescribed-cascade problem.

234 Separation of the favorable fractional part and the finite lower-order budget

The raw two-correction calculation of Part 67 showed that the top-order fractional shape symbol is universal. On the certified compact-frequency corridor of Parts 64–67, write the signed fractional contribution in the energy convention as

$$\boxed{\Re\langle h, \mathcal{L}^{\mathrm{frac}}_{N,\alpha} h\rangle_{\mathcal{J}_{M+2}} \le -\lambda_N^{\mathrm{frac}} \|\Pi_{\mathrm{sh}} h\|^2_{\mathcal{J}_{M+2}} - c_{N,\alpha} \|\Pi_{\mathrm{sh}} h\|^2_{\mathcal{H}^\alpha_N},} \tag{1975}$$

with $c_{N,\alpha} > 0$ and with the amplitude zero mode controlled by the outer-corridor restoring coefficient $\lambda_N^{\mathrm{cor}} > 0$ from Part 63. Consequently the natural zeroth-order damping margin is

$$\boxed{\lambda_N^0 := \min\{\lambda_N^{\mathrm{cor}}, \lambda_N^{\mathrm{frac}}\} > 0.} \tag{1976}$$

All remaining terms in (1974) are finite-order bridge terms. Define their symmetric-part budget by

$$\boxed{\begin{aligned}\mathfrak{C}^{\mathrm{low}}_{N,M} := \sup_{\tau\in[\tau_0,\tau_e]} \sup_{h\neq 0} \frac{1}{\|h\|^2_{\mathcal{J}_{M+2}}} \Big| \Re\langle h, 2\frac{Q_N'}{Q_N} b_N h - D_b \mathcal{E}_N[h] \\ + 2G(1+d_{V,N})\mathfrak{S}_N h - D_S \mathcal{E}_N[\mathfrak{S}_N h]\rangle_{\mathcal{J}_{M+2}} \Big|.\end{aligned}} \tag{1977}$$

This constant is finite once (1951) and the raw bridge jet bounds are certified. The signed stability question is therefore reduced to the single finite inequality

$$\boxed{\mathfrak{C}^{\mathrm{low}}_{N,M} < \lambda_N^0.} \tag{1978}$$

Under (1978), (1974)–(1977) give

$$\boxed{\Re\langle h, \mathcal{L}^b_N h\rangle_{\mathcal{J}_{M+2}} \le -\lambda_N^b \|h\|^2_{\mathcal{J}_{M+2}} - c_{N,\alpha} \|\Pi_{\mathrm{sh}} h\|^2_{\mathcal{H}^\alpha_N},} \tag{1979}$$

where

$$\boxed{\lambda_N^b := \lambda_N^0 - \mathfrak{C}^{\mathrm{low}}_{N,M} > 0.} \tag{1980}$$

Thus (1964) is replaced, for the actual prescribed-cascade reconstruction, by the scalar coercivity criterion (1978). This is strictly more faithful to the raw architecture.

235 A canonical principal bridge removes the artificial Q_N defect

The second difficulty in Part 70 was the unspecified vector trajectory U_N^{pr}. Since Q_N is prescribed exactly, there is no reason to approximate it in the principal bridge. Set

$$\boxed{Q_N^{\mathrm{pr}} := Q_N, \qquad b_N^{\mathrm{pr}} := B_{*,N} s^{-\beta}, \qquad S_N^{\mathrm{pr}} := S_N[Q_N, b_N^{\mathrm{pr}}], \qquad \beta = 2\alpha - 1.} \tag{1981}$$

The continuity equation is then satisfied exactly by the principal bridge, so its continuity defect vanishes identically. Its only dynamical defect is the scalar residual

$$\boxed{F_{N,b}^{\mathrm{pr}} := \partial_\tau b_N^{\mathrm{pr}} - \mathscr{K}_N(b_N^{\mathrm{pr}}).} \tag{1982}$$

Using the exact coefficient matching defining $B_{*,N}$, the Lambert sensitivity expansion, and the raw reconstruction decomposition, this defect splits algebraically as

$$\boxed{F_{N,b}^{\mathrm{pr}} = F_N^{\mathrm{Lam}} + F_N^{\mathrm{frac}} + F_N^{\mathrm{rec}} + F_N^{\mathrm{forc}}.} \tag{1983}$$

The previously established terminal expansions identify the candidate decay rates

$$\boxed{F_N^{\mathrm{Lam}} = O_N(e^{-2^{-k_N}\tau}), \qquad F_N^{\mathrm{rec}} + F_N^{\mathrm{forc}} = O_N(e^{-4\alpha\tau/7}),} \tag{1984}$$

while the fractional contribution has the positive-power decay proved only on the localized conormal tube. Therefore (1984) by itself does not prove a pre-entry estimate for F_N^{frac}. The noncircular bridge requirement is precisely

$$\boxed{\|F_N^{\mathrm{frac}}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^{\mathrm{frac,br}} e^{-\eta_N^{\mathrm{frac,br}}\tau} \quad \text{on the outer bridge.}} \tag{1985}$$

If (1985) is established from the raw bounded-jet corridor, then

$$\boxed{\|F_{N,b}^{\mathrm{pr}}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^{\mathrm{pr}} e^{-\eta_N^{\mathrm{pr}}\tau}, \qquad \eta_N^{\mathrm{pr}} = \min\left\{2^{-k_N}, \frac{4\alpha}{7}, \eta_N^{\mathrm{frac,br}}\right\} > 0.} \tag{1986}$$

Hence the ambiguity in (1965) is removed: the principal bridge can be chosen canonically by (1981), and the remaining defect is a scalar, explicitly split quantity.

236 Evolution-family estimate after the scalar reduction

Let $w_N = b_N - b_N^{\mathrm{pr}}$. Taylor expansion of the exact reduced field gives

$$\boxed{\partial_\tau w_N = \mathcal{L}_N^b(\tau) w_N + \mathcal{N}_N^b(w_N) - F_{N,b}^{\mathrm{pr}}(\tau), \qquad \|\mathcal{N}_N^b(w)\| \le C_{N,M}\|w\|^2.} \tag{1987}$$

If (1978) holds, the linear evolution family satisfies

$$\boxed{\|\mathcal{U}_N^b(\tau, r)\|_{\mathcal{J}_{M+2}\to\mathcal{J}_{M+2}} \le M_N e^{-\lambda_N^b(\tau - r)}, \qquad \tau \ge r.} \tag{1988}$$

If in addition (1986) holds and the initial bridge perturbation is within the standard quadratic stability radius, Duhamel and the usual bootstrap give

$$\boxed{\|w_N(\tau)\|_{\mathcal{J}_{M+2}} \le C_N e^{-\eta_N^b \tau}, \qquad \eta_N^b < \min\{\lambda_N^b, \eta_N^{\mathrm{pr}}\}.} \tag{1989}$$

The slaving estimate (1971) then gives the same rate for $S_N - S_N^{\mathrm{pr}}$, while $Q_N - Q_N^{\mathrm{pr}} = 0$ exactly. Therefore

$$\boxed{\|U_N - U_N^{\mathrm{pr}}\|_{\mathcal{J}_{M+2}} \le \widetilde{C}_N e^{-\eta_N^b \tau}.} \tag{1990}$$

This is exactly the raw-jet decay requested in (1898), now derived from the correct constrained state space rather than from the overlarge system (1963).

237 What is proved and what remains finite

The present audit resolves two structural issues. First, the prescribed Lambert variable must not be varied as an independent dynamical component; therefore the correct stability operator is the scalar Schur complement (1974). Second, the principal bridge may use the exact prescribed Q_N and the exact continuity solver, so the vector defect (1965) reduces to the scalar defect (1982)–(1983).

What remains is finite but still substantive. The source does not yet supply a global return-map estimate proving (1951), so $K^S_{N,M}$ is not yet a certified number. Nor do the displayed raw formulas alone prove the strict margin (1978) on the whole outer bridge. Finally, the localized fractional positive-power estimate cannot be reused before entry; the outer-bridge fractional defect estimate (1985) must be proved independently.

Theorem 64 (Correct constrained reduction of the pre-entry stability problem)**.** *Fix $N < \infty$ and the prescribed finite-N Lambert cascade. Suppose the characteristic continuity inverse satisfies (1951), the finite symmetric-part margin (1978) holds, and the outer-bridge fractional defect satisfies (1985). Then the canonical principal bridge (1981) has exponentially decaying defect (1986), its scalar linearization generates the exponentially stable evolution family (1988), and sufficiently small bridge perturbations satisfy (1989)– (1990). Consequently the pre-entry raw-jet estimate (1898), and hence the complete residual estimate (1899), follow without invoking post-entry conormal persistence.*

free perturbation δQ_N:	removed; Q_N is prescribed, (1968),
continuity correction δS_N:	slaved to δb_N, (1970)–(1971),
correct linearized stability operator:	scalar Schur complement $\mathcal{L}^b_N$, (1974),
principal fractional sign:	favorable as inherited from Parts 67–68, (1975),
complete finite lower-order budget:	$\mathfrak{C}^{\mathrm{low}}_{N,M}$, (1977),
correct signed stability test:	finite margin (1978),
principal bridge:	canonically fixed by exact Q_N and continuity, (1981),
principal defect:	scalar split (1982)–(1983),
Lambert and reconstruction/forcing decay:	positive-power rates in (1984),
outer-bridge fractional defect:	still requires the noncircular estimate (1985),
evolution decay (1927):	follows in the constrained space from (1978), giving (1988),
principal-defect decay (1928):	follows from (1984)–(1985), giving (1986),
pre-entry raw-jet decay (1898):	follows by Duhamel, (1989)–(1990),
first crossing:	follows after the three finite certifications (1951), (1978), (1985).

(1991)

The next calculation is consequently narrower than (1964). One must certify the characteristic inverse (1951), insert its norm into the explicit lower-order budget (1977), and verify the strict scalar margin (1978). In parallel one must prove the outer-bridge fractional defect estimate (1985) without using local conormal smallness. Those three finite estimates are now the direct noncircular route to (1927)–(1928) for the actual prescribed finite-N reconstruction.

Part XV
Characteristic-Inverse Certification and Outer-Bridge Fractional Obstruction Audit

238 The finite flow-box form of the continuity inverse

We now address the three finite certifications isolated in (1991), without using the post-entry conormal tube. The first point is that (1951) is a finite-flow statement rather than a Fourier small-divisor statement. Write

$$\mathcal{B}_N\Sigma = X_N\Sigma + c_N\Sigma, \qquad X_N = v_N\cdot\nabla,$$

with the actual transport direction v_N given by (1939). Let $\Phi_N(\ell;X_0)$ denote the characteristic flow of X_N and let Σ_N^{in} be the normalization value on a transverse incoming section. Then the exact variation-of-constants formula is

$$\boxed{\begin{aligned}\Sigma_N(\Phi_N(\ell;X_0)) &= e^{-\int_0^\ell c_N(\Phi_N(a;X_0))\,da}\Sigma_N^{\mathrm{in}}(X_0)\\ &\quad+\int_0^\ell e^{-\int_r^\ell c_N(\Phi_N(a;X_0))\,da}F_N(\Phi_N(r;X_0))\,dr.\end{aligned}} \tag{1992}$$

Fix the actual finite bridge $\mathfrak{B}_N$ and suppose it is covered by a finite family of flow boxes $\mathfrak{U}_{N,\mu}$ whose incoming and outgoing sections are uniformly transverse to v_N. Define

$$\boxed{L_N^{\mathrm{fl}} := \max_{\mu}\sup_{X_0\in\Sigma_{N,\mu}^{\mathrm{in}}}|\ell_{N,\mu}(X_0)|, \qquad C_N^c := \|c_N\|_{L^\infty(\mathfrak{B}_N)}.} \tag{1993}$$

For fixed N, smoothness and compactness give finite C_N^c; finiteness of L_N^{fl} is exactly the absence of an unnormalized trapped homogeneous orbit in the chosen bridge cover. From (1992),

$$\boxed{\sup_{X_0,\ell}\int_0^{|\ell|}e^{-\int_r^{|\ell|}c_N(\Phi_N(a;X_0))\,da}\,dr \le L_N^{\mathrm{fl}}e^{C_N^cL_N^{\mathrm{fl}}}.} \tag{1994}$$

Thus the second bound in (1951) is certified on every such finite flow-box bridge.

239 Differentiated flow and an explicit characteristic constant

Let

$$V_{N,j} := \|D^j v_N\|_{L^\infty(\mathfrak{B}_N)}, \qquad 1 \le j \le M+1.$$

The first variational equation is

$$\frac{d}{d\ell} D\Phi_N = Dv_N(\Phi_N) D\Phi_N,$$

so Gronwall gives

$$\boxed{\|D\Phi_N(\ell)\| \le e^{V_{N,1}|\ell|} \le e^{V_{N,1} L_N^{\mathrm{fl}}}.} \tag{1995}$$

Higher derivatives satisfy triangular Faà di Bruno variational equations. Inductively there are explicit universal polynomials P_j such that

$$\boxed{\sup_{|\ell| \le L_N^{\mathrm{fl}}} \|D^j \Phi_N(\ell)\| \le P_j\big(L_N^{\mathrm{fl}}, V_{N,1}, \dots, V_{N,j}\big)\, e^{j V_{N,1} L_N^{\mathrm{fl}}}, \quad 1 \le j \le M+1.} \tag{1996}$$

The same estimate applies to the section/return maps provided the section transversality angle has a positive lower bound $\vartheta_N^{\mathrm{tr}} > 0$. Consequently (1951) holds with a finite constant depending only on

$$\boxed{\mathfrak{K}_{N,M}^{\mathrm{fl}} := \mathfrak{K}_{N,M}^{\mathrm{fl}}\left(L_N^{\mathrm{fl}}, (\vartheta_N^{\mathrm{tr}})^{-1}, C_N^c, \|c_N\|_{C^M}, \|v_N\|_{C^{M+1}}\right) < \infty.} \tag{1997}$$

In particular the slaving operator satisfies

$$\boxed{\|\mathfrak{S}_N h\|_{\mathcal{J}_{M+2}} \le K_{N,M}^S \|h\|_{\mathcal{J}_{M+2}}, \qquad K_{N,M}^S \le C_{N,M}^S \mathfrak{K}_{N,M+2}^{\mathrm{fl}}.} \tag{1998}$$

This proves the analytic implication (1951)–(1952) on a finite normalized flow-box bridge. What must still be checked for the particular prescribed bridge is the geometric input $L_N^{\mathrm{fl}} < \infty$ and $\vartheta_N^{\mathrm{tr}} > 0$; these are finite characteristic quantities, not small-divisor estimates.

240 Insertion into the complete lower-order budget

Define the raw finite coefficient bounds

$$\boxed{\begin{aligned} A_{N,M}^{(b)} &:= \sup_\tau \left\| 2\frac{Q_N'}{Q_N} b_N - D_b \mathcal{E}_N \right\|_{\mathcal{J}_{M+2} \to \mathcal{J}_{M+2}}, \\ A_{N,M}^{(S)} &:= \sup_\tau \left\| 2G(1+d_{V,N}) - D_S \mathcal{E}_N \right\|_{\mathcal{J}_{M+2} \to \mathcal{J}_{M+2}}. \end{aligned}} \tag{1999}$$

Then Cauchy–Schwarz in (1977), followed by (1998), gives the completely explicit upper bound

$$\boxed{\mathfrak{C}_{N,M}^{\mathrm{low}} \le A_{N,M}^{(b)} + A_{N,M}^{(S)} K_{N,M}^S \le A_{N,M}^{(b)} + A_{N,M}^{(S)} C_{N,M}^S \mathfrak{K}_{N,M+2}^{\mathrm{fl}}.} \tag{2000}$$

Therefore a directly checkable sufficient form of (1978) is

$$\boxed{A^{(b)}_{N,M} + A^{(S)}_{N,M} C^{S}_{N,M} \mathfrak{K}^{\mathrm{fl}}_{N,M+2} < \lambda^{0}_{N}.} \tag{2001}$$

This is the requested insertion of the characteristic inverse norm into the signed lower-order budget. It is important, however, not to confuse the finiteness of every term in (2001) with its strict sign. The raw identities available in the manuscript do not provide numerical upper bounds for the left side or a quantitative lower bound for λ^{0}_{N} strong enough to verify (2001). Hence the strict margin (1978) is reduced to the finite inequality (2001), but is not proved merely by compactness.

241 Why bounded outer-bridge jets do not imply (1985)

We next audit the independent fractional requirement. The point is subtle: finite-N smoothness and a bounded-jet corridor give tame bounds for the fractional block, but they do not force its *defect from the principal shape* to decay in τ.

To see the logical distinction, let $\mathfrak{F}_{N,\alpha}[a]$ denote the normalized fractional product block evaluated on a bounded shape coefficient $a(\tau,\cdot)$. Tame pseudodifferential calculus on the compact bridge gives for bounded a

$$\boxed{\|\mathfrak{F}_{N,\alpha}[a]\|_{\mathcal{J}_{M+2}} \le C_{N,M,R},} \tag{2002}$$

but (2002) contains no positive power of $s = e^{-\tau}$. For example a bounded coefficient perturbation

$$\boxed{a(\tau,\cdot) = a_{\mathrm{pr}}(\cdot) + \varepsilon \sin\tau\, \psi(\cdot)} \tag{2003}$$

remains in a bounded finite-jet corridor for sufficiently small fixed ε, while its linearized fractional defect contains

$$\varepsilon \sin\tau\, D\mathfrak{F}_{N,\alpha}[a_{\mathrm{pr}}]\psi, \tag{2004}$$

which does not decay as $\tau \to \infty$ unless the displayed derivative vanishes in that direction. Thus no estimate of the form (1985) can follow from bounded jets alone.

This yields the following noncircular obstruction statement.

Theorem 65 (Outer-corridor boundedness is insufficient for fractional defect decay)**.** *A uniform finite-N bound on finitely many outer-bridge jets of the raw reconstruction implies a finite tame bound such as (2002), but does not imply an exponentially decaying defect of the form (1985). To prove (1985) one needs an additional asymptotic shape statement, or an exact structural cancellation making the principal fractional defect vanish independently of shape convergence.*

242 The weakest additional hypothesis that actually proves (1985)

The required extra information can be stated without assuming the full local conormal tube. Let $\mathbf{a}_N(\tau)$ collect precisely the normalized raw coefficients entering the fractional product

block on the principal bridge, and let $\mathbf{a}_N^*$ denote their limiting principal values. Assume only the outer-bridge coefficient convergence

$$\boxed{\max_{0\le j\le M+2} \|E^j(\mathbf{a}_N-\mathbf{a}_N^*)\|_{\mathcal{X}_N} \le C_{N,M}^{a} e^{-\eta_N^a \tau}, \qquad \eta_N^a>0.} \tag{2005}$$

This is weaker than conormal closeness of the complete solution: it concerns only the finite coefficient list entering the fractional block, evaluated on the canonical principal bridge (1981).

On a compact coefficient set separated from all algebraic denominators, the fractional product map is C^1 between the bridge spaces used in Parts 67–68. The fundamental theorem of calculus therefore gives

$$\boxed{\begin{aligned} F_N^{\mathrm{frac}} &= \mathfrak{F}_{N,\alpha}[\mathbf{a}_N]-\mathfrak{F}_{N,\alpha}[\mathbf{a}_N^*] \\ &= \int_0^1 D\mathfrak{F}_{N,\alpha}[\mathbf{a}_N^*+\theta(\mathbf{a}_N-\mathbf{a}_N^*)](\mathbf{a}_N-\mathbf{a}_N^*)\,d\theta. \end{aligned}} \tag{2006}$$

Hence (2005) implies

$$\boxed{\|F_N^{\mathrm{frac}}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^{\mathrm{frac,br}} e^{-\eta_N^a\tau}.} \tag{2007}$$

Thus (1985) follows with

$$\boxed{\eta_N^{\mathrm{frac,br}} = \eta_N^a.} \tag{2008}$$

No post-entry conormal smallness has been used in this implication. The new question is whether (2005) follows from the prescribed raw coefficients $Q_N, S_N[Q_N, b_N^{\mathrm{pr}}]$ themselves. The present source gives positive powers for the Lambert sensitivity and for the reconstruction/forcing block, but it does not yet give (2005) for the complete fractional coefficient list on the whole outer bridge.

243 A useful exact special case

There is one situation in which the preceding obstruction disappears. In the strict generalized affine one-phase architecture, the manuscript already records the exact multiplier identity

$$(-\Delta_{\mathbb{T}^3})^{\alpha} F(s_N) = \kappa^{2\alpha}(-\partial_{ss})^{\alpha} F(s_N),$$

and, when the same fractional realization is used in the finite-N equation and profile equation, the corresponding fractional bridge defect is exactly zero. Therefore

$$\boxed{F_N^{\mathrm{frac}} \equiv 0 \quad\Longrightarrow\quad \text{(1985) holds with arbitrary } \eta_N^{\mathrm{frac,br}}>0.} \tag{2009}$$

But (2009) may be invoked for the present raw two-correction bridge only after proving that its fractional block remains in that exact strict one-phase subclass. The later raw reconstruction contains a nontrivial continuity response, so that identification cannot be made merely from the earlier one-phase formula.

244 Result of the three requested certifications

The audit therefore gives a mixed conclusion. The characteristic inverse can be certified analytically on a normalized finite flow-box bridge, with the explicit bound (1997)–(1998). Its norm enters the complete lower-order budget through (2000), reducing the strict signed test to the finite numerical inequality (2001). The sign of that inequality is not determined by the raw bounds presently supplied. Independently, the outer-bridge fractional decay (1985) does not follow from bounded jets; it follows from the strictly weaker coefficient-convergence condition (2005), or from an exact one-phase cancellation if that architecture is proved to apply to the raw bridge.

characteristic formula:	exact, (1992),
finite flight-time integral:	bounded by (1994),
differentiated flow/return map:	bounded by (1995)–(1997) under finite flow-box transversality,
continuity slaving norm:	explicit finite bound (1998),
condition (1951):	certified on a normalized finite flow-box bridge once $L_N^{\mathrm{fl}} < \infty$ and $\vartheta_N^{\mathrm{tr}} > 0$ are checked,
lower-order budget:	bounded by (2000),
strict coercivity (1978):	equivalent to the finite sufficient test (2001); sign not yet certified from displayed constants,
bounded outer jets $\Rightarrow$ (1985):	false in general; obstruction (2003)–(2004),
noncircular sufficient condition for (1985):	coefficient convergence (2005), giving (2007),
strict one-phase special case:	fractional defect vanishes exactly if the raw bridge is proved to remain in that subclass, (2009),
first crossing:	not yet unconditional.

(2010)

The remaining task is therefore sharper than (1991): compute the actual finite flow-box quantities L_N^{fl} and $\vartheta_N^{\mathrm{tr}}$ and the raw coefficient norms in (1999), so that (2001) can be decided; independently, prove (2005) for the canonical principal bridge, or prove that the raw fractional architecture falls under the exact cancellation (2009). Without one of these two fractional mechanisms, (1985) cannot be obtained from a bounded outer corridor alone.

Part XVI
Exact Flow-Box Constants and the Canonical One-Phase Raw Branch

245 The characteristic flight time and section transversality are exact

The quantities left abstract in (1993)–(1997) can be evaluated more sharply for the actual raw transport field. From (1935)–(1939),

$$\boxed{q_N + r_N = 2.} \tag{2011}$$

Introduce on the universal cover of the torus the transverse coordinate

$$\boxed{\omega := y + z.} \tag{2012}$$

Along every characteristic of the continuity transport field X_N,

$$\boxed{\frac{d\omega}{d\ell} = q_N + r_N = 2.} \tag{2013}$$

Hence the slab $0 \le \omega \le 2\pi$ is crossed in the exact characteristic flight time

$$\boxed{L_N^{\mathrm{fl}} = \pi.} \tag{2014}$$

This conclusion is independent of the x component p_N and therefore does not require the false full-lattice transversality condition ruled out in Part 70.

For the section $\omega = \text{const}$, take the unit normal $n_\omega = (0,1,1)/\sqrt{2}$. Then

$$\boxed{v_N \cdot n_\omega = \sqrt{2}.} \tag{2015}$$

Thus the unnormalized section flux has the exact lower bound $\sqrt{2}$. If $\vartheta_N^{\mathrm{tr}}$ denotes the normalized angle used in the return-map estimate, then

$$\boxed{\vartheta_N^{\mathrm{tr}} = \inf_{\mathfrak{B}_N} \frac{\sqrt{2}}{|v_N|} \ge \frac{\sqrt{2}}{\sqrt{(p_N^+)^2 + 4}}, \qquad p_N^+ := \|p_N\|_{L^\infty(\mathfrak{B}_N)}.} \tag{2016}$$

In particular $\vartheta_N^{\mathrm{tr}} > 0$ on every nondegenerate finite bridge. Equations (2014)–(2016) close the geometric part of (1951) without any Diophantine assumption.

Substitution in (1994) gives the explicit weighted-flight estimate

$$\boxed{\sup_{X_0,\ell} \int_0^{|\ell|} e^{-\int_r^{|\ell|} c_N(\Phi_N(a;X_0))\,da}\, dr \le \pi e^{\pi C_N^c}.} \tag{2017}$$

Consequently one may replace the abstract flow-box constant in (1997) by

$$\boxed{\mathfrak{K}_{N,M}^{\mathrm{fl}} = \mathfrak{K}_{N,M}^{\mathrm{fl}}\left(\pi, \frac{\sqrt{(p_N^+)^2+4}}{\sqrt{2}}, C_N^c, \|c_N\|_{C^M}, \|v_N\|_{C^{M+1}}\right).} \tag{2018}$$

246 What the raw norms in (1999) actually reduce to

The raw reconstruction error is (1825). Therefore its Fréchet derivatives with respect to the free scalar variable and the slaved continuity variable are finite differential operators whose coefficients are algebraic combinations of

$$\boxed{\mathbf{J}^{\mathrm{raw}}_{N,M} := \left(Q_N, S_N, b_N, A_N^{-1}, C_N^{-1}, R_N^{-1}, \partial^{\le M+3} Q_N, \partial^{\le M+3} S_N, \partial^{\le M+3} b_N\right).} \tag{2019}$$

More explicitly, tame product and composition estimates give finite constants $P^{(b)}_{N,M}$ and $P^{(S)}_{N,M}$, polynomial in the positive bridge gaps and in the displayed jets, such that

$$\boxed{\begin{aligned} A^{(b)}_{N,M} &\le 2\left\| \frac{Q_N'}{Q_N} b_N \right\|_{C^{M+2}} + P^{(b)}_{N,M}(\mathbf{J}^{\mathrm{raw}}_{N,M}), \\ A^{(S)}_{N,M} &\le 2\|G(1+d_{V,N})\|_{C^{M+2}} + P^{(S)}_{N,M}(\mathbf{J}^{\mathrm{raw}}_{N,M}). \end{aligned}} \tag{2020}$$

The characteristic inverse is now completely explicit at the geometric level, so (2001) becomes

$$\boxed{2\left\| \frac{Q_N'}{Q_N} b_N \right\|_{C^{M+2}} + P^{(b)}_{N,M} + \left(2\|G(1+d_{V,N})\|_{C^{M+2}} + P^{(S)}_{N,M}\right) C^S_{N,M}\, \mathfrak{K}^{\mathrm{fl}}_{N,M+2} < \lambda^0_N.} \tag{2021}$$

Here $\mathfrak{K}^{\mathrm{fl}}_{N,M+2}$ is evaluated with (2018).

Equation (2021) is as far as the raw prescribed formulas alone can take the strict margin. The quantities $P^{(b)}_{N,M}$ and $P^{(S)}_{N,M}$ contain the actual pre-entry jets of the unknown scalar solution b_N and of its slaved continuity correction. The source prescribes $b_N(z,0) = 3+\sin z$, but does not provide a closed formula for those jets throughout the outer bridge. Therefore no numerical value of the left side of (2021), and hence no proof of its strict inequality with λ^0_N, follows from the prescribed data alone. This is not a missing flow-box constant: (2014)–(2018) have removed that ambiguity. It is a genuine dynamical estimate still required for the complete lower-order budget.

247 The canonical normalized strict one-phase solution of raw continuity

The fractional alternative in (2009) can, however, be settled structurally. Use the distinguished affine phase already fixed in the manuscript,

$$\boxed{C_1 = C_2 = 0, \qquad C_3 = C_4 = 1, \qquad \xi = t - z - \theta_{*,N}.} \tag{2022}$$

Assume $Q_N = Q_N(\xi)$ and $R_N = R_N(\xi)$, as in the prescribed Lambert cascade. Seek the normalized continuity correction in the same one-phase class. Since the raw algebraic reconstruction satisfies

$$\boxed{\frac{C_N^2}{A_N} = R_N,} \tag{2023}$$

the one-phase divergence equation (1877) reduces to

$$\boxed{\partial_z C_N = 0.} \tag{2024}$$

Thus C_N is constant along the phase. Write this positive constant as c_N. Equations (2023) and $V_N = (A_N + C_N)/2 = Q_N + S_N$ then give

$$\boxed{C_N(\xi) = c_N, \qquad A_N(\xi) = \frac{c_N^2}{R_N(\xi)}, \qquad S_N(\xi) = \frac{1}{2}\left(c_N + \frac{c_N^2}{R_N(\xi)}\right) - Q_N(\xi).} \tag{2025}$$

These fields satisfy the raw continuity equation identically and also satisfy the two-correction algebraic architecture, because (2023) is precisely the algebraic relation equivalent to (1820) on the positive branch.

Impose the canonical zero-mean normalization used in Part 69, $\langle S_N \rangle = 0$. Put

$$a_N := \left\langle \frac{1}{R_N} \right\rangle, \qquad q_N^0 := \langle Q_N \rangle. \tag{2026}$$

Then c_N is the positive root of

$$a_N c_N^2 + c_N - 2q_N^0 = 0, \tag{2027}$$

namely

$$\boxed{c_N = \frac{-1 + \sqrt{1 + 8a_N q_N^0}}{2a_N}} \tag{2028}$$

whenever $a_N > 0$ and $q_N^0 > 0$. Thus the zero-mode prescription selects a unique positive strict one-phase raw branch.

248 Exact fractional cancellation on the canonical branch

On (2025), every nonconstant velocity factor entering the fractional product block is a function of the single affine phase ξ; $C_N = c_N$ is constant. The torus fractional Laplacian is a Fourier multiplier with symbol $|k|^{2\alpha}$. Restricting to the one-dimensional Fourier lattice generated by the phase vector therefore gives exactly

$$\boxed{(-\Delta_{\mathbb{T}^3})^\alpha f(\xi) = (-\partial_{\xi\xi})^\alpha f(\xi)} \tag{2029}$$

for the distinguished phase. This is the same exact one-phase reduction used earlier in the manuscript. The standard Fourier-series definition of the torus fractional Laplacian likewise confirms that no transverse multiplier remainder is created by this restriction.

If the canonical principal bridge uses the same one-dimensional fractional realization, its raw fractional bridge defect is therefore

$$\boxed{F_N^{\mathrm{frac}} \equiv 0.} \tag{2030}$$

Consequently (1985) holds identically, with arbitrary positive $\eta_N^{\mathrm{frac,br}}$, and coefficient convergence (2005) is unnecessary on this canonical strict one-phase branch.

This conclusion must be kept separate from the general raw solution of the three-dimensional continuity equation. A different normalized solution $S_N(x, y, z, t)$ with genuine transverse dependence need not satisfy (2030). What has been proved is that the manuscript's distinguished affine phase plus the zero-mean normalization admits the explicit raw two-correction branch (2025)–(2028), and on that branch the exact one-phase fractional cancellation is valid.

249 Updated finite certification ledger

flow-box coordinate:	$\omega = y + z$, exact, (2012),
flight time:	$L_N^{\mathrm{fl}} = \pi$, exact, (2014),
section flux:	$v_N \cdot n_\omega = \sqrt{2}$, exact, (2015),
normalized transversality:	positive lower bound (2016),
weighted characteristic integral:	explicit bound $\pi e^{\pi C_N^c}$, (2017),
characteristic inverse:	geometrically certified by (2018),
raw norms in (1999):	reduced to the actual bridge jets by (2020),
strict lower-order margin (2001):	becomes (2021); not numerically certified because the required pre-entry solution jets are not supplied by the prescribed initial datum alone,
canonical zero-mean raw continuity branch:	explicit, (2025)–(2028),
raw fractional architecture on that branch:	strict one-phase,
outer-bridge fractional defect:	$F_N^{\mathrm{frac}} \equiv 0$, (2030),
condition (2005):	unnecessary on the canonical strict one-phase branch,
remaining finite certification:	prove the dynamical jet bounds entering (2021) and compare the resulting number with λ_N^0,
first crossing:	still conditional on that strict lower-order margin.

(2031)

The requested calculation therefore closes two of the three pieces exactly: the characteristic geometry has $L_N^{\mathrm{fl}} = \pi$ with uniform transverse section $y + z = \mathrm{const}$, and the canonical zero-mean two-correction branch is strictly one-phase, so its fractional bridge defect vanishes exactly. The remaining obstruction is no longer fractional and no longer geometric. It is the quantitative signed comparison (2021), whose left side depends on the actual pre-entry jets of the complete scalar solution. Those jets must be bounded dynamically before the strict coercive margin can be declared.

Part XVII
Actual Pre-Entry Jet Completion and the One-Phase Invariance Test

250 Why the final margin must be tested on the actual reconstructed evolution

Part 73 closed the flow-box geometry exactly and exhibited a canonical strict one-phase solution of the raw continuity constraint. The final step cannot, however, identify that auxiliary continuity branch with the actual solution of the complete reconstructed scalar equation without first checking that the complete evolution preserves the strict one-phase class. This check is decisive because the prescribed forcing in (1822) is

$$\boxed{G(x,y,z) = A_0 + (\sin x \sin y \sin z)^2.} \tag{2032}$$

It is genuinely three-dimensional. Thus the exact cancellation (2030) is an identity on the canonical one-phase submanifold, but it is not automatically an identity along the actual forced trajectory.

251 Direct invariance test at the prescribed initial datum

The prescribed scalar datum is

$$\boxed{b_N(x,y,z,0) = 3 + \sin z.} \tag{2033}$$

Assume at the initial instant that the reconstructed coefficients are on the canonical branch (2025), so in particular $C_N = c_N > 0$. The forcing contribution to the scalar equation is then

$$\boxed{F_N^{\mathrm{forc}}(x,y,z,0) = 2c_N \left[A_0 + \sin^2 x \, \sin^2 y \, \sin^2 z\right].} \tag{2034}$$

Its transverse derivatives are

$$\boxed{\begin{aligned} \partial_x F_N^{\mathrm{forc}} &= 4c_N \sin x \cos x \, \sin^2 y \, \sin^2 z, \\ \partial_y F_N^{\mathrm{forc}} &= 4c_N \sin y \cos y \, \sin^2 x \, \sin^2 z. \end{aligned}} \tag{2035}$$

These functions are not identically zero on $\mathbb{T}^3$. Therefore the complete vector field is not tangent to the strict one-phase manifold unless the remaining terms of the complete scalar equation produce an exact pointwise cancellation of (2035).

For the canonical branch, the raw reconstruction error (1825) vanishes because $C_N = c_N$ is constant:

$$\boxed{\mathcal{E}_N = 0.} \tag{2036}$$

Moreover the Lambert coefficient is a function of the distinguished phase only, and the strict one-phase fractional block has no transverse derivative. Consequently there is no term in those blocks that cancels the nonzero x- and y-derivatives in (2035). Hence, at generic points,

$$\boxed{\partial_x \partial_t b_N(0) \neq 0, \qquad \partial_y \partial_t b_N(0) \neq 0.} \tag{2037}$$

The actual solution generated by the prescribed complete forcing therefore leaves the strict one-phase scalar class immediately.

252 Correction to the interpretation of (2030)

Equation (2030) remains mathematically correct on the canonical one-phase branch, but it cannot be used as the outer-bridge estimate (1985) for the actual prescribed forced solution. The correct statement is

$$\boxed{F_N^{\mathrm{frac}} \equiv 0 \quad \text{on the invariant strict one-phase model only;} \qquad F_N^{\mathrm{frac}}[b_N^{\mathrm{actual}}] \text{ requires a transverse-shape estimate.}} \tag{2038}$$

Thus the "fractional bridge obstruction is closed" conclusion of Part 73 must be qualified: it is closed for the auxiliary canonical branch, not for the actual trajectory driven by (2032).

253 Exact transverse-jet equation

Let

$$Z_N^{\perp} := (\partial_x b_N, \partial_y b_N) \tag{2039}$$

and differentiate the complete raw scalar equation (1824). Schematically, but without discarding any term, one obtains

$$\boxed{\mathcal{D}_N Z_N^{\perp} = D_b \mathscr{H}_N[b_N, S_N] Z_N^{\perp} + D_S \mathscr{H}_N[b_N, S_N]\, \partial_{\perp} S_N + \mathscr{F}_N^{\perp},} \tag{2040}$$

where $\mathscr{F}_N^{\perp}$ contains the explicit transverse derivative of $2C_N G$ together with commutators generated by differentiating the transport operator. At the canonical initial state its forcing component is exactly (2035). In particular

$$\boxed{\|\mathscr{F}_N^{\perp}(0)\|_{H^M} > 0} \tag{2041}$$

for every M for which the nonzero trigonometric mode is measured.

If the reduced evolution has damping rate $\lambda_N^b > 0$, Duhamel gives the sharp structural estimate

$$\boxed{\|Z_N^{\perp}(\tau)\|_{H^M} \leq M_N e^{-\lambda_N^b(\tau-\tau_0)} \|Z_N^{\perp}(\tau_0)\|_{H^M} + M_N \int_{\tau_0}^{\tau} e^{-\lambda_N^b(\tau-r)} \|\mathscr{F}_N^{\perp}(r)\|_{H^M}\, dr.} \tag{2042}$$

The second term is the unavoidable forced transverse response. If $\mathscr{F}_N^{\perp}$ merely remains $O(1)$, (2042) gives an $O(1)$ response, not exponential convergence to the one-phase branch. Exponential transverse convergence requires a genuinely decaying transverse forcing/commutator or a different principal bridge that incorporates the forced transverse profile.

254 The apparent circularity in using (2021) to prove its own jet bounds

The requested final comparison cannot be obtained by first assuming the coercivity generated by (2021), using that coercivity to bound the actual pre-entry jets, and then inserting those bounds back into (2021). Such an argument would be circular. To close the margin one needs an independent absorbing estimate. A sufficient noncircular statement is the following.

Suppose there is a radius $R^{\mathrm{abs}}_{N,M}$, obtained directly from the raw finite-N equation before using (2021), such that

$$\boxed{\sup_{\tau_0\le\tau\le\tau_e}\|\mathbf{J}^{\mathrm{raw}}_{N,M}(\tau)\|\le R^{\mathrm{abs}}_{N,M}.} \tag{2043}$$

Define the corresponding completely finite coefficient envelope

$$\boxed{\begin{aligned}\mathfrak{B}_{N,M}(R):=2\sup_{\|\mathbf{J}\|\le R}\left\|\frac{Q_N'}{Q_N}b_N\right\|_{C^{M+2}}+\sup_{\|\mathbf{J}\|\le R}P^{(b)}_{N,M}(\mathbf{J})\\ +\left[2\|G(1+d_{V,N})\|_{C^{M+2}}+\sup_{\|\mathbf{J}\|\le R}P^{(S)}_{N,M}(\mathbf{J})\right]C^{S}_{N,M}\mathfrak{K}^{\mathrm{fl}}_{N,M+2}.\end{aligned}} \tag{2044}$$

Then the desired strict lower-order margin follows from the genuinely checkable inequality

$$\boxed{\mathfrak{B}_{N,M}(R^{\mathrm{abs}}_{N,M})<\lambda^0_N.} \tag{2045}$$

This is the noncircular completion criterion corresponding to (2021).

255 What the raw energy estimate actually supplies

On any finite preterminal interval on which the algebraic gaps A_N, C_N, R_N stay positive, standard finite-N quasilinear energy estimates give, for sufficiently large m,

$$\boxed{\frac{d}{dt}\|b_N(t)\|^2_{H^m}+\nu\|b_N(t)\|^2_{H^{m+\alpha}}\le\mathcal{C}_{N,m}(t)\big(1+\|b_N(t)\|^2_{H^m}\big).} \tag{2046}$$

Consequently

$$\boxed{\|b_N(t)\|^2_{H^m}\le\left(\|b_N(0)\|^2_{H^m}+\int_0^t\mathcal{C}_{N,m}(r)\,dr\right)\exp\left(\int_0^t\mathcal{C}_{N,m}(r)\,dr\right).} \tag{2047}$$

Together with the characteristic inverse for the continuity correction, (2047) yields a finite $R^{\mathrm{abs}}_{N,M}$ whenever the coefficient integral is independently bounded. It does not, however, imply the smallness required by (2045). Finiteness and strict coercive domination are different claims.

256 Can the final strict comparison be decided from the present prescribed data?

The answer supplied by the raw formulas is now definitive. The manuscript contains neither a numerical lower bound for

$$\lambda_N^0 = \min\{\lambda_N^{\mathrm{cor}}, \lambda_N^{\mathrm{frac}}\} \tag{2048}$$

nor an independently derived absorbing radius $R_{N,M}^{\mathrm{abs}}$ small enough to evaluate (2045). Moreover the actual forcing (2032) immediately generates transverse shape, so the exact one-phase cancellation cannot be used to remove the fractional defect along the actual trajectory. Therefore the strict comparison requested after (2021) cannot be proved from the currently prescribed formulas without an additional quantitative estimate. Declaring it proved would assume precisely the stability that the comparison is meant to establish.

The remaining alternative is finite and explicit. Either route (A) derives an independent absorbing bound (2043), evaluates (2044), verifies (2045), and controls the forced transverse fractional response in (2042); or route (B) replaces the principal bridge by the actual forced transverse profile and repeats the compact-frequency coercivity test about that profile. In symbols,

$$\boxed{\text{(A): independent absorbing estimate and transverse control} \quad \text{or} \quad \text{(B): forced-profile bridge.}} \tag{2049}$$

Route (B) avoids demanding convergence to a one-phase profile that the prescribed forcing does not preserve.

257 Corrected completion ledger

flow-box geometry:	closed exactly by (2012)–(2018),
canonical raw continuity branch:	explicit by (2025)–(2028),
one-phase fractional cancellation:	exact on that auxiliary branch, (2030),
invariance under the prescribed forcing:	fails generically by (2035)–(2037),
actual transverse response:	governed by the forced Duhamel estimate (2042),
raw pre-entry jets:	finite under (2046)–(2047), but no independently proved small absorbing radius is presently available,
noncircular lower-order test:	(2045), with envelope (2044),
strict comparison with λ_N^0:	not certified by the current data,
first crossing:	remains conditional on (2045) plus control of the forced transverse fractional response,
post-entry conormal propagation:	unchanged once entry is actually obtained,
finite-time scalar characteristic law:	unchanged conditionally after entry.

(2050)

The final audit therefore resolves the requested "finish" in the strict logical sense: the remaining margin cannot be promoted to a theorem from the present data, and the reason is now explicit rather than hidden in an unspecified constant. The prescribed three-dimensional forcing immediately creates transverse scalar modes, while the available preterminal energy estimate yields finiteness rather than the smallness needed in (2045). The next theorem-level closure must control that forced transverse response or use its actual forced profile as the bridge.

Part XVIII
Forced-Transverse Principal Bridge and the Remaining Quantitative Margin

258 Why the forced-profile route is the correct next reduction

Part 74 shows that the prescribed forcing does not preserve the strict one-phase submanifold. The appropriate principal bridge must therefore contain the transverse modes which are generated at leading order by $2C_N G$. This section constructs that bridge at the level justified by the raw equations and isolates the exact finite inequalities which remain.

Write

$$\boxed{G = A_0 + G_\perp, \qquad G_\perp = \sin^2 x \, \sin^2 y \, \sin^2 z.} \tag{2051}$$

Using $\sin^2 \theta = (1 - \cos 2\theta)/2$,

$$\boxed{G_\perp = \frac{1}{8}(1 - \cos 2x)(1 - \cos 2y)(1 - \cos 2z).} \tag{2052}$$

Hence the forcing contains only the finite Fourier set

$$\boxed{\mathcal{K}_G = \{(2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3) : \varepsilon_j \in \{-1, 0, 1\}\},} \tag{2053}$$

with the zero mode separated into the longitudinal amplitude. In particular, the initially generated transverse response is finite dimensional.

259 Projection onto longitudinal and forced transverse sectors

Let $\Pi_\parallel$ denote the projection onto the distinguished one-phase sector and let

$$\boxed{\Pi_G := \sum_{k \in \mathcal{K}_G \setminus \{0\}} \Pi_k, \qquad \Pi_R := I - \Pi_\parallel - \Pi_G.} \tag{2054}$$

For the scalar field write

$$\boxed{b_N = b_N^\parallel + b_N^G + b_N^R.} \tag{2055}$$

At the prescribed initial datum $b_N(0) = 3 + \sin z$,

$$\boxed{b_N^G(0) = 0, \qquad b_N^R(0) = 0.} \tag{2056}$$

The first nonzero transverse derivative is therefore produced by the explicit forcing, exactly as in (2035)–(2037).

260 Canonical forced transverse principal profile

Let $\mathcal{L}_N^b(\tau)$ be the reduced scalar linearization already used in (1979)–(1990), evaluated along the longitudinal principal bridge. Define the forced transverse principal profile B_N^G as the solution of

$$\boxed{\partial_\tau B_N^G = \Pi_G \mathcal{L}_N^b(\tau) B_N^G + \Pi_G\big(2C_N^{\mathrm{pr}} G_\perp\big), \qquad B_N^G(\tau_0) = 0.} \tag{2057}$$

Thus, whenever the evolution family $\mathcal{U}_N^G$ exists,

$$\boxed{B_N^G(\tau) = \int_{\tau_0}^{\tau} \mathcal{U}_N^G(\tau, r)\Pi_G\big(2C_N^{\mathrm{pr}}(r) G_\perp\big)\, dr.} \tag{2058}$$

This definition does not ask the actual solution to converge to a submanifold which the forcing fails to preserve. The forcing is built into the principal bridge itself.

Define

$$\boxed{b_N^{\mathrm{FP}} := b_N^{\parallel} + B_N^G, \qquad S_N^{\mathrm{FP}} := S_N[Q_N, b_N^{\mathrm{FP}}],} \tag{2059}$$

where $S_N[Q_N, b]$ is the normalized raw continuity solution from Parts 68–73. The corresponding raw velocity coefficients are

$$\boxed{V_N^{\mathrm{FP}} = Q_N + S_N^{\mathrm{FP}}, \quad A_N^{\mathrm{FP}} = V_N^{\mathrm{FP}} - D_N(V_N^{\mathrm{FP}}), \quad C_N^{\mathrm{FP}} = V_N^{\mathrm{FP}} + D_N(V_N^{\mathrm{FP}}).} \tag{2060}$$

261 Exact defect after absorbing the forced modes

Let $\mathscr{K}_N$ denote the complete reduced raw scalar vector field. The forced-profile defect is

$$\boxed{\mathfrak{D}_N^{\mathrm{FP}} := \partial_\tau b_N^{\mathrm{FP}} - \mathscr{K}_N(b_N^{\mathrm{FP}}).} \tag{2061}$$

By construction, the component which is linear in the explicit transverse forcing cancels in $\Pi_G \mathfrak{D}_N^{\mathrm{FP}}$. What remains consists of

$$\boxed{\mathfrak{D}_N^{\mathrm{FP}} = \mathfrak{D}_N^{\parallel} + \mathfrak{D}_N^{\mathrm{coeff}} + \mathfrak{D}_N^{\mathrm{nl}} + \mathfrak{D}_N^{\mathrm{frac}},} \tag{2062}$$

where $\mathfrak{D}_N^{\mathrm{coeff}}$ is the error from replacing the actual raw coefficients by the forced-profile coefficients, $\mathfrak{D}_N^{\mathrm{nl}}$ contains quadratic and higher transverse interactions, and $\mathfrak{D}_N^{\mathrm{frac}}$ is the difference between the complete fractional block and its linearization included in (2057). Thus the nondecaying source identified in (2042) has been removed from the remainder equation.

262 Remainder equation about the forced profile

Set

$$\boxed{w_N^{\mathrm{FP}} := b_N - b_N^{\mathrm{FP}}.} \tag{2063}$$

Taylor expansion of the complete reduced field gives

$$\boxed{\partial_\tau w_N^{\mathrm{FP}} = \mathcal{L}_N^{\mathrm{FP}}(\tau) w_N^{\mathrm{FP}} + \mathcal{N}_N^{\mathrm{FP}}(\tau, w_N^{\mathrm{FP}}) - \mathfrak{D}_N^{\mathrm{FP}},} \tag{2064}$$

with

$$\boxed{\mathcal{L}_N^{\mathrm{FP}}(\tau) := D\mathscr{K}_N(b_N^{\mathrm{FP}}(\tau)), \qquad \|\mathcal{N}_N^{\mathrm{FP}}(\tau, w)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^{\mathrm{FP}} \|w\|_{\mathcal{J}_{M+2}}^2.} \tag{2065}$$

The theorem-level stability condition is therefore the signed estimate

$$\boxed{\Re\langle w, \mathcal{L}_N^{\mathrm{FP}}(\tau) w\rangle \le -\lambda_N^{\mathrm{FP}} \|w\|_{\mathcal{J}_{M+2}}^2, \qquad \lambda_N^{\mathrm{FP}} > 0.} \tag{2066}$$

This is the forced-profile replacement for the unavailable one-phase margin (2045).

263 A finite sufficient condition for the forced-profile margin

Let λ_N^0 be the principal positive gap from (2048). Define the forced-profile lower-order budget

$$\boxed{\mathfrak{C}_{N,M}^{\mathrm{FP}} := \sup_{\tau_0 \le \tau \le \tau_e} \left\| D\mathscr{K}_N(b_N^{\mathrm{FP}}(\tau)) - \mathcal{L}_{N,\mathrm{prin}}(\tau) \right\|_{\mathcal{J}_{M+2} \to \mathcal{J}_{M+2}},} \tag{2067}$$

where $\mathcal{L}_{N,\mathrm{prin}}$ contains the already isolated principal fractional and corridor damping. Then

$$\boxed{\mathfrak{C}_{N,M}^{\mathrm{FP}} < \lambda_N^0 \quad \Longrightarrow \quad \lambda_N^{\mathrm{FP}} := \lambda_N^0 - \mathfrak{C}_{N,M}^{\mathrm{FP}} > 0.} \tag{2068}$$

Unlike (2045), every coefficient in (2067) is evaluated on the explicitly defined forced principal bridge (2057)–(2060), rather than on the unknown actual solution. This removes the circular dependence on an absorbing radius for the actual pre-entry jets.

264 Size of the forced transverse profile

If the projected evolution family satisfies

$$\boxed{\|\mathcal{U}_N^G(\tau, r)\| \le M_N^G e^{-\lambda_N^G(\tau - r)}, \qquad \lambda_N^G > 0,} \tag{2069}$$

then (2058) gives

$$\boxed{\sup_{\tau \ge \tau_0} \|B_N^G(\tau)\|_{\mathcal{J}_{M+2}} \le \frac{M_N^G}{\lambda_N^G} \sup_{r \ge \tau_0} \|\Pi_G(2C_N^{\mathrm{pr}}(r) G_\perp)\|_{\mathcal{J}_{M+2}}.} \tag{2070}$$

Because $G_\perp$ has only the finite Fourier set (2053), its Sobolev norms are explicit finite constants. Thus the only non-explicit factor in (2070) is the projected bridge stability ratio M_N^G/λ_N^G and the already prescribed coefficient C_N^{pr}.

The forcing is not small merely because it is smooth. Therefore (2070) gives a finite forced-profile radius, not automatically a perturbatively small one. The inequality (2068) still has to be checked with that radius.

265 Fractional block about the forced profile

The exact one-phase cancellation (2030) is no longer invoked. Instead use the smooth finite-N product identity

$$\boxed{\mathfrak{V}_{N,\alpha}(b,A) = AL_\alpha\left(\frac{b}{A}\right) + \frac{b}{A}L_\alpha A.} \tag{2071}$$

At the forced profile $(b_N^{\mathrm{FP}}, A_N^{\mathrm{FP}})$, the fractional contribution is therefore an explicitly evaluable smooth periodic function. Its linearization retains the universal order-2α principal symbol $|\xi|^{2\alpha}$ established in Part 67; the new transverse Fourier modes modify only the finite compact-frequency and lower-order coefficients. Hence there is no new principal-symbol obstruction. The required finite check is precisely their contribution to $\mathfrak{C}_{N,M}^{\mathrm{FP}}$ in (2067).

266 Conditional closure theorem for the forced-profile route

Theorem 66 (Forced-profile pre-entry closure)**.** *Fix $N < \infty$, $1 \le \alpha < 1+\delta$, and a nondegenerate raw bridge on which the normalized continuity inverse of Parts 68–73 exists. Construct b_N^{FP} by (2057)–(2060). Assume the projected estimate (2069), the finite strict margin (2068), and*

$$\boxed{\|\mathfrak{D}_N^{\mathrm{FP}}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^D e^{-\eta_N^D \tau}, \qquad \eta_N^D > 0.} \tag{2072}$$

Then, for sufficiently small remainder at τ_0,

$$\boxed{\|b_N(\tau) - b_N^{\mathrm{FP}}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^{\mathrm{FP}} e^{-\eta_N^{\mathrm{FP}}\tau}, \quad \eta_N^{\mathrm{FP}} < \min\{\lambda_N^{\mathrm{FP}}, \eta_N^D\}.} \tag{2073}$$

Consequently the pre-entry jet estimate needed by the first-crossing argument holds relative to the forced principal bridge.

Proof. Apply Duhamel to (2064), use (2066), (2072), and the quadratic estimate (2065), and close the standard small bootstrap. No one-phase invariance is used. □

267 What can and cannot be certified from the current formulas

The forced-profile construction removes the logical obstruction identified in Part 74: the persistent transverse source is now part of the principal bridge, and the lower-order budget is evaluated on a prescribed bridge rather than on the unknown actual solution. Nevertheless, the current manuscript does not supply numerical values for λ_N^0, M_N^G/λ_N^G, or the complete finite coefficient norm in (2067). It also does not yet prove the decay (2072) of the nonlinear forced-profile defect. Therefore the strict inequality (2068) cannot honestly be declared verified from the displayed formulas alone.

The improvement is that the remaining task is no longer an unknown absorbing estimate for the actual solution. It is a finite calculation on the explicit forced bridge:

$$\boxed{\text{compute } B_N^G \implies (A_N^{\mathrm{FP}}, C_N^{\mathrm{FP}}) \implies \mathfrak{C}_{N,M}^{\mathrm{FP}} \implies \text{test } \mathfrak{C}_{N,M}^{\mathrm{FP}} < \lambda_N^0,} \tag{2074}$$

followed by the explicit defect estimate (2072).

268 Updated completion ledger

flow-box geometry:	closed exactly,
forced transverse Fourier support:	explicit finite set (2053),
forced principal bridge:	defined canonically by (2057)–(2060),
persistent transverse forcing in the remainder:	removed by construction,
continuity response on the forced bridge:	inherited from the normalized raw inverse,
fractional principal symbol:	unchanged, $\lvert\xi\rvert^{2\alpha}$,
forced-profile lower-order margin:	finite test (2068), not numerically certified,
forced-profile defect:	must satisfy (2072),
pre-entry convergence:	follows from (2068), (2069), (2072) by (2073),
first crossing:	conditional on those finite forced-bridge checks,
post-entry conormal and scalar characteristic conclusions:	unchanged conditionally after entry.

(2075)

The forced-profile route therefore resolves the incompatibility between the three-dimensional forcing and the one-phase principal bridge. What remains is quantitatively sharp: a finite projected stability calculation and a finite defect estimate on the explicitly defined forced bridge. These are strictly weaker logical demands than assuming small raw jets of the unknown actual solution, but they are still genuine estimates and are not replaced by smoothness alone.

Part XIX
Spectral Completion of the Forced Bridge and a Rigorous Finite Certification Test

269 An exact Fourier norm for the prescribed transverse forcing

The finite Fourier statement in Part 75 can be made quantitative. Write

$$g(\theta) = \sin^2\theta = \frac{1}{2} - \frac{1}{4}e^{2i\theta} - \frac{1}{4}e^{-2i\theta}, \qquad G_\perp(x,y,z) = g(x)g(y)g(z).$$

If exactly j components of $k \in \mathbb{Z}^3$ are nonzero, then $|k|^2 = 4j$, there are $\binom{3}{j}2^j$ such modes, and the modulus of the corresponding Fourier coefficient is 2^{-3-j}. Consequently, for every $s \geq 0$,

$$\boxed{\|G_\perp\|^2_{H^s(\mathbb{T}^3)} = \sum_{j=0}^{3} \frac{\binom{3}{j}}{64\,2^j}(1+4j)^s.} \tag{2076}$$

After removing the zero mode, the exact transverse norm is

$$\boxed{\|\Pi_\perp G_\perp\|^2_{H^s} = \sum_{j=1}^{3} \frac{\binom{3}{j}}{64\,2^j}(1+4j)^s.} \tag{2077}$$

Thus no unspecified Sobolev constant is needed for the forcing itself.

270 A correction to the finite-mode reduction in Part 75

There is, however, an important distinction between the Fourier support of $G_\perp$ and that of the actual source $2C_N^{\mathrm{pr}}G_\perp$. If C_N^{pr} is not spatially constant, multiplication is convolution in Fourier space:

$$\boxed{\widehat{C_N^{\mathrm{pr}}G_\perp}(k) = \sum_{\ell \in \mathcal{K}_G} \widehat{C}_N^{\mathrm{pr}}(k-\ell)\widehat{G}_\perp(\ell).} \tag{2078}$$

Hence the finite set $\mathcal{K}_G$ in (2053) is the support of $G_\perp$ itself, but it is not in general the complete support of the forced source. The finite-dimensional projected equation (2057) is therefore exact only when C_N^{pr} is constant (as on the auxiliary branch of Part 73), or when the complementary convolution is retained in the principal bridge. This observation is essential for a nondecaying forcing: the omitted complement cannot be placed in an exponentially decaying defect merely from smoothness.

271 The spectrally complete forced principal bridge

The clean correction is to absorb the entire transverse source. Let $\Pi_\perp = I - \Pi_\parallel$ and define B_N^{CF} by

$$\boxed{\partial_\tau B_N^{\mathrm{CF}} = \Pi_\perp \mathcal{L}_N^b(\tau) B_N^{\mathrm{CF}} + \Pi_\perp\big(2C_N^{\mathrm{CF}} G_\perp\big), \qquad B_N^{\mathrm{CF}}(\tau_0) = 0,} \tag{2079}$$

where

$$\boxed{b_N^{\mathrm{CF}} = b_N^\parallel + B_N^{\mathrm{CF}}, \qquad S_N^{\mathrm{CF}} = S_N[Q_N, b_N^{\mathrm{CF}}], \qquad (A_N^{\mathrm{CF}}, C_N^{\mathrm{CF}}) = \mathcal{A}_N(Q_N, S_N^{\mathrm{CF}}).} \tag{2080}$$

Equation (2079) is a nonlinear fixed-point equation because the raw continuity reconstruction feeds B_N^{CF} back into C_N^{CF}. Unlike the finite projection (2057), it contains the full persistent forced transverse response.

Let $\mathcal{U}_N^\perp(\tau, r)$ be the evolution family generated by the transverse restriction of the principal damped operator. If

$$\boxed{\|\mathcal{U}_N^\perp(\tau, r)\|_{H^{M+2} \to H^{M+2}} \le M_N^\perp e^{-\lambda_N^\perp(\tau - r)}, \qquad \lambda_N^\perp > 0,} \tag{2081}$$

and the reconstruction map is Lipschitz on a ball of radius R with constant $L_{N,M}^C(R)$, then Duhamel and the algebra property give

$$\boxed{\|B_N^{\mathrm{CF}}\|_{L^\infty H^{M+2}} \le \frac{2M_N^\perp}{\lambda_N^\perp} \|G_\perp\|_{H^{M+2}} \left(C_{N,0}^{\mathrm{pr}} + L_{N,M}^C(R) R\right).} \tag{2082}$$

A sufficient contraction condition for the complete forced bridge is

$$\boxed{\frac{2M_N^\perp}{\lambda_N^\perp} \|G_\perp\|_{H^{M+2}} L_{N,M}^C(R) < 1,} \tag{2083}$$

with the right side of (2082) not exceeding R. Because the forcing norm in (2083) is given exactly by (2076), the only remaining quantities are raw finite-N reconstruction and stability constants.

272 How λ_N^0 is to be certified rather than guessed

The manuscript defines

$$\lambda_N^0 = \min\{\lambda_N^{\mathrm{cor}}, \lambda_N^{\mathrm{frac}}\}.$$

Neither term has been assigned a numerical lower bound. The correct finite certification is a two-scale spectral calculation. Choose a Fourier cutoff $K \ge 1$ and let P_K project onto $|k| \le K$. On the finite-dimensional space $P_K \mathcal{J}_{M+2}$ form the Hermitian part

$$\boxed{H_{N,K}(\tau) := -\frac{1}{2} P_K \left(\mathcal{L}_{N,\mathrm{prin}}(\tau) + \mathcal{L}_{N,\mathrm{prin}}(\tau)^*\right) P_K.} \tag{2084}$$

Then define the computable compact-frequency gap

$$\boxed{\lambda_{N,K}^{\mathrm{comp}} := \inf_{\tau\in[\tau_0,\tau_e]} \lambda_{\min}(H_{N,K}(\tau)).} \tag{2085}$$

For $|k| > K$, the fractional principal symbol supplies the explicit tail coercivity

$$\boxed{\lambda_{N,K}^{\mathrm{tail}} := \inf_{\tau} \inf_{|k|>K} \left(\nu|k|^{2\alpha} - \mathfrak{r}_{N,M}(\tau,k)\right),} \tag{2086}$$

where $\mathfrak{r}_{N,M}$ is the rigorously bounded lower-order symbol. Thus an actual certified lower bound is

$$\boxed{\underline{\lambda}_N^0(K) := \min\{\lambda_{N,K}^{\mathrm{comp}}, \lambda_{N,K}^{\mathrm{tail}}\}.} \tag{2087}$$

The statement $\underline{\lambda}_N^0(K) > 0$ is a finite matrix eigenvalue check plus an analytic high-frequency tail estimate. It is stronger and more reproducible than assigning a decimal value to λ_N^0 without specifying N, α, the bridge interval, and the coefficient fields.

273 A directly computable replacement for the unresolved margin

Evaluate the complete raw linearization on the spectrally complete bridge and write

$$\boxed{\mathcal{L}_N^{\mathrm{CF}} = \mathcal{L}_{N,\mathrm{prin}} + \mathcal{K}_N^{\mathrm{CF}}.} \tag{2088}$$

For the same cutoff K, define

$$\boxed{\begin{aligned} \mathfrak{C}_{N,M,K}^{\mathrm{comp}} &:= \sup_{\tau} \left\| P_K \frac{\mathcal{K}_N^{\mathrm{CF}} + \mathcal{K}_N^{\mathrm{CF},*}}{2} P_K \right\|, \\ \mathfrak{C}_{N,M,K}^{\mathrm{tail}} &:= \sup_{\tau} \sup_{|k|>K} \left|\Re\,\sigma(\mathcal{K}_N^{\mathrm{CF}})(\tau,k)\right|. \end{aligned}} \tag{2089}$$

The finite-N stability margin is certified if

$$\boxed{\max\{\mathfrak{C}_{N,M,K}^{\mathrm{comp}}, \mathfrak{C}_{N,M,K}^{\mathrm{tail}}\} < \underline{\lambda}_N^0(K).} \tag{2090}$$

This is the rigorous numerical/analytic form of (2068). Every object in (2090) is determined once N, α, the bridge interval, viscosity, forcing amplitude A_0, and the normalized raw continuity solution are specified.

274 The complete forced-profile defect

Because (2079) absorbs the full persistent transverse source, the remaining defect contains only coefficient mismatch, nonlinear Taylor remainder, and principal-profile mismatch. Set

$$\boxed{\mathfrak{D}_N^{\mathrm{CF}} := \partial_\tau b_N^{\mathrm{CF}} - \mathscr{K}_N(b_N^{\mathrm{CF}}).} \tag{2091}$$

After cancellation of the complete source in (2079), Taylor's theorem gives, on a radius-R bridge,

$$\boxed{\|\mathfrak{D}_N^{\mathrm{CF}}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}(R)\left(\|\mathbf{a}_N(\tau)-\mathbf{a}_N^{\mathrm{CF}}(\tau)\|_{\mathcal{X}_N}+\|B_N^{\mathrm{CF}}(\tau)\|_{\mathcal{J}_{M+2}}^2\right)+\|\mathfrak{D}_N^{\parallel}(\tau)\|.} \tag{2092}$$

This formula exposes a second point that cannot be hidden by the bridge construction: a bounded nonzero stationary forced response need not have a decaying quadratic remainder. Therefore exponential defect decay follows only if the complete forced bridge is chosen as an exact (or asymptotically exact) solution of the leading nonlinear forced equation, or if the quadratic stationary part is included in the principal bridge by iteration.

Define the Picard-corrected bridge recursively by

$$\boxed{b_N^{[0]}=b_N^{\parallel},\qquad \partial_\tau b_N^{[m+1]}=\mathscr{K}_{N,\mathrm{prin}}(b_N^{[m+1]})+\Pi_\perp\mathscr{K}_{N,\mathrm{forc}}(b_N^{[m]}),} \tag{2093}$$

with the same normalized continuity reconstruction at every step. Under the contraction condition (2083), this converges to a unique complete forced bridge $b_N^{\mathrm{F},*}$ in the radius-R ball. For that limit the persistent nonlinear transverse terms are part of the bridge, and the residual reduces to the already decaying Lambert/principal-coefficient mismatch:

$$\boxed{\|\mathfrak{D}_N^{\mathrm{F},*}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}^{D,*}e^{-\eta_N^{D,*}\tau}} \tag{2094}$$

provided the previously audited Lambert and raw coefficient convergence is uniform on the chosen bridge. Thus (2072) is not an additional independent assumption once (2083) and that coefficient convergence are certified for the nonlinear complete bridge.

275 Completion theorem in certifiable form

Theorem 67 (Finite-N forced-bridge certification)**.** *Fix all numerical construction data N, $k(N)$, α, ν, A_0 and the finite bridge interval. Suppose the normalized raw continuity inverse satisfies the characteristic estimate of Part 72. If there exist $R>0$ and $K<\infty$ such that the complete-bridge contraction (2083) closes in the radius-R ball, the spectral margin (2090) is strict, and the Lambert/raw coefficient convergence entering (2094) holds, then the nonlinear complete forced bridge $b_N^{\mathrm{F},*}$ exists and*

$$\boxed{\|b_N(\tau)-b_N^{\mathrm{F},*}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}e^{-\eta_N\tau}} \tag{2095}$$

for sufficiently small initial bridge mismatch. Consequently the pre-entry jet hypothesis used by the first-crossing argument is certified for that fixed N.

Proof. The contraction estimate (2083) constructs the complete forced bridge. Equation (2090) gives a positive spectral gap for its linearization. The coefficient convergence gives (2094). Duhamel and the quadratic remainder estimate then close the standard small bootstrap and yield (2095). □

276 What the present manuscript data actually decide

Equations (2076)–(2095) remove two hidden ambiguities from Part 75. First, the product $C_N G_\perp$ is not generally supported on the finite set $\mathcal{K}_G$; the complete convolution must be included. Second, a nondecaying quadratic forced response must be incorporated into the nonlinear principal bridge rather than placed in the defect.

The current source still does not specify one numerical instance containing all of

$$N, \quad k(N), \quad \alpha, \quad \nu, \quad A_0, \quad [\tau_0, \tau_e], \quad \text{and the normalized bridge coefficient tables.}$$

Therefore no numerical value of $\underline{\lambda}_N^0(K)$ or of the left side of (2090) can be extracted honestly from the manuscript as it stands. What is now complete is the certification problem itself: after those data are fixed, (2076), (2083), (2085)–(2090), and (2094) are the finite list of quantities that must be evaluated. No unknown actual-solution absorbing radius and no false finite-support assumption remains.

277 Updated completion ledger

forcing Sobolev norms:	exact formula (2076)–(2077),
finite support of $G_\perp$:	exact, but not preserved by multiplication with nonconstant C_N,
complete forced source:	incorporated by (2079)–(2080),
complete forced bridge:	constructed conditionally by the contraction test (2083),
principal spectral gap:	reduced to finite matrix plus analytic tail, (2084)–(2087),
full lower-order margin:	certifiable by the explicit test (2090),
forced nonlinear defect:	absorbed by the Picard-complete bridge (2093),
defect decay:	follows as (2094) once raw/Lambert coefficient convergence is certified on that bridge,
pre-entry raw-jet convergence:	follows from the finite certification theorem, (2095),
first crossing:	conditional only on these fixed-N certification inequalities,
post-entry scalar conclusions:	unchanged after certified entry.

(2096)

This closes the structural analysis of the forced bridge. The remaining work is no longer a missing derivation: it is a numerical/interval-arithmetic certification for a fully specified finite-N parameter set. Without that parameter set, assigning numerical values to λ_N^0 or to the margin would be unsupported.

Part XX
Interval-Certification Input Closure and Reproducible Finite-N Test

278 Why the next step must begin with a parameter record

Part 76 reduced the pre-entry problem to a finite numerical/analytic certificate, but a certificate is meaningful only after the finite-N problem has been fixed completely. My work contains several numerical experiments with different values of N, different cascade depths, and different regularizations. Those experiments cannot be silently substituted into (2090), because (2090) refers to the *raw spectrally complete forced bridge* and its normalized continuity solution. Accordingly define the certification record

$$\boxed{\mathfrak{P}_N = \left(N, k_N, \alpha, \nu, A_0, \tau_0, \tau_e, M, K, \varepsilon_N^{\mathrm{L}}, \mathfrak{N}_N^{\mathrm{cont}}, \mathbf{T}_N^{\mathrm{br}}\right).} \tag{2097}$$

Here $\mathfrak{N}_N^{\mathrm{cont}}$ records the normalization used for the raw continuity inverse, and $\mathbf{T}_N^{\mathrm{br}}$ denotes the interval-valued coefficient tables for the complete forced bridge. No entry in (2097) may be inherited from an earlier numerical experiment unless it is verified to belong to the same reconstruction and regularization.

279 The bridge table that is actually required

Let $I_j = [\tau_j, \tau_{j+1}]$ be a partition of $[\tau_0, \tau_e]$. On each interval store outward-rounded enclosures for

$$\boxed{\mathbf{T}_{N,j}^{\mathrm{br}} = \left[Q_N, S_N^{\mathrm{F},*}, b_N^{\mathrm{F},*}, A_N^{\mathrm{F},*}, C_N^{\mathrm{F},*}, \partial^\mu Q_N, \partial^\mu S_N^{\mathrm{F},*}, \partial^\mu b_N^{\mathrm{F},*}\right]_{|\mu|\le M+2}.} \tag{2098}$$

The algebraic reconstruction is evaluated interval-wise from

$$\boxed{V_N = Q_N + S_N^{\mathrm{F},*}, \qquad D_N(V) = \frac{-(2V + R_N) + \sqrt{R_N^2 + 8R_N V}}{2},} \tag{2099}$$

$$\boxed{A_N^{\mathrm{F},*} = V_N - D_N(V_N), \qquad C_N^{\mathrm{F},*} = V_N + D_N(V_N).}$$

Every interval must certify the nondegeneracy conditions

$$\boxed{\inf R_N > 0, \qquad \inf V_N > 0, \qquad \inf A_N^{\mathrm{F},*} > 0, \qquad \inf C_N^{\mathrm{F},*} > 0.} \tag{2100}$$

These inequalities prevent interval evaluation of the square root, reciprocal reconstruction, and continuity coefficients from crossing a singular set.

280 Certified continuity inverse on the same table

Part 73 gives the exact flow-box coordinate $\omega = y+z$ and $L_N^{\mathrm{fl}} = \pi$. Hence the only nontrivial interval contribution to the characteristic inverse is the coefficient weight and differentiated flow. For each bridge cell define

$$\boxed{C^{c}_{N,j} := \sup_{I_j\times\mathbb{T}^3} |c_N|, \qquad \mathfrak{K}^{(0)}_{N,j} := \pi e^{\pi C^{c}_{N,j}}.} \tag{2101}$$

Differentiating the characteristic ODE through order $M+1$ and applying interval Gronwall gives recursively computable constants $\mathfrak{K}^{(m)}_{N,j}$, $1 \le m \le M+1$. Set

$$\boxed{\mathfrak{K}^{\mathrm{fl}}_{N,M+1} := \max_j \max_{0\le m\le M+1} \mathfrak{K}^{(m)}_{N,j}.} \tag{2102}$$

Then the range-adapted continuity response of Part 69 has the certified bound

$$\boxed{\|\Theta_N Z\|_{H^M} \le \mathfrak{K}^{\Theta}_{N,M}\|Z\|_{H^M}, \qquad \mathfrak{K}^{\Theta}_{N,M} \le \mathfrak{C}^{\Theta,\mathrm{alg}}_{N,M}\mathfrak{K}^{\mathrm{fl}}_{N,M+1}.} \tag{2103}$$

All factors on the right are interval functions of (2098)–(2100).

281 Interval construction of the complete forced bridge

Let $\mathscr{T}_N$ denote the Duhamel/Picard map defined by (2079) and (2093), now evaluated with outward rounding. For a ball $\mathbb{B}_R \subset L^\infty H^{M+2}$ compute interval bounds

$$\boxed{\mathfrak{q}_N(R) := \frac{2M_N^{\perp}}{\lambda_N^{\perp}}\|G_\perp\|_{H^{M+2}} L^{C}_{N,M}(R),} \tag{2104}$$

$$\boxed{\mathfrak{r}_N(R) := \frac{2M_N^{\perp}}{\lambda_N^{\perp}}\|G_\perp\|_{H^{M+2}}\big(C^{\mathrm{pr}}_{N,0} + L^{C}_{N,M}(R)R\big).} \tag{2105}$$

A machine-checkable existence/uniqueness certificate is

$$\boxed{\overline{\mathfrak{q}_N(R)} < 1, \qquad \overline{\mathfrak{r}_N(R)} \le R.} \tag{2106}$$

Because $\|G_\perp\|_{H^{M+2}}$ is given exactly by (2076), no quadrature is needed for that factor. Once (2106) holds, interval Picard iteration produces an enclosure of $b_N^{\mathrm{F},*}$ and hence closes the table (2098).

282 Low-frequency matrix certificate

For every τ-cell form the interval Hermitian matrix

$$\boxed{\mathbf{H}_{N,K,j} = -\frac{1}{2}P_K\big(\mathbf{L}_{N,\mathrm{prin,j}} + \mathbf{L}^{*}_{N,\mathrm{prin,j}}\big)P_K.} \tag{2107}$$

A rigorous lower eigenvalue bound may be obtained by any verified Hermitian interval method; for example, after choosing a numerical center matrix $H^{c}_{N,K,j}$,

$$\boxed{\lambda^{\mathrm{comp},-}_{N,K,j} := \lambda_{\min}(H^{c}_{N,K,j}) - \|\mathbf{H}_{N,K,j} - H^{c}_{N,K,j}\|_2^{+}.} \tag{2108}$$

Therefore

$$\boxed{\lambda^{\mathrm{comp},-}_{N,K} := \min_{j} \lambda^{\mathrm{comp},-}_{N,K,j}} \tag{2109}$$

is a rigorous lower bound for (2085). This step is finite dimensional; its validity does not rely on floating-point eigenvalues being exact because the interval remainder in (2108) absorbs rounding and coefficient uncertainty.

283 High-frequency analytic certificate

Let the lower-order tail satisfy the verified symbol estimate

$$\boxed{|\mathfrak{r}_{N,M}(\tau,k)| \le \sum_{j=0}^{J} R_{N,j}|k|^{\rho_j}, \qquad \rho_j < 2\alpha.} \tag{2110}$$

Then for $|k| \ge K+1$,

$$\boxed{\lambda^{\mathrm{tail},-}_{N,K} := \nu(K+1)^{2\alpha} - \sum_{j=0}^{J} R_{N,j}(K+1)^{\rho_j}} \tag{2111}$$

is valid whenever the right-hand side of $\nu r^{2\alpha} - \sum_j R_{N,j} r^{\rho_j}$ is increasing for $r \ge K+1$. A sufficient interval-checkable monotonicity condition is

$$\boxed{2\alpha\nu(K+1)^{2\alpha-1} > \sum_{j=0}^{J} \rho_j R_{N,j}(K+1)^{\rho_j-1}.} \tag{2112}$$

Thus the certified principal gap is

$$\boxed{\underline{\lambda}^{0,-}_{N}(K) := \min\{\lambda^{\mathrm{comp},-}_{N,K}, \lambda^{\mathrm{tail},-}_{N,K}\}.} \tag{2113}$$

284 Interval form of the full margin

Evaluate the Hermitian lower-order correction on each compact-frequency cell and its symbol on the tail. Let

$$\boxed{\begin{aligned} \mathfrak{C}^{\mathrm{comp},+}_{N,M,K} &:= \max_{j} \left\| P_K \frac{\mathbf{K}^{\mathrm{F},*}_{N,j} + \mathbf{K}^{\mathrm{F},*,*}_{N,j}}{2} P_K \right\|_2^{+}, \\ \mathfrak{C}^{\mathrm{tail},+}_{N,M,K} &:= \sup_{j} \sup_{|k|>K} \left| \Re\, \sigma(\mathbf{K}^{\mathrm{F},*}_{N,j})(k) \right|^{+}. \end{aligned}} \tag{2114}$$

Define the single certificate number

$$\boxed{\mathfrak{m}^{\mathrm{cert}}_{N,M,K} := \underline{\lambda}^{0,-}_N(K) - \max\{\mathfrak{C}^{\mathrm{comp},+}_{N,M,K}, \mathfrak{C}^{\mathrm{tail},+}_{N,M,K}\}.} \tag{2115}$$

Then the unresolved strict inequality (2090) is proved exactly when

$$\boxed{\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0.} \tag{2116}$$

If the interval in (2115) contains zero, the computation is *inconclusive*, not negative: one must refine the τ partition, increase precision, improve coefficient enclosures, or change K.

285 The raw coefficient-decay certificate

The remaining hypothesis in (2094) can be tested on the same bridge table. Let $\mathbf{a}_N$ collect the raw/Lambert coefficients entering the complete bridge and let $\mathbf{a}^{\mathrm{F},*}_N$ denote their principal forced-bridge values. Define

$$\boxed{\mathfrak{d}_{N,j} := \max_{0\le m\le M+2} \sup_{\tau\in I_j} e^{\eta\tau}\|E^m(\mathbf{a}_N - \mathbf{a}^{\mathrm{F},*}_N)\|_{\mathcal{X}_N}.} \tag{2117}$$

If for some $\eta > 0$

$$\boxed{\mathfrak{d}^{\mathrm{cert}}_N := \max_j \overline{\mathfrak{d}_{N,j}} < \infty,} \tag{2118}$$

then

$$\boxed{\|\mathfrak{D}^{\mathrm{F},*}_N(\tau)\|_{\mathcal{J}_{M+2}} \le C^{D,*}_{N,M}\mathfrak{d}^{\mathrm{cert}}_N e^{-\eta\tau}.} \tag{2119}$$

This is the interval-arithmetic realization of (2094). It uses only pre-entry bridge data and therefore does not invoke the post-entry conormal bootstrap.

286 A complete certificate and its logical consequence

Theorem 68 (Reproducible finite-N interval certificate)**.** *Fix a parameter record $\mathfrak{P}_N$ in (2097). Suppose outward-rounded interval arithmetic verifies (2100), (2106), the differentiated characteristic bounds entering (2102)–(2103), the low-frequency bounds (2108)–(2109), the high-frequency bounds (2110)–(2112), the strict margin (2116), and the coefficient-decay estimate (2118). Then the hypotheses of the finite-N forced-bridge certification theorem of Part 76 hold for that record. In particular, the complete forced bridge exists uniquely in the certified radius-R ball and*

$$\boxed{\|b_N(\tau) - b^{\mathrm{F},*}_N(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M}e^{-\eta_N\tau}.} \tag{2120}$$

Consequently the pre-entry jet hypothesis required by the first-crossing argument is certified for that fixed N.

Proof. Conditions (2100), (2102)–(2103) certify the raw reconstruction and continuity inverse on the bridge. Condition (2106) is the contraction criterion (2083), hence constructs the nonlinear complete forced bridge. Equations (2108)–(2113) give a rigorous lower bound for the principal gap, while (2114)–(2116) prove the full lower-order margin (2090). Finally, (2118)–(2119) give the decaying complete-bridge defect (2094). The theorem of Part 76 then yields (2120). □

287 What can and cannot be filled from the existing manuscript

The structural entries of the certificate are now fixed: the exact forcing norm, exact flow-box flight time, exact transverse section, algebraic raw reconstruction, continuity normalization, compact/tail splitting, and the single margin number (2115). The numerical runs elsewhere in the manuscript do not supply a complete record (2097) for the present raw forced-bridge architecture. In particular, a previously used pair such as (N, k_N) does not determine α, ν, A_0, the precise bridge interval, the cutoff K, or the interval coefficient tables required by (2098).

Therefore the present analysis reaches the strongest conclusion supported by the available data:

$$\boxed{\text{the finite-}N\text{ proof is reduced to the sign of the explicitly defined interval quantity } \mathfrak{m}^{\mathrm{cert}}_{N,M,K},\ \text{together with (2106) and (2118).}} \tag{2121}$$

No numerical sign is asserted until a single complete parameter record is supplied and the corresponding bridge tables are generated with directed rounding. This is a data requirement, not a remaining ambiguity in the mathematical statement of the certification problem.

288 Updated certification ledger

complete parameter record:	defined by (2097); numerical entries still to be fixed,
bridge coefficient table:	exact required contents specified in (2098),
raw nondegeneracy:	finite interval test (2100),
flow-box inverse:	exact $L_N^{\mathrm{fl}} = \pi$ plus interval derivative bounds (2101)–(2103),
forced-bridge existence:	finite contraction/radius test (2106),
compact spectral gap:	rigorous interval eigenvalue test (2107)–(2109),
high-frequency gap:	analytic interval tail test (2110)–(2113),
full stability margin:	single signed certificate (2115)–(2116),
raw/Lambert defect decay:	finite weighted interval test (2117)–(2119),
pre-entry raw-jet convergence:	theorem (2120) once the certificate is positive,
first crossing:	then receives the certified pre-entry hypothesis.

(2122)

The next computational object is therefore not another formal estimate but a single reproducible certificate file containing $\mathfrak{P}_N$, the outward-rounded tables (2098), and the three final outputs $\overline{\mathfrak{q}_N(R)}$, $\mathfrak{m}^{\mathrm{cert}}_{N,M,K}$, and $\mathfrak{d}^{\mathrm{cert}}_N$. A positive certificate can then be checked independently without reconstructing the preceding proof architecture.

289 Concluding comments

Not yet from

$$\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0$$

alone do we have the finite time blowup result.

What we can now conclude precisely is:

$$\boxed{\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0}$$

rigorously establishes the stability inequality (2090). But the manuscript's own finite-N certificate theorem also requires the other certified conditions—especially the forced-bridge contraction (2106), the coefficient-decay estimate (2118), and the raw reconstruction/continuity bounds. With all of those verified, it proves

$$\|b_N(\tau) - b_N^{\mathrm{F},*}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M} e^{-\eta_N \tau},$$

and therefore supplies the pre-entry hypothesis needed by the first-crossing argument.

Then, **if the first-crossing conditions are consequently satisfied**, the earlier theorem gives entry into the certified terminal corridor in finite logarithmic time. Once the established terminal persistence theorem supplies the two-sided law

$$c_- s^{-(2\alpha-1)} \le b_{N,\alpha}(s) \le c_+ s^{-(2\alpha-1)},$$

the exact characteristic equation

$$\dot{s} = 1 - b_{N,\alpha}$$

does give finite physical time. Indeed, for sufficiently small s, $b \ge 2$, so

$$\frac{1}{2} b \le -\dot{s} = b - 1 \le b.$$

Hence

$$\frac{1}{c_+} s^{2\alpha-1} \le \frac{dt}{-ds} \le \frac{2}{c_-} s^{2\alpha-1}.$$

Integration to $s = 0$ gives

$$\boxed{T_{*,N} - t \asymp s^{2\alpha} < \infty}$$

and therefore

$$\boxed{b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}} \longrightarrow \infty \qquad (t \uparrow T_{*,N}).}$$

That finite-time conversion is already explicitly proved in the manuscript.

So the theorem we may legitimately state is:

For a fixed finite N, suppose the complete interval certificate verifies the raw nondegeneracy and continuity bounds, the forced-bridge contraction condition (2106), the strict stability certificate

$$\mathfrak{m}^{\mathrm{cert}}_{N,M,K} > 0,$$

and the raw/Lambert coefficient-decay condition (2118). If these certified estimates supply the hypotheses of the finite-N first-crossing and terminal-persistence theorems, then the corresponding scalar reconstructed quantity $b_{N,\alpha}$ enters the terminal regime and blows up at a finite time $T_{*,N} < \infty$, with

$$b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-(2\alpha-1)/(2\alpha)}.$$

There is one distinction I would preserve very carefully in the manuscript: **this establishes a conditional finite-N scalar blow-up theorem once the complete certificate is actually positive.**

So I would **not** change the conclusion to "$\mathfrak{m}^{\mathrm{cert}} > 0$ alone proves finite-time blow-up." I would change it to the stronger and mathematically accurate implication

$$\boxed{\text{complete positive finite-}N\text{ certificate} \Longrightarrow \text{first crossing} \Longrightarrow \text{terminal persistence} \Longrightarrow T_{*,N} < \infty, \quad b_{N,\alpha}(t) \to \infty.}$$

That is now the appropriate finite-N blow-up conclusion supported by the proof architecture.

290 Conclusion

This work develops a detailed analytical framework for studying finite-time amplification in a reconstructed scalar quantity associated with the three-dimensional incompressible Navier–Stokes equations with fractional dissipation on the periodic torus. The central object is the product

$$b_3 = u_x u_y,$$

whose evolution is obtained directly from the momentum equations rather than introduced as an independent model. The analysis combines the exact product equation, fractional product identities, Lambert-W renormalization, one-phase reductions, Mellin analysis, characteristic dynamics, continuity reconstruction, spectral estimates, and finite-N certification. A principal purpose of the paper has been to determine which parts of the proposed singular mechanism are exact consequences of the equations, which are consequences of finite-N estimates, and which still require quantitative verification.

For the fractional exponent, the analysis is organized on a small supercritical interval

$$1 < \alpha < 1 + \delta.$$

The similarity exponent arising from the terminal balance is not fixed at a special value but varies with α according to

$$r_\alpha = 2\alpha - 1.$$

This leads naturally to the candidate terminal behavior

$$b_3(s) \asymp s^{-(2\alpha-1)}.$$

The associated characteristic law

$$\dot{s} = 1 - b_3$$

then provides the mechanism converting a spatial or phase singularity into a finite physical time singularity. In particular, once two-sided terminal estimates for b_3 are established, the characteristic equation gives

$$T_* - t \asymp s^{2\alpha},$$

and hence

$$b_3(t) \asymp (T_* - t)^{-\frac{2\alpha-1}{2\alpha}}.$$

Thus the finite-time conversion itself is not an additional conjectural step: it follows directly from the terminal power law and the exact characteristic dynamics.

A substantial part of the paper is devoted to establishing the profile theory that underlies this terminal regime. After passing to logarithmic coordinates, the fractional profile equation

is represented through a Mellin multiplier. At the classical anchor $\alpha = 1$, the profile is explicit, the linearized equation can be solved exactly, and its two fundamental solutions can be determined. The apparent Fredholm obstruction that arises from a formal adjoint calculation disappears once the correct weighted spaces are used: the formal dual zero mode does not belong to the relevant dual space. The only admissible kernel direction is generated by translation, and this direction is removed by a phase condition. The resulting phase-fixed operator possesses an explicit inverse.

The Mellin multiplier is then controlled uniformly for α near 1. Gamma-function and digamma estimates yield the required operator bounds and differentiability with respect to α. Introducing a common Mellin graph domain resolves the difficulty caused by the changing order of the fractional operator. The implicit-function argument can consequently be carried out on fixed Banach spaces, producing a positive profile branch for all

$$1 < \alpha < 1 + \delta_{\mathrm{prof}}$$

for some $\delta_{\mathrm{prof}} > 0$. The endpoint asymptotics and positivity of this branch are inherited continuously from the explicit anchor. This establishes the profile component of the proposed mechanism on an interval of fractional exponents rather than at a single distinguished exponent.

The finite-N Lambert construction provides the second major component of the analysis. The renormalized Lambert cascade is kept on the principal real branch, and its contraction properties are used to control the logarithmic sensitivity of the coefficients. The analysis distinguishes carefully between the literal Lambert iteration, the renormalized branch-distance dynamics, the finite-N branch layer, and the coupled limits in N, cascade depth, and regularization. This distinction is essential because the different limits need not commute. The paper therefore avoids identifying a finite-N singular law with a limiting singular solution without an independent compactness argument.

The reconstruction of the complete velocity field introduces additional geometric and analytic constraints. Several stages of the paper audit the relation between the scalar variable b_3, the reconstructed components u_x, u_y, u_z, the continuity equation, and the pressure. This leads to a separation between two reconstruction architectures that had initially been used interchangeably. The later analysis is therefore performed directly for the raw two-correction reconstruction

$$u_y = A_N, \qquad u_z = C_N, \qquad u_x = \frac{b_N}{A_N},$$

with

$$A_N = Q_N + S_N - D_N, \qquad C_N = Q_N + S_N + D_N.$$

This removes the architectural ambiguity and permits the fractional and continuity calculations to be rederived from the actual variables of the construction.

For this raw reconstruction, the viscous product block has the exact form

$$A_N(-\Delta)^{\alpha}\left(\frac{b_N}{A_N}\right) + \frac{b_N}{A_N}(-\Delta)^{\alpha}A_N.$$

Its normalized linearization has universal principal symbol

$$|\xi|^{2\alpha}.$$

Consequently the leading fractional part is coercive at high frequency. The genuinely difficult information is contained in the bounded-frequency sector and in the lower-order response of the continuity reconstruction. This observation converts an initially infinite-dimensional sign problem into a finite compact-frequency certification supplemented by an analytic high-frequency estimate.

The continuity equation is likewise treated as part of the dynamics rather than as an automatically solvable constraint. Linearization produces a first-order transport operator for the correction S_N. The analysis shows that a zero-mean condition alone does not justify an isotropic one-derivative inverse estimate. Instead, the correct inverse is obtained through the characteristic flow of the transport field. For the raw reconstruction, the transverse combination of the transport coefficients satisfies an exact identity that yields a convenient flow-box coordinate. On a 2π transverse slab the flight time is

$$L_N^{\mathrm{fl}} = \pi,$$

and the chosen transverse section is uniformly noncharacteristic. Thus the geometric part of the continuity inverse can be controlled by explicit finite-N characteristic constants.

The prescribed forcing introduces an important further refinement. Although the strict one-phase reconstruction is internally consistent and enjoys exact one-dimensional fractional reduction, the full forcing contains genuine transverse dependence. The actual forced evolution therefore cannot in general remain on the strict one-phase manifold. Rather than treating this as an uncontrolled error, the paper incorporates the persistent transverse forcing into the principal bridge itself. The resulting forced principal bridge contains the complete transverse response, while the remainder is required to decay relative to that bridge. This prevents the terminal argument from assuming convergence toward a manifold that is not invariant under the prescribed forcing.

The stability problem about this forced bridge is then divided into low and high frequencies. On a finite Fourier block, the Hermitian part of the linearized operator is represented by a finite matrix whose smallest eigenvalue can be enclosed by interval arithmetic. Outside that block, the fractional term dominates because its symbol grows like

$$\nu|k|^{2\alpha},$$

whereas all lower-order terms grow with strictly smaller powers of $|k|$. This gives a rigorous analytic tail estimate. The two pieces define a certified principal spectral gap

$$\underline{\lambda}_N^{0,-}(K).$$

The complete lower-order perturbation is then enclosed independently on the compact and high-frequency sectors.

These estimates culminate in the certificate quantity

$$\mathfrak{m}_{N,M,K}^{\mathrm{cert}} = \underline{\lambda}_N^{0,-}(K) - \max\left\{\mathfrak{C}_{N,M,K}^{\mathrm{comp},+}, \mathfrak{C}_{N,M,K}^{\mathrm{tail},+}\right\}.$$

The condition

$$\mathfrak{m}_{N,M,K}^{\mathrm{cert}} > 0$$

is therefore the rigorous finite-N realization of the strict stability margin required in the bridge argument. It is not based on an unverified numerical eigenvalue: the compact-frequency part is enclosed by outward-rounded finite-dimensional computations and the tail is controlled analytically. If an interval enclosure for this quantity contains zero, the result is simply inconclusive and can be refined by increasing precision, changing the Fourier cutoff, or sharpening the coefficient enclosures.

The stability margin is one component of the complete finite-N certificate. The construction also requires raw nondegeneracy, the characteristic continuity bounds, existence of the nonlinear forced bridge through a contraction estimate, and decay of the difference between the raw coefficients and their forced-bridge counterparts. The latter is encoded in a weighted coefficient certificate. When these conditions are simultaneously verified for a single consistent parameter record, the complete forced bridge exists and the actual scalar trajectory satisfies

$$\|b_N(\tau) - b_N^{\mathrm{F},*}(\tau)\|_{\mathcal{J}_{M+2}} \le C_{N,M} e^{-\eta_N \tau}.$$

This is precisely the pre-entry jet control required by the first-crossing argument.

The first-crossing analysis then supplies the connection between the prescribed finite-N dynamics and the terminal profile regime. The normalized scalar equation possesses a restoring sign on every bounded corridor, while the differentiated shape variables are controlled by the certified spectral gap. Once the complete bridge estimates hold, the scalar mismatch and its required Euler derivatives enter the terminal contraction region after a finite logarithmic time. From that point onward, the terminal bootstrap and invariant-region argument propagate the profile asymptotics. Thus the proof architecture forms a continuous chain

$$\text{finite-}N\text{ reconstructed data} \longrightarrow \text{certified forced bridge} \longrightarrow \text{first crossing} \longrightarrow \text{terminal persistence} \longrightarrow \text{finite-time scalar amplification.}$$

For a fixed finite N, the resulting conclusion is particularly clear. Suppose a complete, internally consistent parameter record is chosen and the interval calculations verify the reconstruction and continuity conditions, the forced-bridge contraction, the positive stability margin

$$\mathfrak{m}_{N,M,K}^{\mathrm{cert}} > 0,$$

and the required raw/Lambert coefficient decay. The bridge theorem then supplies the pre-entry estimates, the first-crossing theorem gives entry into the terminal region, and the persistence argument yields

$$c_- s^{-(2\alpha-1)} \le b_{N,\alpha}(s) \le c_+ s^{-(2\alpha-1)}.$$

Since

$$\dot{s} = 1 - b_{N,\alpha},$$

the remaining time satisfies

$$T_{*,N} - t \asymp s^{2\alpha},$$

and consequently

$$b_{N,\alpha}(t) \asymp (T_{*,N} - t)^{-\frac{2\alpha-1}{2\alpha}} \longrightarrow \infty \qquad \text{as } t \uparrow T_{*,N} < \infty.$$

Accordingly, the paper reduces the finite-N scalar blow-up mechanism to a reproducible collection of finite inequalities rather than leaving the decisive bridge and stability assumptions as qualitative hypotheses.

At the same time, the analysis identifies an important boundary on what this conclusion establishes. Blow-up of the reconstructed scalar $b_{N,\alpha}$ at a fixed finite N is not by itself a proof of finite-time singularity for a limiting three-dimensional Navier–Stokes solution. Passing from the finite-N scalar construction to such a statement requires control of the complete reconstructed velocity in the appropriate fixed spatial function spaces, compatibility of all three momentum equations and the pressure throughout the evolution, sufficient compactness to pass the nonlinear and fractional terms to a limit, and preservation of the required estimates under the coupled limiting procedure. The paper therefore separates the finite-N terminal mechanism from the substantially stronger limiting Navier–Stokes assertion.

The overall outcome is thus both constructive and diagnostic. The profile problem near the classical exponent is placed on a common functional domain and continued to a nonempty supercritical interval; the terminal exponent and finite-time conversion are determined explicitly; the Lambert cascade is controlled without confusing distinct limiting regimes; the raw velocity reconstruction and continuity response are incorporated into the same architecture; the persistent transverse forcing is absorbed into a forced principal bridge; and the remaining stability question is reduced to a low-frequency interval matrix calculation together with an analytic fractional tail. Successive audits remove several assumptions that would otherwise have entered the argument circularly and replace them by quantities that can, in principle, be checked independently.

The final mathematical task at the finite-N level is therefore sharply defined. For a chosen parameter record, one must construct the outward-rounded bridge tables and verify the contraction, decay, and positive-margin certificates. A successful certificate establishes the hypotheses leading from the actual reconstructed finite-N dynamics to the terminal scalar regime and hence to finite-time divergence of $b_{N,\alpha}$. Any further claim concerning a genuine singular solution of the full three-dimensional Navier–Stokes system requires the additional velocity–pressure and limiting arguments described above. In this way, the paper ends not with an unresolved formal singular ansatz, but with a precise separation between the analytical mechanism already established, the finite computations required to certify that mechanism for a specified reconstruction, and the additional compactness and compatibility questions that must be resolved before a full Navier–Stokes blow-up theorem can be asserted.