

Global BV Solutions to a System of Balance Laws Arising in Traffic Flow

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Brief History of Traffic Flow

- ▶ Lighthill-Whitham-Richards (LWR) model [1955, 1956]
- ▶ Payne-Whitham (PW) model [1971, 1974]
- ▶ Viscous models studied by Kerner-Konhauser, Kuhne, Beckshulte, and Li [1984-1994, 2008]
- ▶ Aw-Rascle and Zhang's higher continuum (ARZ) models [2000, 2001]

Global BV Solution to a system of balance laws with *nonconcave* flux

- ▶ Goal - Existence and long time behavior of BV solutions to Cauchy problem
- ▶ Context - System of balance laws with nonconcave fundamental diagrams from traffic flow
- ▶ Study of nonconcave flux is an open problem in the framework of Dafermos [3] and are seen in experiment data (See [6] and references there in)
- ▶ One of the characteristic fields of this system is neither linearly degenerate nor genuinely nonlinear

Work Overview

- ▶ We showed in the existence of a global BV solution for a system of balance laws arising in traffic flow in the framework of Dafermos [3]
- ▶ Verified Several Conditions
- ▶ Solved the problem first for concave fundamental diagram and extended results to nonconcave
- ▶ The paper was published in Networks and Heterogeneous Media journal [1]

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Nonlinear Balance Laws

- ▶ Let $U \in \mathbb{R}^n$.
- ▶ $U = (u_1, u_2, \dots, u_n)$, $F(U) = (f_1(u), f_2(u), \dots, f_n(u))$
- ▶ Consider the general conservation form

$$U_t + F(U)_x + P(U) = 0 \quad (1)$$

with initial data

$$U(x, 0) = U_0(x), \quad (2)$$

where $x \in \mathbb{R}$, $t > 0$.

Model

- ▶ To analyze the 2×2 traffic flow model:

$$\begin{aligned}\rho_t + (\rho v)_x &= 0, \\ v_t + \left(\frac{1}{2}v^2 + g(\rho)\right)_x + \frac{v - v_e(\rho)}{\tau} &= 0,\end{aligned}\tag{3}$$

with initial data

$$(\rho(x, 0), v(x, 0)) = (\rho_0(x), v_0(x))\tag{4}$$

where $x \in \mathbb{R}$, $t > 0$, $\tau > 0$,

- ▶ ρ - density, v - velocity, $v_e(\rho)$ - equilibrium velocity.
- ▶ $g(\rho)$ - anticipation factor and satisfies

$$g'(\rho) = \rho(v_e'(\rho)/\theta)^2,\tag{5}$$

where $g'(\rho) \geq 0$, and

$$\rho_0(x) \geq \delta_1 > 0, \quad 0 < \theta < 1\tag{6}$$

for δ_1 constant.

LWR Model

- ▶ The equilibrium flow is described by Lighthill-Whitham-Richards (LWR) model [5, 6]

$$\rho_t + (\rho v_e(\rho))_x = 0, \quad x \in \mathbb{R}, t > 0, \quad (7)$$

with initial data $\rho(x, 0) = \rho_0(x) > 0$.

- ▶ $q(\rho) = \rho v_e(\rho)$ is known as the fundamental diagram.
- ▶ For this study, we let $v_e(\rho)$ such that

$$v_e(0) = b, v_e'(0) = -a, v_e(1) = 0, \quad (8)$$

where $v_e'(\rho) < 0$.

First Transformation

- ▶ We want $P(0) = 0$ we need to do a change of variables $v = u + b$.
- ▶ Now we can rewrite (3) as follows

$$\begin{aligned}\rho_t + (\rho(u + b))_x &= 0 \\ u_t + \left(\frac{1}{2}(u + b)^2 + g(\rho)\right)_x + \frac{u + b - v_e(\rho)}{\tau} &= 0\end{aligned}\tag{9}$$

- ▶ In terms of the general form, we have

$$\begin{aligned}U &= (\rho, v - b) = (\rho, u)^T \\ F(U) &= (\rho(u + b), \frac{1}{2}u^2 + ub + g(\rho))^T \\ P(U) &= (0, \frac{u + b - v_e(\rho)}{\tau})\end{aligned}\tag{10}$$

Eigenvalues Eigenvectors

- ▶ The Jacobian is

$$\begin{bmatrix} u + b & \rho \\ g'(\rho) & u + b \end{bmatrix} \quad (11)$$

- ▶ Using (5), the eigenvalues are

$$\lambda_{1,2} = u + b \pm \rho \frac{v'_e(\rho)}{\theta}. \quad (12)$$

- ▶ The corresponding right eigenvectors are

$$r_{1,2} = \left(\pm \frac{\theta}{v'_e(\rho)}, 1 \right)^T. \quad (13)$$

- ▶ The system (9) is genuinely nonlinear since

$$\nabla \lambda_i \cdot r_i = \frac{q''(\rho)}{v'_e(\rho)} \quad i = 1, 2. \quad (14)$$

- ▶ We note that $q''(\rho)$ changes signs. So the characteristic fields are **not genuinely nonlinear**.

Obstacles

In order to apply Dafermos' theory [3], we had to

- ▶ Search for a convex entropy-entropy flux pair
- ▶ Solve the wave equation with variable coefficients
- ▶ Do perturbation analysis to extend results to nonconcave fundamental diagram
- ▶ Transform system (3) once again into equivalent form and do an *a priori* estimate

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Entropy-Entropy Flux Pair

- ▶ We need to find smooth entropy flux pair $(\eta, q)(U)$ where η is convex and has been normalized by $\eta(0)$, $D\eta(0) = 0$.
- ▶ This is important since admissible solutions U must satisfy the entropy inequality

$$\partial_t \eta(U(x, t)) + \partial_x q(U(x, t)) + D\eta(U(x, t))P(U(x, t)) \leq 0 \quad (15)$$

- ▶ We also want our system to be a **symmetrizable**, which means it needs to be endowed with nontrivial **companion balance laws**.
- ▶ So we also need to solve

$$\begin{aligned} DQ_1(U, X) &= B(U, X)^T DG_1(U, X) \\ DQ_2(U, x) &= B(U, X)^T DG_2(U, X), \end{aligned} \quad (16)$$

where $G_1 = U$, $G_2 = F(U)$, $DQ_i = [\frac{\partial Q_i}{\partial \rho}, \frac{\partial Q_i}{\partial u}]$, $i = 1, 2$.

Simplifying

- ▶ Simplifying (16), we get

$$\begin{aligned}Q_{2,\rho} &= Q_{1,\rho}(u + b) + Q_{1,u}g'(\rho) \\ Q_{2,u} &= Q_{1,\rho}\rho + Q_{1,u}(u + b)\end{aligned}$$

- ▶ Taking partial derivatives and subtracting, we get

$$0 = g'(\rho)Q_{1,uu} - \rho Q_{1,\rho\rho} \quad (17)$$

- ▶ We can go one step further and derive

$$Q_{1,uu} - s^2(\rho)Q_{1,\rho\rho} = 0, \quad (18)$$

- ▶ Note that

$$s(\rho) = -\frac{\theta}{v_e'(\rho)} > 0. \quad (19)$$

Solving the Equation

- ▶ Note that (18) is a second order PDE with **variable coefficients**. We will use Evans' method (section 7.2.5)
- ▶ Let's do a transformation,

$$\tilde{Q}_1(u, \rho) = h(y_1, y_2), \quad (20)$$

where $h(y_1, y_2)$ smooth function, $y_1 = \Phi^1(u, \rho)$, $y_2 = \Phi^2(u, \rho)$.

- ▶ We need to make sure the following equations are satisfied

$$\begin{aligned}(\Phi^1)_{uu} - s^2(\rho)(\Phi^1)_{\rho\rho} &= s(\rho)s'(\rho)\Phi^1_{\rho}, \\ (\Phi^2)_{uu} - s^2(\rho)(\Phi^2)_{\rho\rho} &= s(\rho)s'(\rho)\Phi^2_{\rho}.\end{aligned} \quad (21)$$

- ▶ Solving for Φ^1, Φ^2 , we get

$$\Phi^1 = n_1(u + b + \frac{v_e(\rho)}{\theta}), \quad (22)$$

$$\Phi^2 = n_2(u + b - \frac{v_e(\rho)}{\theta}), \quad (23)$$

where $b, n_1, n_2 > 0$.

General Solution

- ▶ Plugging this in, (18) reduces to

$$h_{y_1, y_2} - \frac{s'(\rho)}{4n_2} h_{y_1} + \frac{s'(\rho)}{4n_1} h_{y_2} = 0. \quad (24)$$

- ▶ After changes of variables and integration factor method twice, we get an explicit solution

$$Q_1(\rho, u) = \tilde{Q}_1(u, \rho) = h(y_1, y_2) = \mu^{-1} \left(\int \mu_2(\mu_1^{-1} f(y_1)) + G(y_2) \right),$$

where f, G are C^2 functions and μ_1, μ_2 are integrating factors.

- ▶ This can be solved under the assumption

$$s''(\rho)s(\rho) + \left(\frac{s'(\rho)}{2}\right)^2 = 0 \quad (25)$$

Special Case

- ▶ Consider the special case $s'(\rho) = 0$ in (21) i.e. $v_e''(\rho) = 0$.
- ▶ In particular, consider

$$\hat{v}_e(\rho) = -a\rho + b, \quad (26)$$

which has an actual *concave* fundamental diagram observed in traffic flow.

- ▶ Under this case, (24) reduces to

$$\hat{h}_{\hat{y}_1, \hat{y}_2} = 0. \quad (27)$$

- ▶ Solving this, we get

$$\hat{h}_{\hat{y}_1, \hat{y}_2} = \hat{\beta}(\hat{y}_1) + \hat{G}(\hat{y}_2), \quad (28)$$

where $\hat{\beta}, \hat{G}$ are arbitrary C^2 functions.

Entropy-Entropy Flux Pair - Special Case

- ▶ From (28), we can construct an explicit solution of a convex entropy-entropy flux pair for the special case.

$$\hat{\eta}(\rho, u) = \hat{Q}_1(\rho, u) = (u - s\rho)^2 + \Gamma(u + s\rho)^2, \quad (29)$$

$$\begin{aligned} \hat{q}(\rho, u) = \hat{Q}_2(\rho, u) = & ((u - s\rho)^2 + \Gamma(u + s\rho)^2)(u + b) \\ & + (1 + \Gamma)(u(s\rho)^2) \\ & + 2(\Gamma - 1)\frac{(s\rho)^3}{3} - \frac{1 + \Gamma}{3}u^3, \end{aligned} \quad (30)$$

where $s = \frac{a}{\theta}$, $\Gamma = \frac{1 + \theta}{1 - \theta} > 1$.

- ▶ With this entropy-entropy flux pair, the convexity conditions are satisfied.

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Perturbation Analysis

- ▶ We will estimate our nonconcave fundamental diagram with the solution of our special case.
- ▶ Assume that

$$|s'(\rho)| \leq \gamma \ll 1, \quad (31)$$

where $\gamma > 0$ is a small constant.

- ▶ This is equivalent to $|v_e''(\rho)| \leq C\gamma$, where $C > 0$ is a constant.
- ▶ From (24) and (27), we derived

$$Q_1(\rho, u) = \beta(\Phi^1(u, \rho)) + G(\Phi^2(u, \rho)) + \mathcal{O}(\gamma), \quad (32)$$

where $\gamma > 0$ is small (31).

- ▶ Finally, we can estimate

$$Q_1(\rho, u) = \hat{Q}_1(\rho, u) + \mathcal{O}(\gamma), \quad (33)$$

$$Q_2(\rho, u) = \hat{Q}_2(\rho, u) + \mathcal{O}(\gamma). \quad (34)$$

Nonconcave Fundamental Diagram

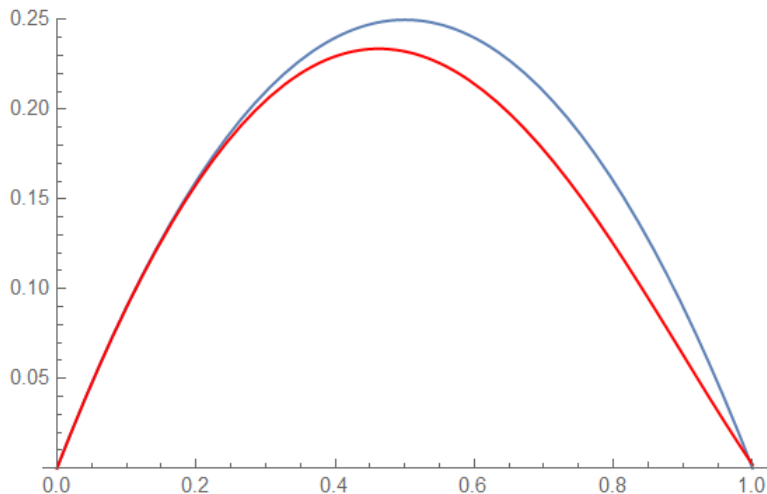


Figure: An Example of non-concave fundamental diagram

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Partial Dissipative Inequality

- ▶ We assume that P is dissipative semidefinite relative to η , i.e.

$$D\eta(U) \cdot P(U) \geq \alpha |P(U)|^2, \quad (35)$$

with $\alpha > 0$.

- ▶ For our system (9), we needed to find a condition such that

$$\begin{bmatrix} \frac{\partial \eta}{\partial \rho} & \frac{\partial \eta}{\partial u} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{u + b - v_e(\rho)}{\tau} \end{bmatrix} \geq \alpha \left(\frac{u + b - v_e(\rho)}{\tau} \right)^2 \quad (36)$$

- ▶ With γ small (31), we require

$$0 < \alpha \leq \Gamma \tau, \quad (37)$$

where

$$\Gamma = n_1^2 + n_2^2 = \frac{1}{\theta} (n_2^2 - n_1^2) > 0. \quad (38)$$

Kawashima Condition

- ▶ The Kawashima condition is given by

$$DP(0)r_i(0) \neq 0, \quad i = 1, \dots, n. \quad (39)$$

- ▶ For our system, we have

$$DP(0)r_i(0) = \begin{bmatrix} 0 \\ \frac{\mp\theta + 1}{\tau} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (40)$$

since $0 < \theta < 1$.

Sub-characteristic condition

- ▶ The sub-characteristic is satisfied when

$$\lambda_1 < \lambda_* < \lambda_2. \quad (41)$$

- ▶ For $v = v_e(\rho)$,

$$\lambda_*(\rho) = -2a\rho + b.$$

- ▶ The sub-characteristic condition is satisfied for (9) since we have

$$v_e(\rho) - \frac{a}{\theta}\rho < -2a\rho + b < v_e(\rho) + \frac{a}{\theta}\rho \quad (42)$$

for $0 < \theta < 1$.

Equivalent Form

- In order to apply Dafermos' theory, we needed to convert (9) into an equivalent form

$$\begin{aligned}\partial_t V + \partial_x G(V, W) &= 0 \\ \partial_t W + \partial_x H(V, W) + C(V, W)W &= 0,\end{aligned}\tag{43}$$

where $x \in \mathbb{R}$, $t > 0$, and $\eta_{WW}C(0, 0) > 0$.

- We had to do the following change of variables

$$V = \rho, \quad W = a\rho + u\tag{44}$$

- This transformed (9) to

$$\begin{aligned}V_t + [V(W - aV + b)]_x &= 0 \\ W_t + \left[\frac{1}{2}(W^2 - a^2V^2) + bW + g(V)\right]_x + C(V, W)W &= 0,\end{aligned}\tag{45}$$

with $C(0, 0) = \frac{1}{\tau} > 0$ and initial conditions

$$Z_0 = (V_0, W_0) = (\rho_0, a\rho_0 + u_0).\tag{46}$$

An *a priori* estimate

- ▶ We show that

$$\rho \geq \frac{1}{2}\delta_1 > 0 \quad (47)$$

for all t under the condition $\rho_0(x) \geq \delta_1 > 0$ for $\delta_1 > 0$ constant.

- ▶ From (45), we have for W

$$\frac{dW}{dt} + \frac{1}{\tau}W = 0, \quad t > 0. \quad (48)$$



$$|W| = |W_0|e^{-t/\tau}, \quad t > 0. \quad (49)$$

- ▶ Modifying the proof of Lemma 16.2 in [2], we can choose $\|W_0\|_{L_\infty}$ small such as

$$\|W_0\|_{L_\infty} \leq \frac{1}{2K}\delta_1 \quad (50)$$

for $K > 0$. Then, we have

$$\rho = V \geq \delta_1 - K\|W_0\|_{L_\infty} \geq \frac{1}{2}\delta_1 > 0, \quad t > 0. \quad (51)$$

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Theorem

Theorem (Admissible BV solution to the Cauchy problem)

Under conditions (6), (25), (31), (37), (50), the system of balance laws (45) is endowed with a convex entropy-entropy flux pair (η, q) (33), satisfies the partial dissipative inequality (35), and satisfies the Kawashima condition (39). consider the Cauchy problem (45) (46). For $\delta_0, \sigma_0 > 0$, suppose that Z_0 decays, as $|x| \rightarrow \infty$, sufficiently fast to render the integral

$$\int_{-\infty}^{\infty} (x^2 + 1) |Z_0(x)|^2 dx = \sigma^2 < \sigma_0^2, \quad (52)$$

$$TV_{(-\infty, \infty)} Z_0(\cdot) = \delta < \delta_0, \quad (53)$$

$$\int_{-\infty}^{\infty} V_0(x) dx = 0, \quad (54)$$

then there exist positive constants $c_0, c_1, c_2, \nu > 0$, so that the Cauchy problem (45) (46) possesses a unique admissible BV solution Z defined on $(-\infty, \infty) \times [0, \infty)$ and

Theorem (Continued)

Theorem

$$\int_{-\infty}^{\infty} |Z(x, t)| \, dx \leq c_0 \sigma, \quad 0 \leq t < \infty, \quad (55)$$

$$TV_{(-\infty, \infty)} Z(\cdot, t) \leq c_1 \sigma + c_2 \delta e^{-\nu t}, \quad 0 \leq t < \infty, \quad (56)$$

$$\int_{-\infty}^{\infty} |Z(x, t)| \, dx \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad (57)$$

$$TV_{(-\infty, \infty)} Z(\cdot, t) \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (58)$$

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