

FTLI: For computing line integrals of conservative VFs
($\mathbb{R}^2$ or $\mathbb{R}^3$)

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{end pt of } C) - f(\text{start pt of } C),$$

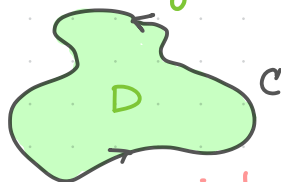
where $\vec{F} = \nabla f$

NEED: $\vec{F}$ a conservative VF

$$\rightarrow Q_x = P_y \text{ or } \text{curl } \vec{F} = \vec{0}$$

To use, need to find a potential function for $\vec{F}$

Green's Theorem: For computing line integrals
($\mathbb{R}^2$ only) of VFs over closed curves



oriented CCW

$$\int_C P dx + Q dy = \iint_D Q_x - P_y dA$$

NEED: C a closed curve in $\mathbb{R}^2$ and $\vec{F}(x,y) = \langle P, Q \rangle$
has continuous partials in D

Hints for when to use:

① If $Q_x - P_y$ is much nicer than $\vec{F}$
e.g. $\vec{F} = \langle y^2 - e^{\sin x}, x - \arctan(y^3) \rangle$
but $Q_x - P_y = 1 - 2y$

② If C has a lot of pieces (would be a pain to parameterize)



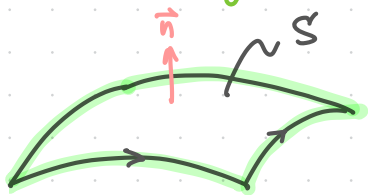
bad!

vs.



not bad

Stokes' Theorem: For computing line integrals
($\mathbb{R}^3$ only)
OR surface integrals of VFs
limited use, see below



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

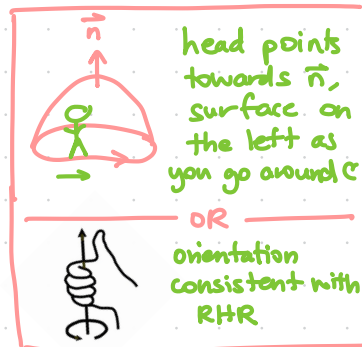
C is the boundary of the surface S

For line integrals:

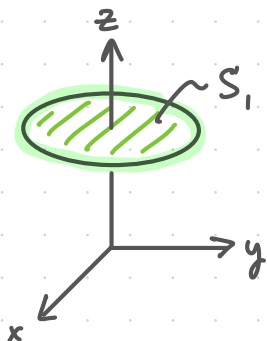
NEED: C is a closed curve in $\mathbb{R}^3$ and $\vec{F}$ has continuous partials on a neighborhood of S

* Remember to check orientation

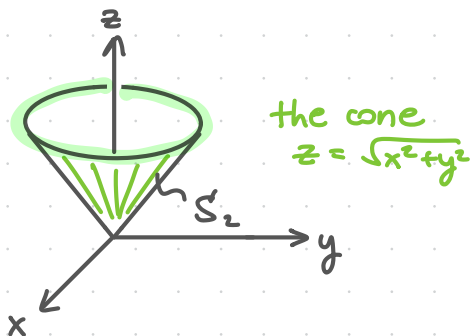
TO USE: Choose S to be a surface whose boundary is the curve C



ex: $\int_C \vec{F} \cdot d\vec{r}$ where C is parameterized by $\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$, $0 \leq t \leq 2\pi$



OR

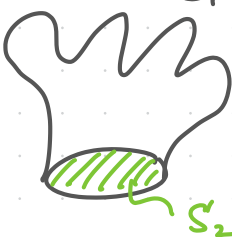


S_1 and S_2 both work

For surface integrals:

If computing $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$, using Stokes' is straight forward. ✓

Bonus Trick: If S_1 and S_2 have the same boundary,


$$\iint_{S_1} \text{curl } \vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl } \vec{F} \cdot d\vec{S}$$

$\int_C \vec{F} \cdot d\vec{r}$

because both equal $\int_C \vec{F} \cdot d\vec{r}$ by Stokes' Thm

If computing $\iint_S \vec{F} \cdot d\vec{S}$, using Stokes' is NOT advisable. ⚡

Would need to find a vector field $\vec{G}$ such that $\text{curl } \vec{G} = \vec{F}$ (i.e. compute an "anticurl")
Hard to do!

Quick check if an anticurl will even exist:

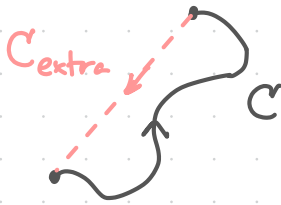
Fact: If $\vec{F}$ has cts 2nd partials,
 $\text{div}(\text{curl } \vec{F}) = 0$

$\Rightarrow$ such a VF $\vec{G}$ does not exist if $\text{div } \vec{F} \neq 0$

MORAL: Use Stokes for $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$ NOT $\iint_S \vec{F} \cdot d\vec{S}$

Bonus Trick for Green's / Stokes':

If C is not closed, close it!

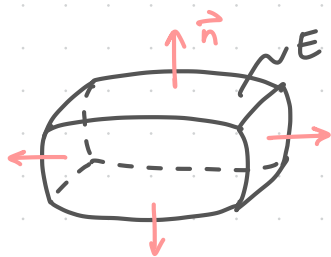


$\tilde{C} = C \cup C_{extra}$ is closed

Compute the integral over $\tilde{C}$ and subtract off the extra:

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{\tilde{C}} \vec{F} \cdot d\vec{r}}_{\substack{\text{compute} \\ \text{using} \\ \text{Green's/Stokes'}}} - \underbrace{\int_{C_{extra}} \vec{F} \cdot d\vec{r}}_{\substack{\text{compute} \\ \text{directly} \\ (\text{parameterize } C_{extra})}}$$

Divergence Theorem: For computing surface integrals of VFs
($\mathbb{R}^3$ only)



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV$$

S is the boundary of the solid E

NEED: A closed surface S , oriented outward and $\vec{F}$ has continuous partials in a neighborhood of E

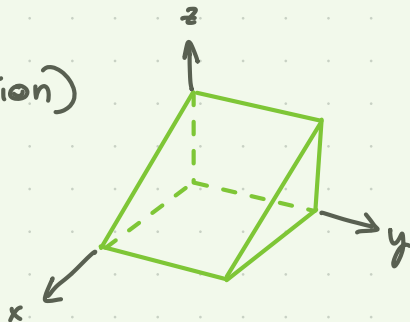
Hints for when to use:

① If $\operatorname{div} \vec{F}$ is much nicer than $\vec{F}$.

e.g. $\vec{F} = \langle xy + \arctan(z), y^2 + e^{xz^3}, \sin(xy) \rangle$

but $\operatorname{div} \vec{F} = y + 2y + 0 = 3y$

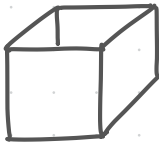
② If S has lots of pieces
(and encloses a solid region)



Bonus Trick for Div. Thm:

If S is not closed, close it!

ex: S is a cube with open top.



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_{\text{cube}} \text{div} \vec{F} dV - \iint_{\text{top}} \vec{F} \cdot d\vec{S}$$