UNIVERSIDADE DE SÃO PAULO
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Notas de Seminários

Fibração de Milnor

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Chapter 1

Introduction

Notes for a Singularity Theory seminar on the Milnor fibration theorem. The seminars took place in the Instituto de Ciências Matemáticas e de Computação of the University of São Paulo in the first semester of 2023 under the supervision of Professor Maria Ruas.

Chapter 2

Teoria de Morse

Estas são anotações referentes ao seminário do Julian sobre teoria de Morse. Em suma, funções de Morse em uma dada variedade codificam e/ou determinam a topologia da variedade. O exemplo mais ilustrativo é a função altura em um toro (posicionado verticalmente).

Defina $M^a = \{x \in M : f(x) \leq a\}$. Temos os seguintes resultados:

1. Suponha $f^{-1}[a, b]$ compacto (em particular, se f é própria) e não contendo nenhum ponto crítico de f.

2. Então M^a é difeomorfo a M^b e, além disso, é um retrato de deformação através da inclusão.

3. Por um argumento também usado no teorema de Ehresmann, há uma fibração associada tal que $f^c \times [a, b] \simeq f^{-1}[a, b]$, onde $c \in [a, b]$. O mais importante corolário é o Teorema de Reeb, que afirma que a existência de uma função de Morse com dois pontos críticos implica que a variedade é uma esfera.

4. É possível mostrar que em valores críticos c não degenerados, as variedades (veja a justificativa abaixo) $M^{c+\delta}$, onde $\delta > 0$ é suficientemente pequeno, são obtidas daqueles de níveis menores através de uma colagem de células cuja dimensão é determinada pelo índice do ponto crítico.

5. O número μ_k de pontos críticos de índice k está relacionado com os números de Betti através de $\sum_k (-1)^k \mu_k \leq \sum_k (-1)^k b_k$.

6. À partir de μ_k e b_k associa-se polinômios de Poincaré e Morse e, sobre variedades compactas, é possível mostrar que são relacionados, generalizando a relação anterior. Funções perfeitas definem a igualdade e foram poderosas ferramentas para a demonstração da conjectura de Poincaré em dimensões altas.

2.1 Homologia de Morse

1. Compacidade da variedade garante que o gradiente da função de Morse é um campo completo que o fluxo converge ao ponto crítico.

2. Variedades estáveis/instáveis são discos abertos. Essas subvariedades decompõe a variedade ambiente no caso de serem de um certo tipo, no mesmo espírito das decomposições celulares anteriores.

3. Exemplo: linhas de fluxo na esfera $\mathbb{S}^2$ com e sem os pólos e no sentido descendo.

4. Direção de linhas de fluxo possuem uma forte relação com o índice do ponto crítico.

5. As interseções das variedades estáveis com superfícies de nível estão em correspondência com linhas de fluxo e daí é possível definir uma homologia com base nesses números de interseção. O resultado principal é que essa homologia coincide com a homologia singular.

2.2 Observações e Perguntas

- Vimos que funções de Morse determinam a topologia das variedades. Uma pergunta natural, seria a determinação da geometria da variedade. Por exemplo, o que se pode dizer da isotopia de uma 2-esfera em $\mathbb{R}^4$ admitindo uma função de Morse com 4 pontos críticos? É uma esfera com ou sem nó? Ou seja, isotópica ou não à canônica? Pesquisar sobre geometria de funções de Morse.

- Fato: Funções de Morse formam um conjunto aberto e denso na topologia C^1. Veja Exercícios 19 e 20 da seção 1.7 do Guillemin e Pollack. [explicar mais]

- Os conjuntos M^a são variedades: a princípio, M^a é a união de um aberto de M, pela continuidade, com uma variedade fechada, pela regularidade de a. Portanto, há ao

menos uma estrutura de CW-complexo. Um argumento completo é dado pelo seguinte resultado:

Lemma 2.2.1 ([10], p.62)**.** *Se M é uma variedade diferenciável e $f : M \longrightarrow \mathbb{R}$ é uma função suave que possui $a \in \mathbb{R}$ como valor regular, então o conjunto $S = f^{-1}(-\infty, a))$ é uma subvariedade suave com bordo $\partial S = f^{-1}(a)$ e com a mesma dimensão de M.*

Proof. O resultado é uma consequência da forma local de submersões numa vizinhança de $f^{-1}(a)$ e fora desses pontos o conjunto é simplesmente um aberto de M. □

Temos ainda a seguinte recíproca:

Proposition 2.2.2 ([10], p.64)**.** *Seja M uma variedade com bordo. Então, existe uma função suave em M não negativa e possuindo 0 como valor regular tal que $f^{-1}(0) = \partial M$.*

- Problema de dessingularização de funções e variedades. Pesquisar sobre.
- Teoria de Morse e geometria do mergulho da variedade.
- Índice do ponto e comportamento das linhas de fluxo. Relação do índice de interseção com a compacidade.
- Homologia não mede conexidade simples: Construa $M = \mathbb{S}^1 \cup \mathbb{S}^1 \cup_{a^2} \mathbb{D}^1 \cup_{b^3} \cup_{(a^3b)^3}$ é tal que $\pi_1(M) = A_5$ e pelo Teorema de Hurewicz $H_1(M) = 0$.

Chapter 3

Prolegomena

3.1 Transversalidade

O conceito de transversalidade generaliza o resultado sobre pré-imagem de valores regulares por funções suaves serem subvariedades. Neste caso, se $Q = f^{-1}(0)$, então $T_xQ = \ker(df_x)$.

Theorem 3.1.1 (Teorema de Transversalidade)**.** *Se $f : X \longrightarrow Y$ é tal que $f \pitchfork Z$ e $f(M) \cap Z \neq \emptyset$, então $Q = f^{-1}(Z)$ é uma subvariedade tal que* $\operatorname{codim}_X(f^{-1}(Z) = \operatorname{codim}_Y(Z)$*. Além disso, $T_xQ = df_x^{-1}(T_{f(x)}Z)$.*

Esquema. A prova consiste em supor, via cartas locais, que f é uma aplicação entre espaços Euclideanos. Além disso, usa-se que localmente P é a pré-imagem por 0 da projeção canônica $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-p}$. Isso é possível pela forma local das imersões.

Temos que $\ker(d\pi_{f(p)}) = T_{f(p)}P$. Defina $h = \pi \circ f$ e note que $h^{-1}(0) = f^{-1}(P)$. Resta mostrar que 0 é um valor regular:

$$\begin{aligned} dh_q(T_q\,\mathbb{R}^m) &= d\pi_{f(q)}\left(df_q(T_q\,\mathbb{R}^m)\right) \\ &= d\pi_{f(q)}\left(df_q(T_q\,\mathbb{R}^m + T_{f(q)}P\right) \\ &= d\pi_{f(q)}(T_0\,\mathbb{R}^n) \\ &= T_0\,\mathbb{R}^{n-p} \end{aligned}$$

□

Quando $Z, X \subset Y$ são subvariedades, a transversalidade entre Z, X é a transversalidade da inclusão $i : X \longrightarrow M$ com relação a Z, ou seja, $Z \cap X = \emptyset$ ou nos pontos de interseção deve

valer $T_xZ + T_xX = T_xY$. Pelos resultados anteriores, segue que $Z \cap X$ é uma subvariedade cujo espaço tangente em x é dado por $T_xX \cap T_xZ$.

Também vale um teorema de transversalidade para variedades com bordo. Nesse caso, é razoável pedir que o bordo de $f^{-1}(Z)$ seja exatamente a interseção da variedade com o bordo de X. Isso nem sempre ocorre: considere, por exemplo, a projeção do semiplano superior na segunda coordenada e a pré-imagem de 0. Para que isso ocorra, a transversalidade deve também ser verificada para a restrição de f a ∂X.

Theorem 3.1.2 (Transversalidade e o bordo)**.** *Sejam X, Y variedades e $f : X \longrightarrow Y$ uma aplicação suave. Suponha que $\partial X \neq \emptyset$ e também que f e sua restrição ∂f sejam transversais a subvariedade $Z \subset Y$. Então, $f^{-1}(Z)$ é uma subvariedade com bordo $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$ e de mesma codimensão que Z.*

Lemma 3.1.3 ([10], p.62)**.** *Se X é uma variedade diferenciável e $f : X \longrightarrow \mathbb{R}$ é uma função suave que possui $a \in \mathbb{R}$ como valor regular, então o conjunto $S = f^{-1}(-\infty, a)$ é uma subvariedade suave com a mesma dimensão de M e bordo $\partial S = f^{-1}(a)$.*

Proof. O resultado é uma consequência da forma local de submersões numa vizinhança de $f^{-1}(a)$ e fora desses pontos o conjunto é simplesmente um aberto de X. □

- Se $F : X \times S \longrightarrow Y$ é transversal a $N \subset Y$, então existe um conjunto denso de índices em S tal que $f_s \pitchfork N$. É sempre possível construir tal F estendendo f e então uma perturbação de f satisfaz a transversalidade.

- Teorema de Thom: Se $W \subset J^k(X, Y)$ é subvariedade, então o conjunto de funções $f \in C^\infty(X, Y)$ tais que $j^k(f) \pitchfork W$ é denso.

- É um cojunto residual, mas $C^\infty(X, Y)$ é um espaço de Baire.

- Seja $\Sigma^r = \{\sigma \in J^1(X, Y) : \text{corank}(d\sigma) = r\}$. Se $Y = \mathbb{R}$, esses conjuntos são não vazios somente se $r = 0$ ou $r = 1$.

- No último caso, a dimensão de Σ^r é nula e a condição de transversalidade $j^1(f) \pitchfork \Sigma^r$ é a condição de ser Morse.

- Consequência: funções de Morse são densas.

- Generalidade da transversalidade: Sejam X, Y variedades e $N \subset Y$ uma subvariedade. Entã, o conjunto de mapas transversais a W é denso em $C^\infty(X, Y)$. Se W é fechada, é também aberto.

Os resultados acima se encontram em [10, Chapter 2] e [9, Chapter 2].

3.2 Campos Vetoriais e Curvas Integrais

O conteúdo desta seção é baseado em [3]. Um campo vetorial em uma variedade suave $X \subset \mathbb{R}^n$ é uma aplicação $V : X \longrightarrow \mathbb{R}^n$ tal que $V(x) \in T_xX$ para todo $x \in X$. Uma curva integral de um campo V é uma curva $\alpha : (a, b) \longrightarrow X$ em X tal que $\alpha'(t) = V(\alpha(t))$. Supondo que $0 \in (a, b)$, dizemos que a curva começa em $\alpha(0) = x$ e tal dependência do ponto inicial é denotada por $\alpha(t) = H_t(x)$.

Pode-se considerar, intuitivamente, uma família de curvas em X tais que todas no tempo 0 avaliadas em um ponto x é o próprio x e que satisfazem uma condição aditiva com relação a ação de $(\mathbb{R}, +)$. Tal família é chamada de fluxo. Por um lado, cada fluxo tem um campo associado e, por outro, a recíproca é verdadeira.

Theorem 3.2.1. *Seja V um campo em X. Então existe um aberto $U \subset X$, $\epsilon > 0$ e um único mapa suave (maximal)*

$$\begin{aligned} H : (-\epsilon, \epsilon) \times U &\longrightarrow X \\ (t, x) &\longmapsto H_t(x) \end{aligned}$$

tal que para cada $x \in U$, $t \mapsto H_t(x)$ é uma curva integral de V, ou seja, $\frac{d}{dt}H_t(x) = V(H_t(x))$ começando em x. Além disso, $H_t(H_s(x)) = H_{t+s}(x)$ sempre que ambos os lados estejam definidos, isto é, o fluxo esteja definido nos tempos t, s e $t + s$.

Se o fluxo é definido em $\mathbb{R} \times X$, então este é chamado fluxo global. Os campos associados são chamados de completos. Em particular, campos suaves em variedades compactas são completos. Fluxos globais definem uma família a 1-parâmetro de difeomorfismos de M. Essa característica permitirá construir difeomorfismos definidos sobre subvariedades de X mapeados, essencialmente, pelas curvas integrais. No caso de curvas integrais definidas sobre $\mathbb{R}$, temos o seguinte.

Proposition 3.2.2.

1. *Se G é um subconjunto fechado de $(\mathbb{R},+)$ que também é um grupo, então, a menos de isomorfismo, $G \simeq \mathbb{R}, G \simeq 0$ ou $G \simeq \mathbb{Z}$.*

2. *Se $t \mapsto H_t(x)$ é uma linha de fluxo definida sobre $\mathbb{R}$, então $G = \{t : H_t(x) = x\}$ é um subgrupo fechado de $(\mathbb{R},+)$.*

3. *Uma linha de fluxo como acima definida sobre $\mathbb{R}$ é uma imersão se, e somente se, $G \neq \mathbb{R}$ e periódica se, e somente se, $G \simeq \mathbb{Z}$. Nesse último caso, $H_t(x)$ possui imagem difeomorfa ao círculo.*

Esquema. Suponha $G \neq 0, \mathbb{R}$. Afirmamos que se G não é discreto, então é denso. Para isso, um aberto $(a,b) \subset F$ centrado em x_0 pode ser transladado para se obter uma vizinhança, contida em F, de qualquer outro $y_0 \in \mathbb{R}$. Como estamos supondo G fechado, G é discreto. O elemento $\alpha = \inf\{x \in \mathbb{R} : x \in G\}$ pertence a G e será um gerador. De fato, por contradição, se $\beta \in G$ e n é o menor inteiro tal que $\beta < n\alpha$, segue que como G é fechado para soma, obtemos uma contradição com a definição de α.

Para o segundo item, utiliza-se a propriedade aditiva do fluxo e a continuidade das curvas integrais. Para o último, se $G \simeq \mathbb{R}$, a linha é constante e sua derivada nula, de maneira que não pode ser imersão. Reciprocamente, se $G \neq \mathbb{R}$, então $G \simeq \mathbb{Z}$ e se a velocidade é nula em algum ponto, existe um intervalo em que a solução é constante, o que contradiz a discretude de G.

Com relação a periodicidade, se $G \simeq \mathbb{Z}$ a linha é uma imersão e então não constante. Considerando $T = t_\epsilon + t_{-\epsilon}$, onde $t_\epsilon \in G$, segue que $H_{t+T}(x) = H_t(x)$ para todo t, de maneira que a linha é periódica. Essas condições devem implicar, pela unicidade, que a curva é difeomorfa a um círculo. Reciprocamente, se a curva é periódica e não constante, $G \neq 0$. Se $G = \mathbb{R}$, a curva é constante. Segue que $G = \mathbb{Z}$. □

A volta do item 3 nem sempre é válida, ou seja, existem imersões injetivas para as quais não há fluxo associado. Campos em variedades distintas podem ser associados via aplicações suaves. Se $f : X \longrightarrow Y$, dois campos V, W em X e Y, respectivamente, são f-relacionados se $W(f(p)) = Df_p(X(p))$.

Lemma 3.2.3. *Dois campos V, W são f-relacionados se, e somente se, $f \circ H_t(x) = G_t \circ f(x)$, onde H, G são os fluxos de V, W respectivamente.*

Lemma 3.2.4. *Se $f : X \longrightarrow Y$ é uma submersão e W é um campo vetorial em Y, então existe um campo vetorial V em X tal que V e W são f-relacionados.*

Proof. Sejam $(x_1, \ldots, x_m)$ e $(y_1, \ldots, y_n)$ coordenadas locais de X e Y, respectivamente, tais que f é uma projeção. Segue que os campos $\frac{\partial}{\partial x_i}$ e $\frac{\partial}{\partial y_i}$ são f-relacionados, pois a derivada também é uma projeção.

Localmente, $W = \sum_{i=1}^{n} W_i \frac{\partial}{\partial y_i}$ e definindo, também localmente, $V = \sum_{i=1}^{n} (W_i \circ f) \frac{\partial}{\partial x_i}$ segue que V e W são f-relacionados. De fato, $W_i \circ f$ é W_i em Y. Agora, usando uma partição da unidade, estende-se V e W a campos definidos globalmente e f relacionados. □

Lemma 3.2.5. *Se $f : X \longrightarrow Y$ é própria e V, W são f-relacionados, então as curvas integrais de V e W são definidas no mesmo intervalo.*

Proof. Fixe $f(x) \in Y$ e suponha $G_t(f(x))$, a curva integral de W, definida em $[0, b)$. Se $H_t(x)$ é definida em $[0, a)$ onde $a < b$, considere o compacto $K = f^{-1}\{G_t(f) : t \in [0, a]\}$. Isso implica que a curva $H_t(x)$ está contida em um compacto, o que contradiz o teorema de existência e unicidade se $a < \infty$. □

Em outras palavras, a igualdade do primeira lema é válida para cada x enquanto um dois lados se verifica.

3.3 Fibrados e o Teorema de Ehresmann

Um fibrado suave (E, π, B, F) consiste dos seguintes objetos:

- Uma variedade diferenciável E, denotada como o espaço total.
- Uma variedade diferenciável B, denotada como espaço base.
- Uma sobrejeção suave $\pi : E \longrightarrow F$, denotada como o mapa de projeção do fibrado.
- Uma variedade diferenciável F, denotada como a fibra, tal que existe uma cobertura $\{U_\alpha\}_\alpha$ tal que para cada α existe um difeomorfismo

$$\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$$

  satisfazendo $\text{proj}_1 \circ \phi_\alpha^{-1}(x, f) = x$. As vizinhanças U_α são as trivializações locais.

Tem-se ainda a seguinte definição para a equivalência de fibrados.

Definition 3.3.1 (Equivalência de Fibrados)**.** *Dois (E, π, B, F) e (E', π', B, F') fibrados com mesma base são equivalentes se existe um difeomorfismo $\Theta : E \longrightarrow E'$ tal que $\pi' \circ \Theta = \pi$. Em particular, as fibras são difeomorfas.*

Example 3.3.2.

- Se B, F são variedades, a projeção $\pi : B \times F \longrightarrow B$ define uma fibração com fibra F.
- Fibrados vetoriais: são fibrados em que a fibra F é um espaço vetorial. Por exemplo, a projeção $\pi : TM \longrightarrow M$ define um fibrado, pois um aberto $U \subset M$ suficientemente pequeno é tal que $U \simeq \mathbb{R}^m$ e então $TU \simeq \mathbb{R}^{2m}$ e a projeção se reduz a projeção na segunda coordenada. Em particular, a projeção do cilindro $T\mathbb{S}^1 \simeq \mathbb{S}^1 \times \mathbb{R}$ sobre $\mathbb{S}^1$.
- Espaços de recobrimento: São fibrações em que a projeção é um homeomorfismo local. Isso implica que a fibra F é um espaço discreto. Dois exemplos clássicos são:

$$\mathbb{Z}_2 \hookrightarrow \mathbb{S}^n \longrightarrow \mathbb{RP}^n$$
$$\mathbb{Z} \hookrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1$$

Justificaremos em breve a razão de fibrações serem importantes, ao menos do ponto de vista topológico, de maneira que é desejável obtê-las. O Teorema de Ehresmann fornece condições suficientes para que isso ocorra. Aqui, a teoria da seção anterior será útil ao considerarmos uma família de difeomorfismos da variedade que será usada para construir um difeomorfismo para a trivialização local.

Theorem 3.3.3 (Ehresmann)**.** *Seja M uma variedade diferenciável com bordo e N uma variedade diferenciável sem bordo e $f : M \longrightarrow N$ uma sobrejeção própria. Se $f_{|M^\circ} : M^\circ \longrightarrow N$ e $\partial f : \partial M \longrightarrow N$ são submersões, então f é um fibrado.*

Proof. Suponha inicialmente que $\partial M = \emptyset$. Precisamos verificar a condição de trivialização local. Pela definição, podemos supor $N = \mathbb{R}^n$.

A ideia da demonstração é mostrar que as fibras de $f^{-1}(\mathbb{R}^n)$ são levadas difeomorficamente a $f^{-1}(0)$ através de curvas integrais de campos vetoriais adequados. Essa é exatamente a situação ideal de $f^{-1}(\mathbb{R}^n)$ como um cilindro ao longo de $f^{-1}(0)$ o qual é projetado sobre $\mathbb{R}^n$ e então cada ponto tendo uma fibra sobre si.

Tal transporte ocorrerá em M mas será codificado em $\mathbb{R}^n$ através de campos f-relacionados. A existência será uma consequência de f ser submersão. A condição de ser própria permitirá concluir que os campos levantados em M são completos.

Considere os campos vetoriais canônicos $\frac{\partial}{\partial x_i}$, cujas curvas integrais são as retas com direções canônicas e_i. Note que tais campos são completos. Pelos Lemas 3.2.4 e 3.2.5, existem campos vetoriais campletos $V_1, \ldots, V_n$ em N, visto que f é própria. Isso implica que os fluxos H^i associados são definidos em $\mathbb{R}$. Agora, definimos o seguinte mapa:

$$\begin{aligned} \phi : f^{-1}(\mathbb{R}^n) &\longrightarrow \mathbb{R}^n \times f^{-1}(0) \\ z &\longmapsto (f(z), H^n_{-y_n} \circ \cdots \circ H^1_{-y_1}), \end{aligned}$$

onde $f(z) = (y_1, \ldots, y_n)$. Se G^i são os fluxos dos campos em N, temos que:

$$f \circ H^n_{-y_n} \circ \cdots \circ H^1_{-y_1}(z) = G^n_{-y_n} \circ \cdots \circ G^1_{-y_1} \circ f(z).$$

Mas esta última composição é 0, pois em cada direção e_i o vetor *retorna* y_i unidades para a origem. Portanto, o mapa é bem definido. Já o seu mapa inverso é dado por:

$$\begin{aligned} \phi^{-1} : \mathbb{R}^n \times f^{-1}(0) &\longrightarrow f^{-1}(\mathbb{R}^n) \\ (y_1, \ldots, y_n) &\longmapsto H^1_{y_1} \circ \cdots \circ H^n_{y_n}(x) \end{aligned}$$

Além disso, como para cada instante y_j as soluções $H^i_{y_j}$ dependem suavemente de z, os mapas são suaves. Por fim, a condição $\text{proj}_1 \circ \phi(z) = f(z)$ é imediata.

Destacamos que os seguintes pontos devem ser observados para concluir o resultado para variedades com bordo:

1. Forma local de submersões: que segue de um argumento análogo ao estender a função sobre um aberto do semiplano H^m a um aberto de $\mathbb{R}^m$ e então considerar sua restrição.

2. Existência e unicidade do fluxo: o teorema é válido para variedades com bordo se o campo é tangente ao bordo, ou seja, $V_{|\partial M}$ é um campo de ∂M (veja [12, Theorem 9.34]). Usando o item anterior, temos que os campos levantados satisfazem essa condição nas coordenadas adequadas.

3. Extensão das curvas integrais por aplicações próprias: veja [12, Lemma 9.33], o qual garante que para uma subvariedade D com bordo de M, se um campo V de M é tangente a ∂D, as curvas integrais de V que começam em D, ali permanecem.

4. A pré-imagem $g^{-1}(Z)$ de uma subvariedade Z por uma submersão g nem sempre é uma subvariedade devido a sua interseção com ∂M. Entretanto, as hipóteses permitem aplicar o Teorema 3.1.2.

□

Example 3.3.4.

- Condição de ser própria é essencial: a restrição $\pi_1 : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}$ da projeção π_1 na primeira coordenada é tal que para todo U, $\pi_1^{-1}(U)$ é conexo, mas $\pi^{-1}(0) \times U$ não. De maneira que obtemos um exemplo de submersão sobrejetiva que não é um fibrado.

- Segundo exemplo: considere o conjunto

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |y| \geq |x|\}.$$

  A projeção na primeira coordenada π_1 restrita a Ω é tal que $\pi_1^{-1}(0) \times U$, para qualquer intervalo aberto $U \subset \mathbb{R}$ contendo a origem, é uma variedade topológica, enquanto $\pi_1^{-1}(U)$ não.

- A recíproca não vale: fibrados vetoriais não são próprios e, portanto, nem toda fibração é como no enunciado do teorema.

Corollary 3.3.5. *Sob as condições acima, as fibras de uma submersão própria são todas difeomorfas entre si.*

O mesmo argumento da demonstração nos permite concluir o seguinte resultado.

Corollary 3.3.6 (Lema Básico da Teoria de Morse)**.** *Seja $f : M \longrightarrow \mathbb{R}$ própria. Suponha que f é regular em $(a, b) \subset \mathbb{R}$. Então, $f^{-1}(a, b) \simeq (a, b) \times f^{-1}(c)$ para todo $c \in (a, b)$.*

Corollary 3.3.7. *Se uma n-variedade fechado M admite uma função de Morse com exatamente dois pontos críticos, então M é homeomorfa a n-esfera.*

O lema básico é usado na prova do Teorema de Reeb essencialmente para mostrar que M é homeomorfa a um cilindro com dois discos anexados, ou seja, a uma esfera.

3.4 Fibrações e Topologia dos Espaços

Nesta seção queremos justificar o estudo de fibrados do ponto de vista topológico. Mais especificamente, como fibrações podem descrever a topologia de um espaço? A motivação para esta pergunta é entender como o Teorema de Fibração de Milnor pode ser interpretado como um resultado sobre a estrutura topológica da singularidade. Uma direção para esta resposta recai no conceito de levantamentos, dentro do contexto da teoria de espaços de recobrimento e se generaliza para as chamadas *fibrações*. Recorde inicialmente que, para caminhos no espaço base, existem (e são únicos) levantamentos que são caminhos no espaço total. Os resultados aqui enunciados podem ser encontrados em [6, Sections 6.5, 9.8].

Theorem 3.4.1. *Sejam $\pi : E \longrightarrow B$ um espaço de recobrimento e X um espaço localmente conexo. Dadas $f, g : X \longrightarrow B$ com levantamentos $\tilde{f}, \tilde{g} : X \longrightarrow E$, cada homotopia H entre f, g possui um único levantamento $\tilde{H}$ entre $\tilde{f}, \tilde{g}$.*

Definition 3.4.2. *Um mapa $\pi : E \longrightarrow B$ possui a propriedade de levantamento com relação ao espaço X se para cada homotopia H em B tal que $H(., 0) = f : X \longrightarrow B$ e um levantamento $\tilde{f}$ de f, existe um levantamento $\tilde{H}$ de H tal que $\tilde{H}(., 0) = \tilde{f}$. Uma fibração é um mapa π que satisfaz essa propriedade para todo disco $\mathbb{D}^n$.*

Segue que espaços de recobrimento são fibrações. Ainda, é possível mostrar que fibrados com espaço base paracompacto também são fibrações. Por fim, observamos que as fibras de uma fibração são todas homotopicamente equivalente. Como consequência, temos o seguinte resultado, o qual justifica o estudo de fibrações do ponto de vista topológico.

Theorem 3.4.3. *Seja $\pi : E \longrightarrow B$ uma fibração e $F = \pi^{-1}(b_0)$.*

1. *O mapa π induz um isomorfismo $\pi_n(E, F) \simeq \pi_n(B)$ para todo n.*
2. *A sequência abaixo é exata:*

$$\ldots \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \ldots$$

Remark 3.4.4. *Do ponto de vista topológico, a escolha da fibra não é essencial, pois em fibrações todas elas são homotópicas.*

Chapter 4

Milnor Fibration Theorem

4.1 Introduction

The main references are Milnor's book itself [13], the Brazilian notes [4], and the survey [1]. We begin with an example. This first part is from [4, Capítulo I].

Example 4.1.1. Consider $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by $f(z_1, z_2) = z_1^p + z_2^q$. We would like to study the set $V = f^{-1}(0)$ from a topological point of view. To do so, one may consider the intersection $K_\epsilon = V \cap \mathbb{S}_\epsilon$, where the radius ϵ is enough small. Using polar coordinates, set $z_1 = re^{i\theta}$ and $z_2 = se^{i\omega}$. We obtain the following equations:

$$r^p = s^q$$
$$r^2 + s^2 = \epsilon^2$$

Therefore, s_0, r_0 are determined and $K_\epsilon \subset \mathbb{S}_{r_0} \times \mathbb{S}_{s_0}$. In such a way that the angles θ, ω are varying. On the other hand, $e^{i\theta p} = e^{i\omega q}$ implies $\theta p = \omega q + \pi n$. If $\theta/p = t \in \mathbb{R}$, we may ignore n and K_ϵ is parametrized by the curve:

$$\alpha \longmapsto \left(r_0 e^{iqt}, s_0 e^{i(t+\pi/q)}\right).$$

Such an embedding is called a *torus knot* of type (p, q) and the condition $(p, q) = 1$ is to guarantee no redundancies.

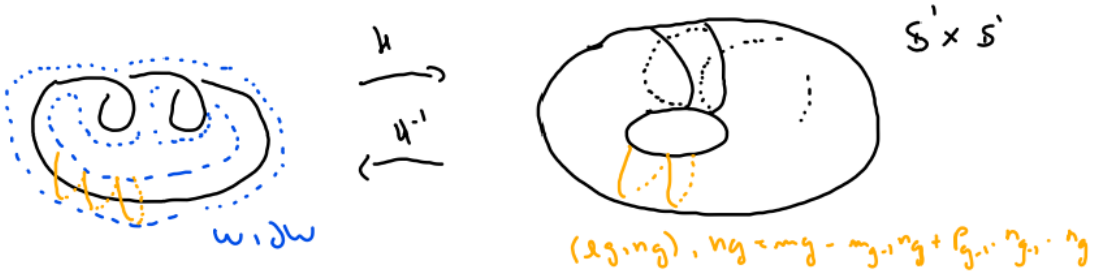
The procedure above can be used to study more general varieties defined by complex analytic maps. First, some definitions.

Definition 4.1.2. *A complex analytic function $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is analytically irreducible if $(f) \subset \mathbb{C}[\![z_1, z_2]\!]$ is prime.*

From the definition, if f has z as a critical point, it must be isolated, which is a generalization of Study's Lemma.

Theorem 4.1.3 (Brauner - 1928)**.** *If f is an irreducible analytic polynomial such that $f(0) = 0$, then for a sufficiently small ϵ, the intersection $K_\epsilon = \mathbb{S}_\epsilon \cap V$ is a knot torus of type K_g, where g is determined by the Puiseuax series.*

From the Puiseuax series, we obtain a sequence of pairs of natural numbers $(p_1, q_1), \ldots, (p_n, q_n)$ from which we construct by iterations, knots in the torus. More specifically, given the knot K_{g-1}, we consider a tubular neighborhood W such that ∂W is diffeomorphic to a torus, and so form a knot torus from the pair (p_g, q_g). We pull back this curve by the inverse diffeomorphism. Now, the question is partially closed by the following result.



Theorem 4.1.4 (Burau-Zariski - 1928-32)**.** *If $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies $f(0) = 0$ and is analytically irreducible, then its topological type is completely determined by the knot K_g.*

This discussion justifies the local study of V by its intersection with a small sphere, which is the approach considered by Milnor. Before, we will show that this procedure also gives the local topological behavior of V.

4.2 Conical Structure Theorem

Suppose that $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a complex polynomial with 0 as the single singularity of a small neighborhood $U \subset \mathbb{C}^n$. That is, f has 0 as an isolated singularity. It follows that $f_{U\setminus\{0\}}$ is a submersion and taking its pre-image at 0, $V \setminus \{0\}$ is a manifold of dimension $2n - 2$.

Theorem 4.2.1. *For a sufficiently small ϵ, $V \pitchfork \mathbb{S}_\epsilon$ and the link $K_\epsilon = V \cap \mathbb{S}_\epsilon$ is a $(2n-3)$-manifold.*

Proof. Consider the restriction $\tilde{r}$ to $V - \{0\}$ of the distance function $r(z) = \|z\|^2$. We claim that $\Sigma(\tilde{r})$ is an algebraic variety. Write $f(x,y) = (u(x,y), v(x,y))$, where u, v are the real and imaginary vectors, respectively. The point $z = (x,y)$ is critical for $\tilde{r}$ if for every vector $v = (v_1, v_2) \in \mathbb{R}^{2n} \simeq \mathbb{C}^n$ in the tangent space of V, which is equal to $\ker(d(f_U)_z)$, it holds that $d\tilde{r}_z(v) = 0$. As the restriction of the distance function, $d\tilde{r}_z(v) = \langle z, v\rangle = 0$ (real dot product) and we obtain the following equations:

$$\begin{aligned} u_x v_1 + v_x v_2 &= 0 \\ u_y v_1 + v_y v_2 &= 0 \\ x v_1 + y v_2 &= 0 \end{aligned}$$

In other words, the matrix $\begin{pmatrix} u_x & v_x \\ u_y & v_y \\ x & y \end{pmatrix}$ is zero at a $2n-2$-dimensional space $(T_z(V - \{0\}))$ and $u(x,y) = v(x,y) = 0$. That is, the matrix has rank ≤ 2 and $u = v = 0$ define the algebraic conditions on $\Sigma(\tilde{r})$.

By Whitney's result, $\Sigma(\tilde{r})$ has a finite number of connected components which are also smooth manifolds. At each component, $\tilde{r}$ must be a constant, and the critical values are finite. Thus, a small ϵ is a regular value for $\tilde{r}$ and $\tilde{r}^{-1}(\epsilon^2) = V \cap \mathbb{S}_\epsilon$ is a manifold. This intersection is transversal, on the other hand, $T_z V \subset T_z \mathbb{S}_\epsilon$, and so z satisfies the above equations, yielding a contradiction. □

Remark 4.2.2. In 1966, before Milnor's book publication, Brieskorn discovered that links of analytic polynomials arise as *exotic spheres*, that is, a manifold homeomorphic to the standard sphere but not diffeomorphic. In 1956, Milnor discovered the 28 non-equivalent structures on $\mathbb{S}^7$. Later, it was proved that the following singularities give these structures:

$$f(z) = z_1^2 + \cdots + z_4^{6k-1} + z_5^3, \ k = 1, \ldots, 28.$$

Theorem 4.2.3. *For a sufficient small ϵ, the following pairs are hommeomorphic:*

$$(B_\epsilon, V \cap B_\epsilon) \simeq (C(\mathbb{S}_\epsilon), C(K_\epsilon)).$$

Idea. It is clear that the cone over the sphere is the ball. This is formalized by considering the radial vector field and its integral curves, which are the lines through the origin. Since a point in the sphere determines the direction and a number within $[0, 1)$ identifies the point in the ball, it remains to attach the origins properly.

The idea is that we can apply this procedure preserving V. We need to construct a vector field locally and then apply partition of unity. For we consider outside V the radial vector field and on V we define w to "point away" from the origin. The compactness of B_ϵ will imply that w is complete and the flow gives a diffeomorphism.

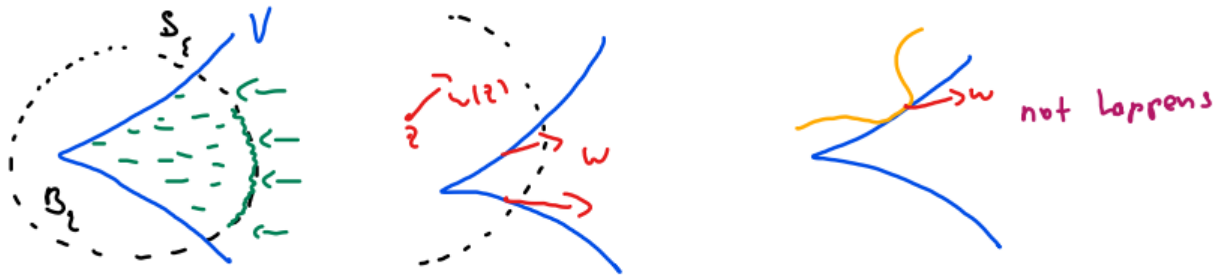


Figure 4.1: Deforming the cylinder and preserving V

□

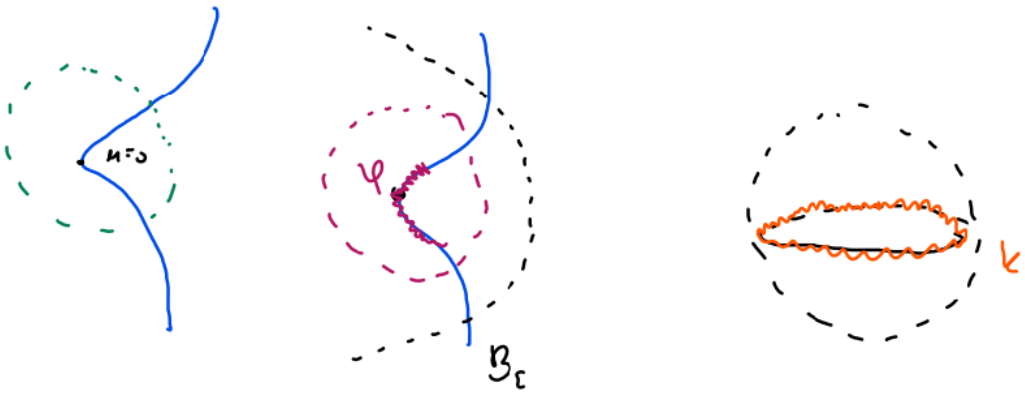
For regular points, nothing unexpected happens.

Proposition 4.2.4. *If the origin is a regular point, the link is an unknotted sphere.*

Proof. There exist a local coordinate system $u_1, \ldots, u_{2n}$ in U such that the distance function $r(z) = \|z\|^2$ has the form $r = \sum_i u_i^2$. On the other hand, since $V \setminus \{0\}$ is a manifold, V is described (in some other coordinates) by $\{\phi_1 = \phi_2 = 0\}$. Composing $r \circ \phi$, we obtain that $r^{-1}(\epsilon^2) = K_\epsilon$ are the points $\sum_{i=3} \phi_i^2 = \epsilon^2$, which is a sphere. □

To study the general case, that is, non-isolated singularities of holomorphic functions, we remark the following.

- The topology of V is codified by the topology of the link.
- For analytical varieties, one can obtain the same result by smooth decompositions from Whitney and Hironaka's works.



- For non-isolated singularities, V is a stratified space, and for each stratum, the intersection with a small sphere is transversal. Under this condition, using a Thom-Mather controlled vector field, the conical structure remains valid.

4.3 Gradients

For f analytic we would like to associate some other real functions. First, we may suppose that 0 is an isolated critical value and the origin is not necessarily an isolated singularity. Recall that U is a neighborhood of the origin. If $f(z) = |f(z)|e^{i\theta(z)}$, restricting the angles considered, we have the following real functions:

$$\theta : U \setminus V \longrightarrow \mathbb{R}$$
$$\log|f| : U \setminus V \longrightarrow \mathbb{R}$$

And $\log(f) = \log|f| + i\theta$. So

$$\theta = \mathrm{Im}(\log f) = \mathrm{Re}(-i\log f)$$
$$\log|f| = \mathrm{Re}(\log f).$$

The real gradients will be of interest because the gradients will translate orthogonality conditions associated with some hypersurfaces. Recall that the gradient of a real function g is defined to be the vector associated with the linear functional:

$$d_z(v) = \langle \mathrm{grad} g(z), v\rangle.$$

Since we are working with $\mathbb{C}^n$, we consider the Hermitian product $\langle v, w\rangle = \sum_i v_i\overline{w_i}$. This space is a usual Euclidean space if we take the real part of this product. Then we take the

same definition for the gradient of an analytic function and we obtain, by considering the canonical basis, the vector:

$$\operatorname{grad}(f(z)) = \left(\overline{\frac{\partial f}{\partial z_1}}, \ldots, \overline{\frac{\partial f}{\partial z_n}} \right).$$

Direct computations yield the following result.

Proposition 4.3.1. *It holds that:*

1. $\frac{d\theta(p(t))}{dt} = \operatorname{Re}\langle p'(t), i \operatorname{grad} \log(f(p(t))) \rangle$.
2. $\frac{d \log |f(p(t))|}{dt} = \operatorname{Re}\langle p'(t), \operatorname{grad} \log(f(p(t))) \rangle$.
3. $\operatorname{grad} \theta = i \operatorname{grad} \log(f)$.
4. $\operatorname{grad} \log |f| = \operatorname{grad} \log f$.

The level hypersurfaces will be denoted by:

$$\theta^{-1}(\theta_0) = f^{-1}(\mathcal{L}^+_{\theta_0}) = E_{\theta_0}$$
$$\log |f|^{-1}(\log \delta) := f^{-1}(\partial \mathbb{D}_\delta) = N(\delta)$$

where $\mathcal{L}^+_{\theta_0}$ is the semi ray defined by θ_0 and $\delta > 0$ a positive number. The intersection $N(\delta) \cap B_\epsilon = N(\delta, \epsilon)$.

4.4 Fibration Theorem

Theorem 4.4.1. *If ϵ is sufficiently small, the map*

$$\phi := \frac{f(z)}{|f(z)|} : \mathbb{S}_\epsilon \setminus K_\epsilon \longrightarrow \mathbb{S}^1$$

is a fiber bundle.

We prove the result in two steps: first, the map is a submersion to conclude the fibers are manifolds, and after the local trivialization.

4.4.1 ϕ is a submersion

Let $\Phi = \pi \circ f : B_\epsilon \setminus V \longrightarrow \mathbb{S}^1$ be a extension of ϕ, where $\pi(z) = z/|z|$ is the radial projection. Since 0 is an isolated critical value, f is a submersion in $B_\epsilon \setminus V$ and so is Φ. Its fibers are:

$$\Phi^{-1}(\theta_0) = f^{-1}(\pi^{-1}(\theta_0)) = f^{-1}(\mathcal{L}^+_{\theta_0}) = E_{\theta_0}.$$

We conclude that E_{θ_0} is a codimension 1 submanifold. Now we characterize the critical points of ϕ.

Lemma 4.4.2. *It holds that*

$$\Sigma(\phi) = \{z \in \mathbb{S}_\epsilon \setminus K : i \operatorname{grad}\log(f(z)) = \lambda z \text{ for some real } \lambda\}.$$

Proof. Observe that $\ker d\phi_z \subset \ker d\Phi_z$ and if z is a critical point, $T_z(\mathbb{S}_\epsilon \setminus K) \subset \ker d\phi_z$. Since $\ker d\Phi_z = T_z E_{\theta_0}$ has dimension $2n-1$, it follows that:

$$T_z E_{\theta_0} = T_z(\mathbb{S}_\epsilon \setminus K)$$

as well as their normal spaces (in the Euclidean meaning). Considering that $\operatorname{grad}\theta = i \operatorname{grad}\log$, we conclude. □

Now, we must show that there exists ϵ_0 such that for all $\epsilon < \epsilon_0$ the previous situation does not happen and ϕ will be a submersion there. Also under this condition, it holds that $\mathbb{S}_\epsilon \setminus V \pitchfork E_\theta$, since they are codimension 1 submanifolds with distinct and intersecting tangent spaces.

Lemma 4.4.3. *There exists $\epsilon_0 > 0$ such that for all $z \in U \setminus V$ with $\|z\| = \epsilon < \epsilon_0$, $i \operatorname{grad}\log(f)$ is linear independent to z over $\mathbb{R}$.*

A stronger result will be applied here and after.

Lemma 4.4.4. *There exists $\epsilon_0 > 0$ such that for all $z \in U \setminus V$ with $\|z\| = \epsilon < \epsilon_0$ one of the following conditions hold:*

1. $\operatorname{grad}\log f(z) \notin \langle z \rangle$ *or*
2. $i \operatorname{grad}\log f(z) = \lambda z$, *where* $|\arg\lambda| < \pi/4$. *In particular,* $\operatorname{Re}(\lambda) > 0$.

Since the linear independence over $\mathbb{C}$ implies the same for $\mathbb{R}$, the first condition implies the lemma, and we claim that the second condition also does. On the contrary, if $\lambda' \in \mathbb{R}$ satisfies $i \operatorname{grad} \log f = \lambda' z$, it follows that z is real, because $i \operatorname{grad} \log f$ is real. Then

$$\operatorname{Im}(i \operatorname{grad} \log f) = \operatorname{Re}(\operatorname{grad} \log f) = 0.$$

However, $\operatorname{Re}(\lambda z) \neq 0$, because $\operatorname{Re}(\lambda) > 0$.

To prove Lemma 8.3.16, Milnor used the Curve Selection Lemma in the context of algebraic varieties. Later, the statement was proved for analytic sets and we state this general version.

Lemma 4.4.5 (Selection Curve)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open neighborhood of the origin, $g_1, \ldots, g_p$ analytic functions, and A an analytic set. Define also $U = \{z \in \Omega : g_i(z) > 0 \ \forall \ i\}$. If $0 \in \overline{A \cap U}$, then there exists an analytic curve $\gamma(t) : [0, \delta) \longrightarrow \Omega$ such that $\gamma(0) = 0$ and $\gamma(t) \in A \cap U$ for all $t \neq 0$.*

Remark 4.4.6. *A very useful application of the previous lemma is to prove that a singularity is non-isolated through the existence of an analytical curve with suitable properties.*

Lemma 4.4.4. By contradiction, suppose the existence of points $z \in B_\epsilon \setminus V$ arbitrarily close to the origin such that there exists $\lambda_z \neq 0$ satisfying $\operatorname{grad} \log f(z) = \lambda_z z$ and $|\arg(\lambda)| \geq \pi/4$. For the case $|\arg(\lambda)| > \pi/4$ define the following sets:

$$A = \{z : \exists\, \lambda_z \in \mathbb{C} \text{ with } \operatorname{grad} f(z) = \lambda_z z\}$$
$$U^{\pm} = \{z : \operatorname{Re}(1 \pm i)\lambda > (<) 0\}$$

That is, the set A implies the linear dependency, and the open sets $U^{\pm}$ imply the condition on the argument of λ. To apply the Curve Selection Lemma, we need to see that A is analytic. For this, recall that

$$\operatorname{grad} \log f(z) = \frac{\operatorname{grad} f(z)}{\overline{f(z}}.$$

Then, if $z \in A$, $\operatorname{grad} \log(f(z) = \lambda z/\overline{f(z)}$ and $\arg(\lambda) = \arg < \operatorname{grad} f(z), z >$. Defining $\lambda' =< \operatorname{grad} f(z), z >$ we obtain an analytic description of A. Note also that we may replace A by:

$$A = \{w : \exists\, \alpha_z \in \mathbb{C} \text{ with } \operatorname{grad} \log f(z) = \alpha_z w\}.$$

This remark is very important because Milnor proved the lemma under the assumption of A as a semialgebraic set.

Now, the hypothesis states that $0 \in \overline{A \cap (U^+ \cup U^-)}$ and the Selection Lemma gives a curve satisfying:

$$\operatorname{grad} \log f(\lambda(t)) = \lambda(t)p(t).$$

It is possible to show that in this case $|\arg(\lambda')| \to 0$ which gives a contradiction.

If $\lambda = 0$ (which happens for an isolated z) or $|\arg(\lambda)| = \frac{\pi}{4}$, the following set captures these conditions:

$$B = \{z : (\operatorname{Re}(1+i)\lambda')\,(\operatorname{Re}(1-i)\lambda') = 0\}.$$

As an open set, consider $U = \{z : f(z) \neq 0\}$, since we are interested in $B_\epsilon \setminus V$. The analytic set $A \cap B$ and U are the sets used to apply the Selection Lemma. □

Now, the fibers have the following description:

$$\phi^{-1}(\theta) := F_\theta = \Phi^{-1}(\theta) \cap \mathbb{S}_\epsilon = E_\theta \cap \mathbb{S}_\epsilon.$$

We recall that the intersection above is transversal (since the normal vectors are linear independent over $\mathbb{R}$) and we conclude that the fiber has dimension $2n-2$.

4.4.2 ϕ is locally trivial

The first attempt to obtain a local triviality condition is by considering Ehresmann Theorem, however, the map ϕ is not necessarily proper, since $\mathbb{S}_\epsilon \setminus K$ is an open set of the sphere. Nonetheless, the proof will follow the same idea.

Let $u = ie^{i\theta}$ the rotational vector field in $\mathbb{S}^1$, also given by $u = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Suppose we are given a complete lifting $w(z)$ through ϕ. The desired result will follow.

Theorem 4.4.7. *The map $\phi := f/|f| : \mathbb{S}_\epsilon \setminus K \longrightarrow \mathbb{S}^1$ has all the fibers diffeomorphic and it is also the projection of a fiber bundle.*

Proof. Since $w(z)$ is complete, the flow $H_t : \mathbb{S}_\epsilon \setminus K \longrightarrow \mathbb{S}^1$ defines a diffeomorphism. We claim that $H_t(F_\theta) = F_{\theta+t}$. Indeed, if $G(t)$ is the flow associated to u, since this pair is ϕ-related, we have:

$$\phi \circ H_t(z) = G_t \circ \phi(z) = e^{i(\theta+t)},$$

because the integral curves of $G(t)$ are linear paths on $\mathbb{S}^1$. More specifically, the integral curves are projected by ϕ to paths with positive direction on $\mathbb{S}^1$. Fix $\theta \in \mathbb{S}^1$ and let W be an open interval centered at θ with length less than 2π. The following pair of diffeomorphisms shows the local triviality:

$$\begin{aligned} \psi : W \times F_\theta &\longrightarrow \mathbb{S}^1 \\ (e^{i(\theta+t)}, z) &\longmapsto H_t(z) \\ \psi^{-1} : \phi^{-1}(W) &\longrightarrow W \times F_\theta \\ z &\longmapsto (\phi(z), H_{-\phi(z)}(z)) \end{aligned}$$

We note that the condition on W is necessary for the inverse ψ^{-1} to be well defined, since on the contrary, one would have the same point $z_0 \in \phi^{-1}(W)$ projected to two angles in W. In other words, the *monodromy* on the fiber would not be defined (see the figure below).

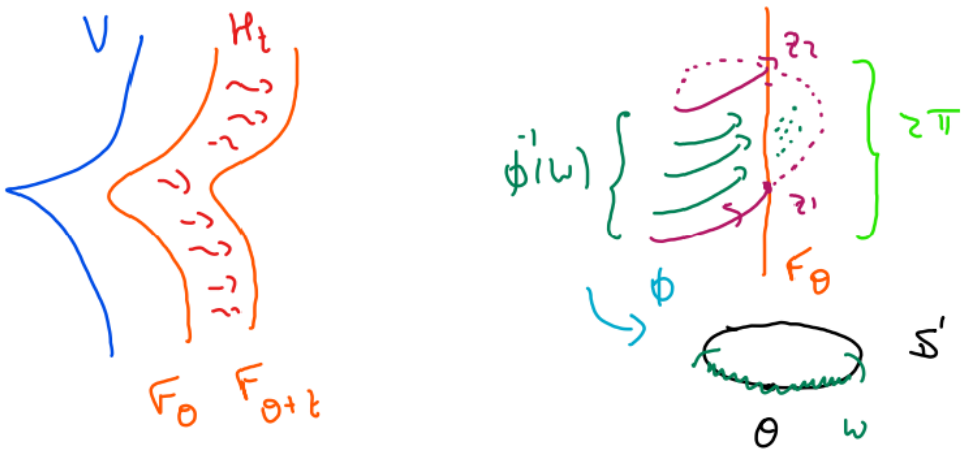


Figure 4.2: Diffeomorphic fibers and the monodromy

□

It remains to discuss the existence of the vector field $w(z)$. For completeness, we need to control the behavior of the integral curves close to $K = \mathbb{S}_\epsilon \cap V$. That is, for any finite limit t_0 we have $\lim_{t \to t_0} p(t) \notin K$, equivalently:

$$f(p(t)) \nrightarrow 0 \iff \log|f(p(t))| \nrightarrow -\infty.$$

Enough condition for this is:

$$\left| \frac{d}{dt} \log|f(p(t))| \right| < 1.$$

We may rewrite it as

$$\begin{aligned}\left|\frac{d}{dt}\log|f(p(t))|\right| &= \left|\mathrm{Re}\left(\frac{d}{dt}\log f(p(t))\right)\right| \\ &= \left|\mathrm{Re} < \frac{d}{dt}(p(t)), \mathrm{grad}\log(f(p(t)) >\right| \\ &= |\mathrm{Re} < w(z), \mathrm{grad}\log(f(p(t)) >| < 1\end{aligned}$$

Moreover, $w(z) \in T_z(\mathbb{S}_\epsilon \setminus K)$, so the second condition is

$$\mathrm{Re} < w(z), z >= 0.$$

Finally, $w(z)$ must be a lifting and we have seen that

$$\theta(p(t)) = t + c_\theta,$$

where c_θ is constant. Therefore, the third condition is $\frac{d}{dt}\theta(p(t)) = 1$. Alternatively,

$$\begin{aligned}\frac{d}{dt}\theta(p(t)) &= \frac{d}{dt}\mathrm{Re}\,(-i\log f) \\ &= \mathrm{Re}\,\langle w(z), -i\,\mathrm{grad}\log(f(z)\rangle \\ &= \mathrm{Re}\,\langle w(z), i\,\mathrm{grad}\log(f(z)\rangle \\ &= 1\end{aligned}$$

We state a sharper result whose proof will contain exactly the needed vector.

Lemma 4.4.8. *There exists a smooth vector field $w(z)$ on $B_\epsilon \setminus V$ such that for every $\epsilon \le \epsilon_0$ and every $z \in \mathbb{S}_\epsilon \setminus V$, the vector $w(z)$ is tangent to $\mathbb{S}_\epsilon \setminus V$ at z and the complex inner product*

$$\alpha = \langle w(z), i\,\mathrm{grad}\log(f(z))\rangle$$

is non zero and $|\arg(\alpha)| \le \pi/4$.

Proof. By Lemma 4.4.4, for each $z_0 \in \mathbb{S}_\epsilon \setminus V$, with small ϵ, we have two possibilities. For each one, we construct the vector fields locally and then apply the partition of unity.

1. $\mathrm{grad}\log f(z_0) \notin \langle z_0\rangle$: this condition is local and it is enough to solve the following equations:

$$\begin{aligned}\langle w(z), z\rangle &= 0 \\ \langle w(z), i\,\mathrm{grad}\log f(z)\rangle &= 1\end{aligned}$$

If $\beta = \langle w(z), \operatorname{grad} \log f(z) \rangle$, then the second condition shows that $\operatorname{Re}(\beta) = \operatorname{Im}(i\beta) = 0$.

An approach to construct this vector is to take $w(z_0) \in T_z \mathbb{S}_\epsilon \cap N(\epsilon_0, |f(z_0|))$. This intersection is transversal because the normal vectors are linearly independent also over $\mathbb{R}$. We would obtain

$$\frac{d}{dt} \log |f(p(t))| = |\operatorname{Re} < w(z), \operatorname{grad} \log(f(p(t)) >| = 0$$

because $\operatorname{grad} \log f(p(t))$ is normal (in the Euclidean meaning) to $N(\epsilon_0, |f(z_0)|)$. It remains to show the lifting condition.

2. $\operatorname{grad} \log f(z_0) = \lambda z_0$ with $|\arg \lambda| < \pi/4$. Set

$$w(z) = \frac{iz}{\operatorname{Re}(\overline{\lambda})|z|^2}.$$

For z_0, $w(z_0)$ satisfies:

$$\kappa = \langle w(z_0), i \operatorname{grad} \log f(z_0) \rangle = \frac{\overline{\lambda}}{\operatorname{Re}(\overline{\lambda})}.$$

If $\lambda = a + ib$, $\kappa = \frac{a-ib}{a} = a - i\frac{b}{a}$, which also has the modulo of the argument less than $\pi/4$. Also $\operatorname{Re}(\kappa) = 1$ and $\operatorname{Im}(\kappa) < 1$. Now, since these conditions occur in an open neighborhood of z_0 by continuity, $w(z)$ as above is the desired local field. This shows the result. □

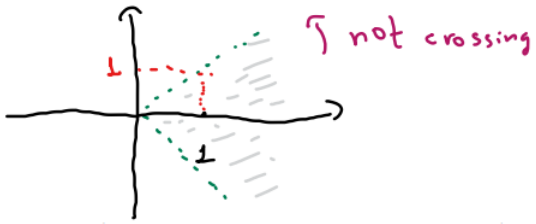


Figure 4.3: Real and imaginary parts of κ

Proposition 4.4.9. *The vector field construct above is transversal to the fiber.*

Proof. Since $\phi^{-1}(\theta) = F_\theta$, we have that $T_z F_\theta = \ker(d\phi_z)$, but $w(z) \notin \ker(d\phi_z)$ since it is a lifting by ϕ of the rotational vector field. Alternatively, if $w(z) \in T_z F_\theta = T_z(E_\theta \cap \mathbb{S}_\epsilon)$, $w(z) \in T_z E_\theta$. So

$$\begin{aligned} 0 &= \operatorname{Re}\langle w(z), \operatorname{grad} \theta \rangle \\ &= \operatorname{Re}\langle w(z), i \operatorname{grad} \log f \rangle \end{aligned}$$

since the gradient of θ is orthogonal in the Euclidean meaning to the level hypersurface E_θ and this contradicts the third condition. □

For regular points, the fiber is an Euclidean space.

Proposition 4.4.10. *If the origin is a regular point, the fiber F_θ is diffeomorphic to $\mathbb{R}^{2n-2}$.*

Proof. Set $W = f^{-1}(\mathbb{R})$ and fix for this manifold coordinates $u_1, \dots, u_{2n-1}$ such that $V = \{u_1 = 0\}$ and $\tilde{r} := r_{|W}$ has the form $u_1^2 + \cdots + u_{2n-1}^2$.

To finish, we must verify that the fiber lies on the sphere and in a specific part of the complement of V, that is, where $u_1 > 0$. For this $\phi^{-1} = f^{-1}(\mathcal{L}_\theta^+) \cap \mathbb{S}_\epsilon$, that is, we are looking for the pre-image of the semi-ray. On the other hand, $f^{-1}(\mathbb{R})$ has three connected components corresponding to $u_1 = 0$ (or V), and $\pm u_1 > 0$. Finally, since V is not part of the domain of ϕ, we get the following description:

$$\phi^{-1}(1) = \{u : \tilde{r}(u) = \epsilon^2 \text{ and } u_1 > 0\}$$

which gives a hemisphere of the $2n-2$ sphere in $\mathbb{C}^{2n-1}$. □



Figure 4.4: Connected components of $f^{-1}(\mathbb{R})$

It will imply that the fiber bundle is trivial by the following result:

Theorem 4.4.11. *A vector bundle over $\mathbb{S}^1$ with total space orientable is trivial.*

Coments. The open subset $\mathbb{S}_\epsilon \setminus K$ of the sphere is clearly oriented. On the other hand, the isomorphism class of vector bundles over $\mathbb{S}^1$ is in bijection with the homotopy classes $[\mathbb{S}^0, \mathrm{GL}^+(n)]$, where $\mathrm{GL}^+(n)$ is one of the two connected components of the general linear group of $\mathbb{R}$ (see [11, Proposition 1.14]). Thus there is only one class. Finally, the trivial bundle always represents a class.

Alternatively, a formal reference is given by Milnor [13, p.23]. Essentially, the bundle is determined by the diffeomorphisms of the fiber under the monodromy (since the base is the circle). But up to isotopy, there are two classes of such diffeomorphisms of a Euclidean space (identity and reflection) and they induce the same bundle. In some way, such conditions imply others to the whole total space. □

Some links of reference:

- Isomorphism class of vector bundles over $\mathbb{S}^1$.
- Vector bundles over $\mathbb{S}^1$.

Chapter 5

Conservation of the Milnor Number

A proof for the preservation of the Milnor number under small perturbations is given by a differential topology approach through the degree of a map.

5.1 Introduction

The goal of these notes is to prove the conservation of the Milnor number under small perturbations of an analytic finitely determined germ. More formally, we consider a representative f of the germ defined on a neighborhood $V \subset \mathbb{C}^n$ of the origin and a deformation $F : U \times V \longrightarrow \mathbb{C}$, where $U \subset \mathbb{C}^n$, such that every $F(t, x) = f_t(x)$ has critical points within the interior of a compact subset $V_0 \subset V$ in which f has 0 as an isolated critical point.

Theorem 5.1.1. *Suppose f is a representative of a germ function and a deformation F as before. Then, it holds that*

$$\mu(f) = \sum_{x_i \in \Sigma(f_t)} \mu_{x_i}(f_t),$$

where f_t is a fixed deformation and $\Sigma(f_t)$ is the set of critical points of f_t.

In other words, the Milnor number splits into the Milnor numbers of a small deformation at its critical points. The proof follows [14, Appendices E and D] and is based on the notion of the *degree* of a map. The Cohen-Macaulay theory is applied to connect this real notion to the analytical setting. There is also another and algebraic proof (see [7, Proposition 7.1.1]). The following example will give a motivation.

Example 5.1.2. Consider the A_k singularity type $f = x^2 + y^{k+1}$ with Jacobian ideal $J_f = \langle 2x, (k+1)y^k \rangle$. Consider the deformation $f_t(x, y) = x^2 + y^{k+1} + ty$. While f has isolated singularity, if we consider f as a complex germ function, for each $t \neq 0$, there exist k distinct critical points. This translates the idea that Milnor number splits into the Milnor numbers at critical points of a deformation.

5.2 The degree of a map

Let $f : M \longrightarrow N$ be a differentiable map between two compact orientable manifolds of same dimensions. Let $y \in N$ be a regular value for f, under the conditions above, it implies that $f^{-1}(y)$ is a finite set of points. Now, we sum up these points considering the signs of the orientation:

$$\deg(f) = \sum_{x_i \in f^{-1}(y)} \operatorname{sign}(df_{x_i}).$$

This sum can be interpreted as the intersection number of the map with the manifold N at the point y.

Theorem 5.2.1 ([10], p.110)**.** *Let M be an oriented compact manifold with boundary of dimension $n+1$. The restriction to the boundary ∂M of every differentiable map $f : M \longrightarrow \mathbb{S}^n$ has degree zero:*

$$\deg(f_{|\partial M}) = 0.$$

The theorem remains valid if we consider an n-dimensional oriented compact manifold with boundary instead only the sphere. This is essentially due to the classification of oriented 1-dimension compact manifolds with boundary. The pre-image $f^{-1}(M)$ consists of intervals each one with a pair of extreme points weighted with ± 1 and $f^{-1}(\partial M)$ will be the sum of these weighted points.

Definition 5.2.2. *Let $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be a differentiable map with the origin as an isolated zero. Let U be a closed neighborhood of the origin with it the single zero. We define the index of f as:*

$$\operatorname{ind}_0 f = \deg\left(\frac{f}{|f|}\right),$$

where $\left(\frac{f}{|f|}\right) : \mathbb{S}^{n-1}_\epsilon(0) \longrightarrow \mathbb{S}^{n-1}$ and $\mathbb{S}^{n-1}_\epsilon = \partial U$.

The previous theorem implies that the radius ϵ is irrelevant provided we consider neighborhoods where the origin is the single zero. Now, to obtain the desired result, we must set f as the gradient vector field defined by a function germ, and then an isolated zero means an isolated singularity.

Lemma 5.2.3. *Let U be a limited neighborhood with boundary ∂U and $f : \partial\overline{U} \longrightarrow \mathbb{R}^n$ with no zeros in ∂U and with isolated zeros in the interior of U. Then*

$$\sum_{x \in U \cap f^{-1}(0)} \operatorname{ind}_x(f) = \deg\left(\frac{f}{|f|} : \partial U \longrightarrow \mathbb{S}^{n-1}\right).$$

Proof. Take small closed balls U_x centered at each x for which f has x as the single zero. If W denotes $U \setminus \bigcup \operatorname{int} U_x$, then the boundary is described by $\partial W = \partial U - \bigcup \partial U_x$ and $\frac{f}{|f|}$ is defined on W. Therefore:

$$\deg_{\partial U} \frac{f}{|f|} = \deg_{\bigcup U_x} \frac{f}{|f|} = \sum_{x_i} \operatorname{ind}_{x_i} f.$$

□

Partial proof of 5.1.1. The degree is preserved by homotopy (see [10, Sections 2.3 and 3.4]), and the deformation F defines a homotopy. Therefore:

$$\begin{aligned} \deg_0\left(\frac{f}{|f|})\right) &= \operatorname{ind}_0 f \\ &= \deg_0\left(\frac{f_t}{|f_t|}\right) \\ &= \sum_{x \in f_t^{-1}(0) \cap U} \operatorname{ind}_x(f_t) \end{aligned}$$

It remains to see that the indexes will be multiplicities of f and then Milnor numbers. □

5.3 Few Remarks on Cohen-Macaulay Rings

Let R be a commutative ring with unity. Recall that the height of a prime ideal p is the maximal length of a chain $0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n \subsetneq p$ of prime ideals. Then one defines the *Krull dimension* of R as the maximum height among the possible lengths.

The first example is the ring of analytic germs $\mathcal{O}_n$, which has dimension n due to the following chain:

$$0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n).$$

Note that between two elements there are no prime ideals. Fix R as a local ring with maximal ideal m and M a R-module.

Definition 5.3.1.

1. *A sequence $(a_1, \ldots, a_n)$ of elements in m is M-regular if a_i is not zero in the quotient $M/(a_1, \ldots, a_{i-1})$ for all $i = 1, \ldots, n$.*

2. *The depth of M, denoted by* $\operatorname{depth}(M)$ *is the largest length of a regular sequence.*

3. *The dimension of M, denoted by* $\dim(M)$*, is the dimension of the ring $R/(\operatorname{Ann}(R)$, where*
$$\operatorname{Ann}(R) = \{z \in R : xz = 0 \ \Rightarrow \ z = 0 \text{ for } x \in M\}.$$

Example 5.3.2.

1. In the ring $R = \mathcal{O}_2$, the ideal $I = \langle x^2, xy, y^2 \rangle$ is not regular.

2. In the ring $R = k[X, Y, Z]$, where k is a field, the sequence $X, Y(1-x), Z(1-X)$ is regular. If we remove X, the resulting sequence is no longer regular.

Theorem 5.3.3. *It holds that* $\dim(M) \geq \operatorname{depth}(M)$*. If the equality holds, the module is called a Cohen-Macaulay R-module.*

Cohen-Macaulay rings allow us to study the ring of holomorphic germ functions and the analytical sets involved. The complex case is important to derive results analogous to the *Nullstellensatz*.

5.4 Connecting multiplicity to the Milnor number

We define the multiplicity of $f : (X, x_0) \longrightarrow (\mathbb{C}^d, 0)$, denoted by $\operatorname{mult}(f)$, where (X, x_0) is an analytic germ set, as the complex dimension of $\mathcal{O}_{X,x_0}/(f_1, \ldots, f_d)$. On the other hand, as we have seen, we can define the degree of a complex map just by considering it as a real map. We derive the following.

Theorem 5.4.1. *If $f : (X, x_0) \longrightarrow (\mathbb{C}^d, 0)$ is a finite surjective map germ, it holds that*
$$\deg(f) \leq \operatorname{mult}(f).$$
We have equality if and only if (X, x_0) is Cohen-Macaulay.

By Ruckert's Nullstellensatz, there is a correspondence between analytic varieties in $\mathbb{C}^n$ and radical ideals of $\mathcal{O}_{\mathbb{C}^n,0}$, which justifies the nomenclature in the statement. The following corollary of this result will show Theorem 5.1.1.

Corollary 5.4.2. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ be analytic such that $f^{-1}(0) = \{0\}$. Then* $\deg(f) = \text{mult}(f)$.

Proof of 5.1.1. We have that

$$\begin{aligned} \text{ind}_x(f_t) &= \deg\left(\frac{f_t}{|f_t|}\right) \\ &= \text{mult}\left(\frac{f_t}{|f_t|}\right) \\ &= \text{mult}(f_t) \end{aligned}$$

Since the map f_t represents the gradient of a function g_t, its multiplicity is the Milnor number of g_t. The result follows. □

A more elementary connection between the multiplicity (and then the Milnor number) and the degree of a map was introduced by Milnor itself [13, Chapter 7]. However, it is not enough to close the algebraic and geometric relation. First, we set the result for Morse singularities and then apply the conservation result to conclude. In the first case, a proof can be found in [4, Exemplo III.12], see also [14, p.260].

Chapter 6

Milnor-Lê Fibration

Starting from an alternative description of the Milnor fibers, we introduce the Milnor-Lê Fibration and its equivalence with the Milnor Fibration under a slight modification.

6.1 Alternative Description of the Fiber

After having studied the topology of the fiber and the link, Milnor gave an alternative description of the fiber, related to the fibers $f^{-1}(c)$ of f for c sufficiently small. This shows that the fibration can be used to study the local behavior of f near the set V. We begin with the same technique of construction of vector fields to obtain integral curves which will define diffeomorphisms.

Lemma 6.1.1. *There exists a vector field $w(z)$ in $U \setminus V$ so that the inner product*

$$\langle w(z), \operatorname{grad} \log f(z) \rangle$$

is real and positive and $\langle w(z), z \rangle$ has positive real part.

Proof. As before, we construct the vector field locally and then glue it through a partition of unity. We need to consider two cases.

1. $\operatorname{grad} \log f(z_0)$ is linearly independent with z_0 over $\mathbb{C}$. In this case, it is enough to solve the following equations in a neighborhood of z_0:

$$\begin{aligned} \langle w(z), \operatorname{grad} \log f(z) \rangle &= 1 \\ \langle w(z), z \rangle &= 1. \end{aligned}$$

Note that if the vectors were not linearly dependent over $\mathbb{C}$, the system would not have a solution at z_0.

2. $\operatorname{grad}\log f(z_0) = \lambda_0 z_0$ with $|\arg\lambda_0| < \pi/4$. Defining the vector field as $w(z) = \operatorname{grad}\log f(z)$, it holds at z_0 that:

$$\begin{aligned}\langle w(z_0), \operatorname{grad}\log f(z_0)\rangle &= |\lambda_0 z_0|^2 \in \mathbb{R}^+\\ \langle w(z_0), z_0\rangle &= \langle \lambda_0 z_0, z_0\rangle = \lambda |z_0|^2\end{aligned}$$

whose real part is positive, since $|\arg\lambda_0| < \pi/4$. By continuity, it must hold for $w(z)$ in a neighborhood of z_0 and we conclude.

□

Corollary 6.1.2. *The integral curves $p(t)$ of the vector field $w(z)$ constructed as in the above proof satisfies:*

1. $\arg(f(p(t)))$ *is constant.*
2. $|f(p(t))|$ *is a strictly monotone function of t.*
3. $p(t)$ *is a strictly monotone function of t.*

In other words, the trajectories are pointing away from the origin and in a direction for which $|f|$ increases.

Proof. Recall that $\langle w(z), \operatorname{grad}\log f(p(t))\rangle$ is real and positive. On the other hand, we have that:

$$\frac{d}{dt}\log f(p(t)) = \langle w(z), \operatorname{grad}\log f(p(t))\rangle$$

Since $\log f(p(t)) = \log|f(p(t))| + i\theta(f(p(t)))$, it follows that $\theta(f(p(t)))$ is constant and we conclude the first item. Moreover, these conditions imply

$$\begin{aligned}\frac{d}{dt}\log|f(p(t))| &= c_0\\ |f(p(t))| &= e^{c_0 t + k_0},\end{aligned}$$

where c_0 is real and positive, so the second item follows. Finally:

$$\begin{aligned}\frac{d}{dt}\langle p(t), p(t)\rangle &= \frac{d}{dt}|p(t)|^2\\ &= 2\operatorname{Re}\langle p'(t), p(t)\rangle\\ &= 2 > 0\end{aligned}$$

which proves the last item. □

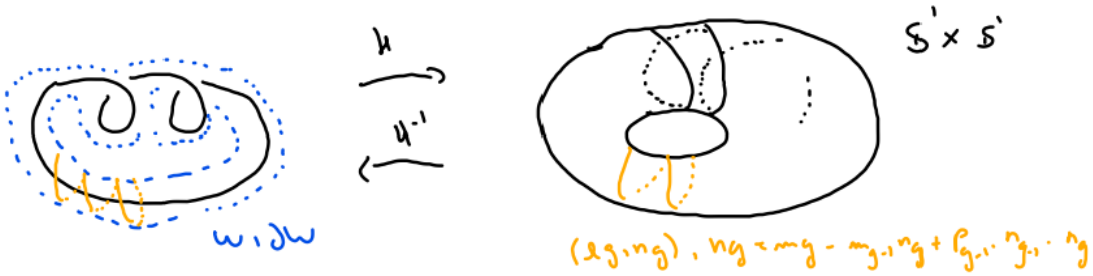


Figure 6.1: Behavior of f on $p(t)$

Corollary 6.1.3. *Let $c = |c|e^{i\theta_0} \in \mathbb{C}$ be a regular value for f with $|c|$ sufficiently small. Then, it holds that:*

$$f^{-1}(c) \cap \mathbb{D}_\epsilon \simeq \{z \in F_{\theta_0} : |f(z)| > c\},$$

where $\mathbb{D}_\epsilon$ denotes the open ball of radius ϵ and center at the origin.

Proof. For each $z_0 \in f^{-1}(c) \cap \mathbb{D}_\epsilon$, we take the integral curve $p(t)$ of $w(z)$ passing through $z_0 = p(t_0)$. The previous result implies that we "travel" away from the origin reaching the sphere at a point z', since the modulo increases. The fact that the point is in the fiber F_θ follows from the first item of the previous result:

$$\theta(p(t_1)) = \theta(p(t_0)) = \theta_0.$$

Also, this correspondence is unique, due to the uniqueness of the integral curves defined by points. Conversely, each point in $F_\theta \subset \mathbb{S}_\epsilon \setminus K$ defines an integral curve that should cross the set $f^{-1}(c)$, since in this direction $|f(p(t))|$ is decreasing and so $p(t)$ approaches to $f^{-1}(c)$, by the monotonicity, $p(t)$ attaches $f^{-1}(c)$. □

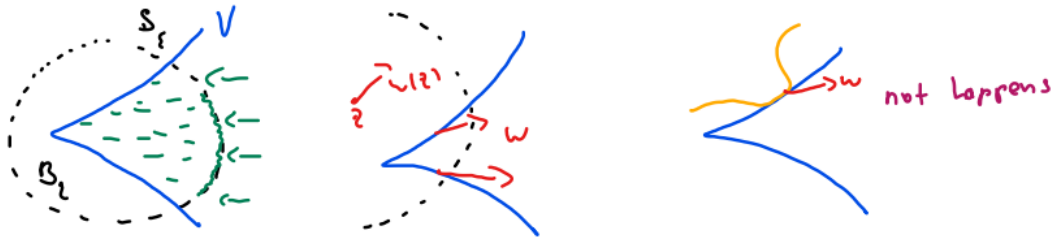


Figure 6.2: A portion of the Milnor fiber and the fiber $f^{-1}(c)$

If $\mathbb{S}_{|c|}$ is the set of complex numbers with modulo $|c|$, it follows that $f^{-1}(\mathbb{S}_{|c|})$ is diffeomorphic to the points z of $\mathbb{S}_\epsilon \setminus K$ with $|f(z)| > c$. Therefore, this result suggests that we can fiber the complement of V in the ball U. It will be the Tubular Filtration.

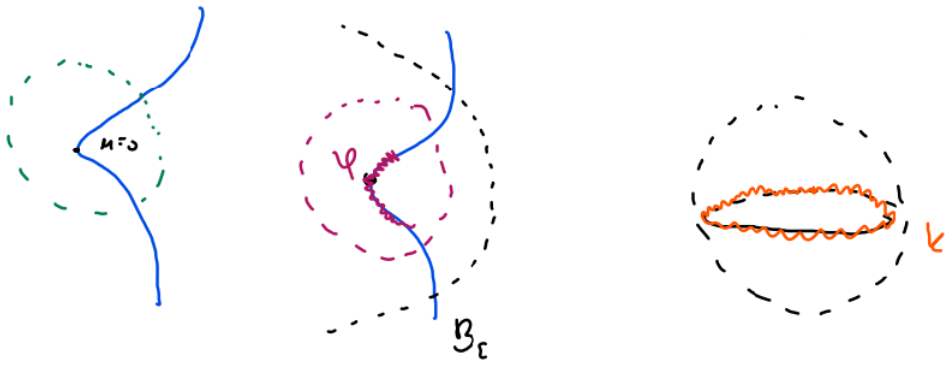


Figure 6.3: A suggestive tubular fibration

For c sufficiently small, we can conclude more by applying the Morse Theory. Before, we state the following result, which shows that the critical points of $|f(z)|$ in F_θ are contained in a compact set.

Lemma 6.1.4. *There exists $\eta_\theta > 0$ such that the critical points of $a_\theta := |f(z)| : F_\theta \longrightarrow \mathbb{R}$ are contained in the compact set $\{z \in F_\theta : |f(z)| \geq \eta_\theta\}$.*

Proof. We proceed by contradiction by applying the Curve Selection Lemma. Suppose the existence of critical values of a_θ whose images are arbitrarily close to zero. Take as the open set the neighborhood U and consider the sets:

$$A = \{z \in \mathbb{S}_\epsilon \setminus K : z \in \Sigma(a_\theta)\}$$
$$B = \{z \in U \setminus K : |f(z)| < 1\}.$$

By the compactness of $\mathbb{S}_\epsilon$ there exists a limit point z_0 and by continuity, $z_0 \in V$, the zero set of f. Moreover, $z_0 \in \overline{A \cap B}$ and the lemma predicts an analytic curve $p(t)$ such that $p(0) = z_0$ and $p(t) \in A \cap B$ for all $t \neq 0$. Along this curve, a_θ is a nonzero constant, as well as $|f|$, on the other hand, the values are going to zero along $p(t)$ and this gives a contradiction. □

Remark 6.1.5. It is important to observe why the set A above is compact. It is enough to see that A is closed in $\mathbb{S}_\epsilon$. But A is a closed subset of a closed submanifold of $\mathbb{S}_\epsilon \setminus K$, so it is

closed in this latter set. By contradiction, if A is not closed, A has limit points in K, which is a contradiction with the continuity $|f(z)|$.

Theorem 6.1.6. *If $|c| \neq 0$ is sufficiently small, it holds that*

$$f^{-1}(c) \cap \mathbb{D}_\epsilon \simeq \{z \in F_\theta : |f(z)| > c\} \simeq F_\theta,$$

where $\mathbb{D}_\epsilon$ is the open ball of radius ϵ and center at the origin.

Proof. Take $|c| < \eta_\theta$. If

$$W = (-|f|))^{-1}(-\infty, -|c|],$$

then $W \simeq f^{-1}(c) \cap \mathbb{D}_\epsilon$ by the previous corollary. On the other hand,

$$F_\theta = (-|f|)^{-1}(-\infty, 0].$$

The statement means W being diffeomorphic to F_θ, but under these descriptions and a result from Morse Theory, it remains to verify that $(|f|)^{-1}[-|c|, 0]$ does not contain critical values of $-|f|$. But in this interval, $|f| < c < \eta_\theta$, by our initial choice. □

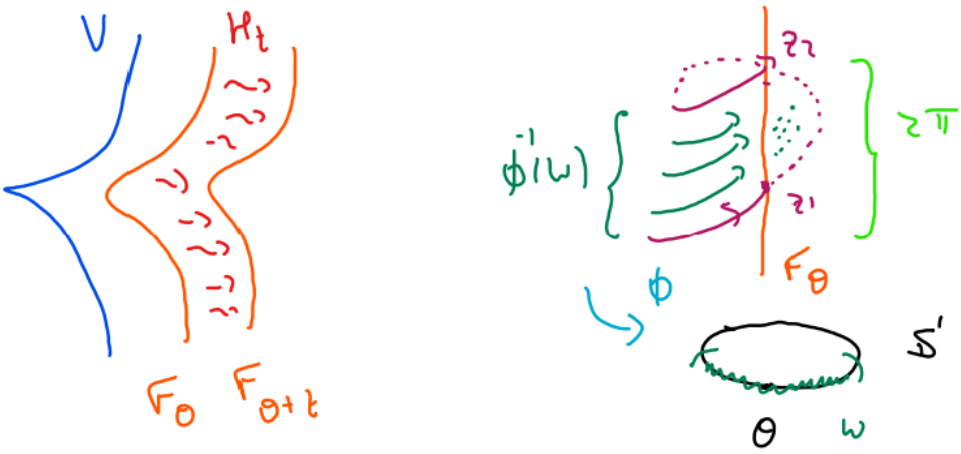


Figure 6.4: Relation between the fibers

Corollary 6.1.7. *Each fiber of the Milnor fiber bundle is diffeomorphic to a complex analytic manifold. In particular, they are orientable manifolds.*

6.2 Thom a_f-condition

As we claimed before, the relation between the Milnor fiber and the fibers of f suggests a fibration on the tube with projection map f itself. Milnor proved this result for the case of an isolated singularity and, in fact, we need some stratification conditions to understand how the fibers $f^{-1}(c)$ touch the sphere $\mathbb{S}_\epsilon$. We then define the Thom a_f-condition on the stratification of V and apply Ehresmann Theorem to conclude. In the next section, we prove that both fiber bundles are *essentially the same*.

Definition 6.2.1. *Given $V = f^{-1}(0)$, take a Whitney stratification of U such that V is a union of strata and $U \setminus V = S_\beta$. Let $S_\alpha \subset \overline{S_\beta}$ and $\{z_n\} \subset S_\beta$ a sequence such that $z_n \to z \in S_\alpha$. Each $z_n \in f^{-1}(c_n)$ for some $c_n \in \mathbb{C}$. If $T = \lim_n T_{z_n}F_n$, the map f satisfies Thom a_f-condition if $T_z S_\alpha \subset T$.*

Remark 6.2.2.

- This is trivially satisfied if the singularity is isolated since the origin has 0-dimensional tangent space, which is contained in every tangent space at it.
- Whitney stratifications always exist for analytic sets.
- On the inclusion $S_\alpha \subset S_\beta$: by the regularity condition (Whitney stratification), if we have $z \in S_\alpha \cap \overline{S_\beta}$, necessarily we obtain that $S_\alpha \subset \overline{S_\beta}$. This property is called *frontier condition* and roughly speaking, it says that the frontiers of the strata are composed of strata as well.
- By the compactness of the Grassmaniann space, we can pass to a subsequence to obtain the convergence of the sequence F_n.
- The property holds trivially for isolated singularities.
- A comparison with the definition of Whitney stratification is that this property is defined for pairs of strata while here, the regularity condition is between f and the Whitney stratification of its zero-set V.

Theorem 6.2.3 (Hironaka)**.** *All complex analytic maps $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ satisfy Thom's a_f-condition.*

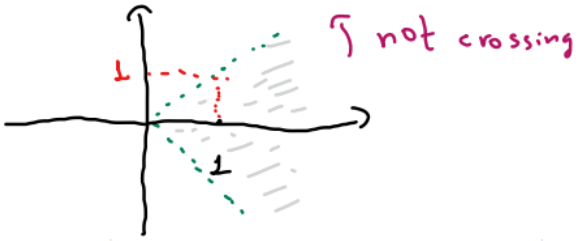


Figure 6.5: Thom a_f-condition

Proposition 6.2.4. *Suppose that $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ has an isolated singularity at the origin. Then, there exists $\delta > 0$ such that all the fibers over the closed disc $\mathbb{D}_\delta$ with radius δ are transversal to $\mathbb{S}_\epsilon$. In other words, the fibers sufficiently close to V are also transverse to $\mathbb{S}_\epsilon$.*

Proof. We consider the restriction of f to $\mathbb{S}_\epsilon$, denoted by $\tilde{f}$. The transversality condition at $z \in \mathbb{S}_\epsilon \cap V = K$ means that $d\tilde{f}_z$ is regular since it also implies $\ker d\tilde{f}_z \not\subset T_z\mathbb{S}_\epsilon$. By continuity, there exists an open neighborhood U_z of z in the sphere such that it remains true. Therefore, since $T_z f^{-1}(c) = \ker d\tilde{f}_z$, $f^{-1}(c) \cap \mathbb{S}_\epsilon = \tilde{f}^{-1}(c)$ is a transversal intersection provided c is small.

Let U be the union of all these neighborhoods. We claim that there exists a sufficiently small δ, such that $f^{-1}(\mathbb{D}_\delta)$ intersect transversally the sphere. That is, $f^{-1}(\mathbb{D}_\delta) \cap \mathbb{S}_\epsilon \subset U$. On the contrary, we obtain a sequence $z_n \in f^{-1}(c_n) \cap \mathbb{S}_\epsilon$, where $c_n \in \mathbb{C}$ goes to zero and the intersection is not transversal. That is, $(z_n) \in \mathbb{S}_\epsilon \setminus U$. By compactness, passing to a subsequence if necessary, there exists a limit point z in this complement. By continuity, $z \in (V \cap \mathbb{S}_\epsilon) \cap \mathbb{S}_\epsilon \setminus U$, which is a contradiction. □

Corollary 6.2.5. *Let $\mathbb{B}_\epsilon$ the closed ball with boundary such that each stratum of V is transverse to $\partial\mathbb{B}_\epsilon = \mathbb{S}_\epsilon$. Then, there exists $\delta > 0$ such that all the fibers over the closed disc $\mathbb{D}_\delta$ are transverse to $\mathbb{S}_\epsilon$. In other words, the fibers sufficiently close to V are also transverse to $\mathbb{S}_\epsilon$.*

Proof. We use a similar argument of that from Proposition 6.2.4 translated to the case of a convergence of a sequence of spaces. Suppose a sequence (z_n) such $z_n \in \mathbb{S}_\epsilon \cap F_n$. Passing to a subsequence if necessary, we assume the convergence of $F_n \to T$ as well as $T_{z_n}\mathbb{S}_\epsilon \to T_a\mathbb{S}_\epsilon$, where $z \to a$. Since $T_{z_n}F_n \subset T_{z_n}\mathbb{S}_\epsilon$, it holds the same inclusion for $T \supset T_a S_\alpha$, for some stratum $S_\alpha \subset V$. But this latter is transversal to the sphere, yielding a contradiction. □

6.3 Milnor-Lê Fibration

We restate Milnor fibration theorem.

Theorem 6.3.1. *If ϵ is sufficiently small, the map*

$$\phi := \frac{f(z)}{|f(z)|} : \mathbb{S}_\epsilon \setminus K_\epsilon \longrightarrow \mathbb{S}^1 \tag{6.1}$$

is a fiber bundle. The fibers are denoted by $F_\theta = \phi^{-1}(e^{i\theta})$.

Recall that $f^{-1}(\partial\mathbb{D}_\delta) \cap B_\epsilon = \mathcal{N}(\epsilon, \delta)$. Also, this hypersurface has $\operatorname{grad} \log f(z)$ as the normal vector at each point. We also have that this tube is compact.

Theorem 6.3.2. *The restriction $f : \mathcal{N}(\epsilon, \delta) \longrightarrow \partial\mathbb{D}_\delta$ is the projection of a fiber bundle.*

Proof. We shall apply Ehresmann Fibration Theorem. The Milnor tube has boundary $\partial\mathcal{N}(\epsilon, \delta) = f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{S}_\epsilon$. The previous result shows that this intersection is transversal and then $f_{\partial\mathcal{N}(\epsilon,\delta)}$ is a submersion. Since $f_{\mathbb{B}_\epsilon \setminus V}$ is a submersion and the tube lies on this complement, f as above is a submersion. Finally, since the tube is compact, the map is proper and we can apply the previously mentioned result. □

6.4 The Equivalence

In order to compare the fiber bundles, we may consider a diffeomorphism between $\partial\mathbb{D}_\delta$ and $\mathbb{S}^1$ through the projection $\pi(z) = z/|z|$. The result can be described by

$$\Phi_{|\mathcal{N}(\epsilon,\delta)} = \pi \circ f : \mathcal{N}(\epsilon, \delta) \longrightarrow \mathbb{S}^1. \tag{6.2}$$

Moreover, it is not possible to compare the tube and $\mathbb{S}_\epsilon \setminus K$. Therefore, we consider $\Gamma := \mathbb{S}_\epsilon \setminus f^{-1}(\mathbb{D}_\delta)$, where $\mathbb{D}_\delta$ is the open disc of radius δ. The map is described by

$$\Phi_{|\mathbb{S}_\epsilon \setminus f^{-1}(\mathbb{D}_\delta)} = \pi \circ f : \mathbb{S}_\epsilon \setminus f^{-1}(\mathbb{D}_\delta) \longrightarrow \mathbb{S}^1. \tag{6.3}$$

The reason is the following. The set Γ is exactly the union of the portions $\{z \in F_\theta : |f(z)| \geq \delta\}$ of all fibers F_θ, and the diffeomorphism constructed from the vector field in 6.1.2 has the image in this set. On the other hand, if we want to study the equivalence with the original fiber bundle 6.2, we must exclude the boundary of the tube, and then generalize the diffeomorphism given by Theorem 6.1.6 to all $\mathbb{S}_\epsilon \setminus K$ (see [13, Remarks, Chapter 11]).

Theorem 6.4.1. *The restriction of the Milnor fiber bundle* (6.3) *and the Milnor-Lê fiber bundle* $f : \mathcal{N}(\epsilon, \delta) \longrightarrow \mathbb{S}^1$ *are equivalent.*

Proof. To obtain the desired diffeomorphism, it is enough to use the correspondence in 6.1.3 restricted to the tube. For interior points, the correspondence sends the interior of the tube to $\mathbb{S}_\epsilon \setminus K$. The boundary $\partial(\mathbb{S}_\epsilon \setminus f^{-1}(\mathbb{D}_\epsilon))$ is exactly $f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{S}_\epsilon$. In order to glue these correspondences, we must verify that they do not interpolate each other. It is enough to see that $w(z)$ is transversal to the tube, that is, the integral curves do not approach $\partial\mathcal{N}(\epsilon, \delta)$. But this is a consequence of the fact that $\mathrm{Re}\langle w(z), \operatorname{grad}\log f(z)\rangle \neq 0$ since $\operatorname{grad}\log f(z)$ is normal to the tube. Moreover, we have that

$$\Phi(z) = \theta(z) = \theta(z') = \Phi(z').$$

Thus, we obtain that the tube is diffeomorphic to

$$\Gamma = \mathbb{S}_\epsilon \setminus f^{-1}(\mathbb{D}_\delta) = \{z \in \mathbb{S}_\epsilon \setminus K : |f(z)| \geq \delta\}.$$

□



Figure 6.6: Diffeomorphism on the interior and transversal integral curves

We state as reference the third fiber bundle.

Theorem 6.4.2. *The Milnor fiber bundle* (6.1) *and the restriction* $\tilde{f} : \mathcal{N}(\epsilon, \delta) \setminus \mathbb{S}_\epsilon \longrightarrow \mathbb{S}^1$ *of the Milnor-Lê fiber bundle are equivalent.*

Remark 6.4.3. The usual correspondence in Section 1 shows that $\mathcal{N}(\epsilon, \delta) \setminus \mathbb{S}_\epsilon$ is diffeomorphic to $\{z \in \mathbb{S}_\epsilon \setminus K : |f(z)| > \delta\}$. For each fiber and small δ, this portion is diffeomorphic to the whole fiber. Therefore, the proof of the previous theorem should consist in considering these diffeomorphisms simultaneously.

Chapter 7

Real Fibrations

We prove a fibration theorem for real analytic mappings $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ with an isolated critical point. The projection map is constructed first over a tube and then extended to the complement of the link in the sphere. We also discuss the projection map of this extension.

7.1 Introduction

Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be an analytic mapping such that $f(0) = 0$. We require that the origin is an isolated critical point, that is, there exists a neighborhood U of the origin such that the derivative df_x has rank k for all $x \in U \setminus \{0\}$. First, we show the existence of a tubular neighborhood of the link $K_\epsilon = V \cap \mathbb{S}_\epsilon^{m-1}$.

Lemma 7.1.1. *The origin is a regular value of the restriction $f : \mathbb{S}_\epsilon^{m-1} \longrightarrow \mathbb{R}^k$.*

Proof. We have proved that the intersection $V \cap \mathbb{S}_\epsilon^{m-1}$ is transversal. Since $T_xV = \ker df_x$, it means that the tangent space of the sphere and the kernel of df_x are transversal spaces, and the result follows. □

The previous result implies the existence of a closed neighborhood $\mathbb{B}_\eta^k$ of the origin consisting only of regular values of f. We then obtain a fiber bundle.

Lemma 7.1.2. *For a sufficiently small δ, the fibers of f over $\mathbb{B}_\delta^k$ intersect the sphere $\mathbb{S}_\epsilon^{m-1}$ transversally.*

Proof. Recall that we already have $V \cap \mathbb{S}_\epsilon^{m-1}$ as a transversal intersection. By continuity, for each point x in the intersection, there exists an open neighborhood U_x such that the derivative of the restriction of f is a submersion over U_x. Set $U := \bigcup U_x$.

Suppose the statement is false, then there exists a sequence $(c_n) \subset \mathbb{R}^k$ converging to zero and a sequence $x_n \in f^{-1}(c_n)$ for which $T_{x_n} f^{-1}(c_n)$ does not intersect the sphere transversally. By compactness and continuity, there exists a limit point $x \in V \cap \mathbb{S}_\epsilon$, up to pass to a subsequence. Since $x \in \mathbb{S}_\epsilon^{m-1} \setminus U$, because this set is closed, this is a contradiction. □

Theorem 7.1.3. *Consider the following set:*

$$T = \{x \in \mathbb{S}_\epsilon^{m-1} : |f(x)| \leq \eta\}.$$

The restriction $f : T \longrightarrow \mathbb{B}_\eta^k$ defines a trivial fiber bundle, that is, $T \simeq K \times \mathbb{B}_\eta^k$. The set T is called a tubular neighborhood of K.

Proof. The previous lemma shows that f is a submersion on $f^{-1}(\mathbb{B}_\eta^k) \cap \mathbb{S}_\epsilon^{m-1}$. Since T is a closed set in a compact, we conclude by Ehresmann Theorem. The triviality is a consequence of the fact that the base space is contractible [11, Corollary 1.8]. □

7.2 Obtaining a Fibration

Let $\mathcal{N}(\epsilon, \eta) := \mathbb{B}_\epsilon^m \cap f^{-1}(\mathbb{S}_\eta^{k-1})$ be a tube inside the ball with radius ϵ and centered at the origin. Similar to the holomorphic case, we have a fibration on this tube.

Theorem 7.2.1. *The restriction $f : \mathcal{N}(\epsilon, \eta) \longrightarrow \mathbb{S}_\eta^{k-1}$ defines a fiber bundle with typical fiber $F_y = \mathbb{B}_\epsilon^m \cap f^{-1}(y)$ with boundary $\partial F_y = \mathbb{S}_\epsilon^{m-1} \cap f^{-1}(y)$ diffeomorphic to K_ϵ.*

Proof. The proof is again an application of Ehresmann Theorem. We have that f is a submersion over the interior part of the tube and for the restriction to the boundary it is a consequence of $f : \mathbb{S}_\epsilon^{m-1} \longrightarrow \mathbb{R}^k$ having $\mathbb{S}_\eta^{k-1}$ as regular value set. It remains to study the last assertion. But this is a consequence of $f^{-1}(y) \cap \mathbb{S}_\epsilon^{m-1}$ being a fiber of the trivial bundle in Theorem 7.1.3. □

Remark 7.2.2. *The difference with the holomorphic function case begins here. Even though we have a stratification for V such that each stratum intersects the sphere transversally, a similar proof we have given for the tubular fibration does not hold because not necessarily the origin is a critical isolated value. That is, we do not have a Bertini-Sard type theorem. it implies that the fibers $f^{-1}(c_n)$, for c_n going to the origin, are not necessarily manifolds.*

7.3 Characterization of the Total Space

As in the holomorphic case, we are interested in a fiber bundle over the sphere. We will show that the tube is indeed diffeomorphic to a portion of the sphere $\mathbb{S}_\epsilon^{m-1}$. This is very similar to the equivalence of the Milnor-Lê fibration with a restriction of the classical Milnor fibration. To prove this, we proceed as before by considering the integral curves of a suitable vector field.

Lemma 7.3.1. *There exists a vector field $w(x) \in \mathbb{B}_\epsilon \backslash V$ such that the inner products $\langle w(x), x\rangle$ and $\langle w(x), \operatorname{grad} |f(x)|^2\rangle$ are positive.*

Proof. The construction is also local and then extended by applying partition of unity. For each x_0 we have two possibilities. If the vectors x_0 and $\operatorname{grad} |f(x_0)|^2$ are linearly independent, it is enough to solve the equations

$$\langle w(x), x\rangle = \langle w(x), \operatorname{grad} |f(x)|^2\rangle = 1$$

around x_0. If $\operatorname{grad} |f(x_0)|^2 = \lambda_0 x_0$, define $w(x) = \lambda_0 x$. We obtain

$$\begin{aligned} \langle w(x_0), \operatorname{grad} |f(x)|^2\rangle &= \lambda_0^2 x_0^2 \\ \langle w(x_0), x_0\rangle &= \lambda_0 x_0^2 \end{aligned}$$

The [13, Corollary 3.4] implies that the two vector fields do not point in opposite directions (since $2x = \operatorname{grad} |x|^2$, the hypothesis is satisfied) so that both the last equation is positive. By continuity the equations above hold for x in a neighborhood of x_0. □

Corollary 7.3.2. *The tube $\mathcal{N}(\epsilon, \eta)$ is diffeomorphic to the complement $\mathbb{S}_\epsilon^{m-1} \setminus \operatorname{int} T$ of an open tubular neighborhood in the sphere.*

Proof. Since $\partial\mathcal{N}(\epsilon, \eta) = f^{-1}(\mathbb{S}_\eta^{k-1}) \cap \mathbb{S}_\epsilon^{m-1} = \partial\left(\mathbb{S}_\epsilon^{k-1} \setminus \operatorname{int} T\right)$, we set the map as the identity over the boundary. Starting from the inner part of the tube, we travel along the integral curves $p(t)$ of $w(x)$. First, note that:

$$\frac{d}{dt}|f(p(t))|^2 = \langle p'(t), \operatorname{grad} |f(p(t))|^2\rangle > 0$$

so that $|f|$ strictly increases along the curve $p(t)$. In a similar way, $|p(t)|$ also increases. Setting the usual correspondence, we obtain the desired diffeomorphism. □

Let denote the normal bundle of a submanifold $Z \subset Y$ by $N(Z;Y)$. This manifold has the same dimension as Y. We will show that the normal bundle can be shrunken to a tubular neighborhood of the zero section Z and the converse is also true for ϵ-neighborhoods [10, Section 2.3]. We consider $Z \subset Y \subset \mathbb{R}^M$ for sufficiently large M.

Theorem 7.3.3. *Let $\pi : N(Z;Y) \longrightarrow Y$ be the normal bundle of $Z \subset Y$. If U is an open neighborhood of Z in Y, then there exists an ϵ-neighborhood $V \subset U$ diffeomorphic to $N(Z;Y)$. Conversely, every ϵ-neighborhood V is diffeomorphic to $N(Z;Y)$.*

Proof. First, the existence of an ϵ-neighborhood in U is a consequence of [10, Exercise 1, Section 2.3]. Here, ϵ is a positive function. Let $f : [0,\infty) \longrightarrow [0,1)$ be a diffeomorphism such that f is the identity on a neighborhood of the origin (this is a bump function, see [10, Chapter 1]). We claim that the following diffeomorphism is the desired one:

$$\psi(\eta) = \epsilon(x) f(|\eta|)\frac{\eta}{|\eta|},\ \eta \in \pi^{-1}(x).$$

Its inverse is

$$\psi^{-1}(v) = \left(\epsilon\left(\pi^{-1}(\pi(v))\right)\right)^{-1/2} f^{-1}(|v|)\frac{v}{|v|}.$$

We observe that the maps are well defined, since in a neighborhood of the zero section Z they are the identity. Moreover, the factor $\epsilon(x)$ is not a problem to smoothness in ψ by the local trivialization, in such a way that we may define it locally and then glue by a partition of unity. □

In some sense, the proof means that we rescale the normal bundle according to the function ϵ. Recall that the closure $\overline{F}_\theta$ of the Milnor fiber in the holomorphic function case had a trivial normal bundle, as well as the link. The same is true here and both conclusions allow us to extend fiber bundles on the sphere arising from the fiber bundle over the tube.

Corollary 7.3.4. *The link K_ϵ has a trivial normal bundle in the sphere.*

Proof. The interior of the tubular neighborhood T in Theorem 7.1.3 is an ϵ-neighborhood, by definition, and then the normal bundle is trivial. □

Proposition 7.3.5. *The fiber bundle over $\mathbb{S}_\epsilon^{m-1} \setminus \operatorname{int} T$ can be extended to $\mathbb{S}_\epsilon \setminus K_\epsilon$ in such way that each fiber is the interior of a compact manifold bounded by the link K_ϵ.*

Idea. The previous proofs consist of *inflate* the tube inside the ball $\mathbb{B}_\epsilon$ to a portion of the sphere. This is done through a certain flow whose image does not cover totally the complementary of the link in the sphere. Thus, we extend these paths into $\mathbb{S}_\epsilon \setminus K_\epsilon$ along a flow, which is possible due to the triviality of the normal bundle of K_ϵ. The projection map is taken to be a submersion over $\operatorname{int} T \setminus K_\epsilon$. □

7.4 On the Projection Map

The reason that the projection map is no longer $f/|f|$ is that the directions $f(x)/|f(x)|$ are not preserved along the flow of $w(x)$, that is, the extension of the projection map may not preserve the previous fiber in such way that the extensions are glued in concordance with $f(x)/|f(x)|$. The following concrete counter-example was given by Milnor in [13, Chapter 11].

Example 7.4.1. Let $f = (x, x^2 + y(x^2 + y^2))$ be an analytic mapping. If $\phi := f/|f|$ is defined on $\mathbb{S}^1_\epsilon$, the expression becomes:

$$\phi(x, y) = \frac{(x, x^2 + y\epsilon^2)}{(x^2 + y^2\epsilon^4 + x^4 + 2x^2y\epsilon^2)}.$$

The fiber of $(0, 1) \in \mathbb{S}^1$ consists of a single point, but the fiber of $(\sqrt{2}/2, \sqrt{2}/2)$ has more than one point. Indeed, we obtain the following equations:

$$\begin{aligned} x &= x^2 + y\epsilon^2 \\ y &= \frac{x - x^2}{\epsilon^2} \\ \epsilon^2 &= x^2 + y^2 \\ 0 &= x^4 - 2x^3 + x^2(\epsilon^4 + 1) - \epsilon^6 \end{aligned}$$

Let denote the last quartic equation by $p(x)$. Considering $\epsilon < 1$, since $p(0) < 0$ and $p(\pm\epsilon) = \epsilon^2(\epsilon^2 \pm 2\epsilon + 1) > 0$, by the intermediate value theorem, there exist at least two real roots with modulo less than ϵ. This implies that there exists more than one point in this fiber and so the map is not a fiber bundle.

On the other hand, we may locally *inflate* the tube through the integral curves corresponding to the vector fields lifted from the radial vector field in $\mathbb{R}^k$ by f, which is a

submersion on $T \setminus K_\epsilon$ by the previous proposition. It follows that the projection map of the fiber bundle may be taken as the usual $f/|f|$ over an open tubular T' neighborhood of K_ϵ. We summarize the results.

Theorem 7.4.2. *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be a real analytic mapping such that $f(0) = 0$ and the origin is an isolated singularity. For a sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that the restriction*

$$f : \mathcal{N}(\epsilon, \delta) \longrightarrow \mathbb{S}^{k-1}$$

is a fiber bundle, where $\mathcal{N}(\epsilon, \delta) = \mathbb{B}_\epsilon \cap f^{-1}(\mathbb{S}_\delta^{kndle,whe-1})$ is the Milnor tube. Moreover, this tube is diffeomorphic to

$$\mathbb{S}_\epsilon^{m-1} \setminus f^{-1}(\mathbb{D}_\delta),$$

where $\mathbb{D}_\delta \subset \mathbb{R}^k$ is the open disc centered at the origin with radius δ. The induced projection on this complement can be extended to a fiber bundle

$$\phi : \mathbb{S}_\epsilon^{m-1} \setminus K_\epsilon \longrightarrow \mathbb{S}^{k-1}$$

which has the expression $f/|f|$ over an open tubular neighborhood T' of K_ϵ.

7.5 Remarks

Besides the question about the expression of the projection map, the real fibration has one more weakness. The assumption of isolated singularity is very strong so explicit examples are not easy to obtain, although Milnor gave some examples [13, p.103]. Also, the holomorphic mappings with isolated singularities arise as examples. On the other hand, we should require non-trivial examples. Indeed, in the case of regular points, the fiber bundle in the holomorphic case had Milnor fiber diffeomorphic to a Euclidean space. Therefore, Milnor suggested that nontrivial examples are those for which the Milnor fibers are not diffeomorphic to balls of suitable dimension.

For some pairs (n, p) such examples do not exist, and for the other cases, Church and Lamotke gave partial answers in the 1960s. At this period, explicit examples were known only for $p = 2$, and more recently others were presented. The problem related to the pairs (n, p) was completely solved only in 2016. Finally, this question motivated the investigations of mixed function resulting in a new line of research in Singularity Theory.

Chapter 8

Examples for the case $p = 2$

These are the notes for the partial studies of [17]. The article introduces the now-called mixed functions and the classification of the classes for which a Milnor fibration exists with the projection map with the form $f/|f|$.

8.1 Introduction

In his classical book, Milnor proved the Fibration Theorem for holomorphic functions $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. Roughly speaking, this result describes the topology of the function near its singular points. A similar result for analytic mappings $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ can be also derived if we suppose the origin to be an isolated critical point. Milnor asked for what he called non-trivial examples, that is, isolated singularities with the fiber not diffeomorphic to a disk.

Such examples were first given by Church and Lamotke based on Looijenga's results. However, concrete examples were not described. For the case $p = 2$, N. A'Campo exhibited an example and others were presented decades later by J. Seade. The main objects studied in the article derive from his construction.

Consider $\langle ., . \rangle$ the usual Hermitian product in $\mathbb{C}^n$ and holomorphic vector fields F and X. We define the following analytic function:

$$\psi_{F,X}(z) = \langle F(z), X(z) \rangle = \sum_i F_i(z)\overline{X}_i(z).$$

The cases $F(z) = (k_1 z_{\sigma_1}^{a_1}, \ldots, k_1 z_{\sigma_n}^{a_n})$ and $X(z) = (t_1 z_{\sigma_1}^{b_1}, \ldots, t_1 z_{\sigma_n}^{b_n})$, where $t_i, k_i \in \mathbb{C}$, $a_i, b_i \in \mathbb{N}$, and σ is a permutation of $\{1, \ldots, n\}$ are of particular interest since they provide the examples mentioned above.

The article classifies the functions as above admitting the isolated critical point condition. This is called *Milnor condition at* 0. A second problem with the real case is the following. Milnor shows that the total space of the fibration is diffeomorphic to the complement of an open tubular neighborhood of the link in the sphere and that this can be extended to the whole sphere minus the link. However, the projection is not necessarily the usual one given by $f/|f|$.

For the functions satisfying the Milnor condition, if the extended projection map is of the form $\psi/|\psi|$, we say that $\psi_{F,X}$ satisfies the *strong Milnor condition at* 0. The article also classifies such functions. Furthermore, results of robustness and topology are presented.

8.2 On Milnor Conditions and Finite Determinacy

Preliminaries

We have seen that the real case presents some obstructions between finite determinacy and isolated singularity. Although this problem, we may conclude C^k-finite determinacy in some cases. It will be related to the isolated singularity condition.

Theorem 8.2.1. *Let $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}$ and $N_{\mathcal{G}}(f)$ equals to:*

1. $N_{\mathcal{R}}(f) = \delta_1^2 + \cdots + \delta_l^2$ *if* $\mathcal{G} = \mathcal{R}$, *where* $\delta_i = \frac{\partial f}{\partial x_i}$;
2. $N_{\mathcal{C}}(f) = f_1^2 + \cdots + f_p^2$ *if* $\mathcal{G} = \mathcal{C}$, *where* f_i *are the coordinate functions of* f;
3. $N_{\mathcal{K}}(f) = N_{\mathcal{C}}(f) + N_{\mathcal{R}}(f)$ *if* $\mathcal{G} = \mathcal{K}$.

If $|N_{\mathcal{G}}(f)| \geq c|x|^{\alpha}$ *for some constants* $c, \alpha > 0$, *then there exists* $r = r(\alpha, k, \mathcal{G})$ *such that* f *is* r-C^k-$\mathcal{G}$-*finite determined for all* $0 \leq k < \infty$.

More specifically, the involved diffeomorphisms of the respective group are of class C^k. In this case, the groups act on the space of C^k-maps and this induces an action on the space of smooth maps and we consider this last action.

Corollary 8.2.2. *Suppose* $|N_{\mathcal{G}}(f)| \geq c|x|^{\alpha}$ *for some constants* $c, \alpha > 0$.

1. *If* $\mathcal{G} = \mathcal{R}$, *then* f *has isolated singularity.*
2. *If* $\mathcal{G} = \mathcal{K}$, *then* f *satisfies* $\Sigma_f \cap f^{-1}(0) = \{0\}$.

In other words, the stability conditions are recovered as in the C^∞-finite-determined case.

Proof. For $\mathcal{G} = \mathcal{R}$ the hypothesis implies that $\mathcal{V}(N_\mathcal{R}(f)) = \mathcal{V}(\langle N_\mathcal{R}(f)\rangle) = \mathcal{V}(J_f) = \{0\}$, since the function vanishes only at the origin. Similarly for $\mathcal{G} = \mathcal{K}$, because:

$$\begin{aligned}\mathcal{V}(J_f + < f_1, \ldots, f_p >) &= \mathcal{V}(N_\mathcal{K}(f)) \\ &= \mathcal{V}(J_f) \cap V(f_1, \ldots, f_p) \\ &= \Sigma_f \cap f^{-1}(0)\end{aligned}$$

□

Proposition 8.2.3. *Let $f : (\mathbb{F}^n, 0) \longrightarrow (\mathbb{F}^p, 0)$, where $n \geq p$ and the coordinate functions f_i are homogeneous of degree d. Suppose that $\Sigma(f) = \{0\}$, Then f is d-C^k-$\mathcal{R}$-finitely determined for all $1 \leq k < \infty$.*

The proof is similar to the function case for the group $\mathcal{G} = \mathcal{R}$ for which a Thom-Levine vector field is constructed. But here, the field has class C^k determined by a control function.

Example 8.2.4. The germ $f = x^5 + y^5$ is 6-$\mathcal{R}$-determined and it is not $\mathcal{R}$-equivalent to $g = x^5 + x^3y^3 + y^5$, although they are C^1-$\mathcal{R}$-equivalent.

Some Results

Theorem 8.2.5. *Let $f \in C(n, p)$ be an analytic germ. Then the Milnor condition at 0 is equivalent to C^l-$\mathcal{R}$-finite determinacy for every $l \in [0, \infty)$.*

Proof. The Milnor condition is equivalent to f satisfy $\mathcal{V}N_\mathcal{R}(f) = \{0\}$, since f is analytic. It is also proved that this is equivalent to f be $C^l - \mathcal{R}$-finitely determined for every $0 \leq l < \infty$, see [16, Proposition 2.4 and Corollary 3.14] and [19, Section 2.6]. □

The result implies the following. Given F, X vector fields and $\psi_{F,X}$ the associated real analytic function, if we add terms of high degree to F and X to form ψ_{F_*,X_*} this new function is C^l-$\mathcal{R}$-equivalent to $\psi_{F,X}$ for every l and thus satisfies the Milnor condition at 0. In particular, they are topologically equivalent [17, Corollary 1.3].

Other types of perturbations are allowed if we restrict to the cases of $\psi_{F,X}$ be weighted homogeneous and F, X be homogeneous. More specifically, the perturbations are described in a more detailed way. Moreover, vector fields of Morse type, that is, vector fields satisfying a tangency condition with spheres, are considered under perturbations of the metric. For this class, the group $\mathcal{K}$ is useful since it involves the singular set and the zero set.

8.3 Classification Theorems

Consider F and X holomorphic vector fields as defined above. We shall classify under which conditions on the exponents a_i, b_i according to the permutation one has that $\psi_{F,X}$ satisfies the Milnor condition at 0. First, one can assume that $t_i = 1$ and then $X = (z_1^{b_1}, \ldots, z_n^{b_n})$.

The Milnor Condition

It is possible to identify $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ through the following vector identities:

$$2x = z + \overline{z}$$
$$2y = z - \overline{z}$$

If $\phi : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a real analytic function and ϕ_1, ϕ_2 are the real and the imaginary parts, respectively, its singular points can be identified by studying the minors of the form

$$\begin{pmatrix} (\phi_1)_{z_1} & (\phi_1)_{\overline{z}_1} \\ (\phi_2)_{z_1} & (\phi_2)_{\overline{z}_1} \end{pmatrix}$$

If σ is a permutation of $\{1, \ldots, n\}$ it has an associated decomposition into cycles $\sigma^1, \ldots, \sigma^r$. Let σ_r be a cycle of length r. Up to a change of coordinates, we can assume that the r-first coordinates of F are

$$(k_1 z_2^{a_1}, \ldots, k_{r-1} z_r^{a_r - 1}, k_r z_1^{a_r}).$$

In these coordinates, we have that

$$\psi(z) = k_1 z_2^{a_1} \overline{z}_1^{b_1} + k_{r-1} z_r^{a_r - 1} + k_r z_1^{a_r} \overline{z}_r^{b_r} + \text{terms in other variables.}$$

Hence, we can analyze the singular points of ψ by looking at its decomposition according to the cycles of σ. In other words, we can restrict ourselves to $F = (k_1 z_2^{a_1}, k_2 z_3^{a_2}, \ldots, k_r z_1^{a_r})$ and $X = (z_1^{b_1}, \ldots, z_r^{b_r})$. One can think that ψ is a sum of ψ_r defined by vector fields in $\mathbb{C}^r$ in the restricted coordinates only, but the explicit form depends on a re-label (linear change) of coordinates, which does not change the study of singular points.

Theorem 8.3.1. *Assume $a_i > 1$ for all $i = 1, \ldots, n$. Then ψ satisfies the Milnor condition at 0 if and only if either one of the following conditions is satisfied.*

1. *The permutation σ is the identity and $a_i \neq b_i$ for all $i = 1, \ldots, n$.*

2. *The permutation $\sigma = \sigma_1 \circ \cdots \circ \sigma_r$ with cycle σ_r of length r is such that:*

   (a) *If $r = 1$, the correspondent exponents satisfy $a_i \neq b_i$.*

   (b) *If $r > 1$, the correspondent exponents $b_i = 1$.*

Remark 8.3.2. *In the canonical form stated above, the associated exponents are $a_1, b_1, \ldots, a_r, b_r$.*

Idea. Under the canonical form, the function has the form

$$\psi_r = k_1 z_2^{a_1} \overline{z}_1^{b_1} + k_2 z_3^{a_2} \overline{z}_2^{b_2} + \cdots + k_r z_1^{a_r} \overline{z}_r^{b_r}.$$

The part of ψ_r containing the variables z_i and $\overline{z}_i$ is

$$k_{i-1} z_i^{a_{i-1}} \overline{z}_{i-1}^{b_i - 1} + k_i z_{i+1}^{a_i} \overline{z}_i^{b_i}.$$

And the Jacobian matrix related to these variables is

$$\begin{pmatrix} a_{i-1} z_i^{a_{i-1}-1} \overline{z}_{i-1}^{b_{i-1}} + b_i \overline{k}_i z_i^{b_i - 1} \overline{z}_{i+1}^{a_i} & b_i k_i z_{i+1}^{a_i} \overline{z}_i^{b_i - 1} + a_{i-1} \overline{k}_{i-1} z_{i-1}^{b_i - 1} \overline{z}_i^{a_{i-1}-1} \\ a_{i-1} z_i^{a_{i-1}-1} \overline{z}_{i-1}^{b_{i-1}} - b_i \overline{k}_i z_i^{b_i - 1} \overline{z}_{i+1}^{a_i} & b_i k_i z_{i+1}^{a_i} \overline{z}_i^{b_i - 1} - a_{i-1} \overline{k}_{i-1} z_{i-1}^{b_i - 1} \overline{z}_i^{a_{i-1}-1} \end{pmatrix}$$

whose determinant is

$$a_{i-1}^2 |k_{i-1}|^2 |z_i|^{2a_{i-1}-2} |z_{i-1}|^{2b_{i-1}} - |k_i|^2 b_i^2 |z_{i+1}|^{2a_i} |z_i|^{2b_i - 2}.$$

Here, the indices i are taken modulo r, the cycle length. If $\lambda_i = |k_i|$ and $\nu_i = |z_i|^2$, we obtain the following system:

$$\begin{aligned} \lambda_1 a_1^2 \nu_2^{a_1 - 1} \nu_1^{b_1} &= \lambda_2 b_2^2 \nu_3^{a_2} \nu_2^{b_2 - 1} \\ &\vdots \\ \lambda_r a_r^2 \nu_1^{a_r - 1} \nu_r^{b_r} &= \lambda_1 b_1^2 \nu_2^{a_1} \nu_1^{b_1 - 1} \end{aligned}$$

If the permutation is the identity, the equation is reduced to

$$a_i^2 |k_i|^2 |z_i|^{2a_i - 2} |z_i|^{2b_i} - |k_i|^2 b_i^2 |z_i|^{2a_i} |z_i|^{2b_i}.$$

The condition becomes $a_i \neq b_i$. If the cycle has length $r = 1$, the determinant is also reduced for this form.

In general, if the rank of $D\psi$ is less than 2, all of these determinants vanish and we obtain a system of equations for which the existence of some $b_j > 1$ implies singularities at the points z with $z_{j-1} = z_j = z_{j+2} = z_{j+3} = 0$. we detail the procedure for the case $r = 3$. The system is

$$\begin{aligned}
\lambda_1 a_1^2 \nu_2^{a_1-1} \nu_1^{b_1} &= \lambda_2 b_2^2 \nu_3^{a_2} \nu_2^{b_2-1} \\
\lambda_2 a_2^2 \nu_3^{a_2-1} \nu_2^{b_2} &= \lambda_3 b_3^2 \nu_1^{a_3} \nu_3^{b_3-1} \\
\lambda_3 a_3^2 \nu_1^{a_3-1} \nu_3^{b_3} &= \lambda_1 b_1^2 \nu_2^{a_1} \nu_1^{b_1-1}
\end{aligned}$$

Since some $b_i > 1$, for instance, $i = 3$, it follows that the points $z = (z_1, 0, 0)$ are solutions and the singularity is not isolated. On the other hand, if $b_i = 1$ for all i and ν is a solution and some $\nu_i = 0$, then all $\nu = 0$. Then, suppose the existence of a critical point which is a solution of

$$\prod [\lambda_i a_i^2 \nu_i^{a_i-1}] = \prod [\lambda_i \nu_i^{a_i-1}]$$

then $a_i = 1$, which gives a contradiction. □

Example 8.3.3. The function $\psi = \sum_{i=1}^n z_1|z_1|^2 + \cdots + z_n|z_n|^2$ satisfies the Milnor condition at the origin since it is formed by the vector fields $F = (z_1, \ldots, z_n)$ and $X = (z_1, \ldots, z_n)$.

Example 8.3.4. On the other hand, the singularities $\psi = \sum_{i=1}^n |z_i|^4$ do not satisfy Milnor condition at 0 since it is formed from a vector field X with exponents $b_i > 1$.

Example 8.3.5. The function $\psi = |z_1|^2 + \cdots + |z_{n-1}|^2 + z_n^2 \overline{z}_n$ does not satisfy the Milnor condition at the origin since it is formed by the vector fields $F = (z_1, \ldots, z_{n-1}, z_n^2)$ and $X = (z_1, \ldots, z_n)$.

Example 8.3.6. The function $\psi = z_2^2 \overline{z}_1 + z_3^2 \overline{z}_2 + \overline{z}_3 z_1^2$ satisfies the condition since it is formed by $F = (z_2^2, z_3^2, z_1^2)$ and $X = (z_1, \ldots, z_n)$.

Under some assumptions, we may consider some of the exponents equal to 1.

Theorem 8.3.7. *Let F be as above except we allow $a_i \geq 1$ and X is the radial vector field. The function $\psi_{F,X}$ satisfies the Milnor condition if and only if the following conditions hold:*

1. *For each cycle of σ with odd length, there is at least one $a_i > 1$.*

2. *For each cycle of σ with even length, either there is at least one $a_i > 1$, or else the corresponding coefficients satisfy*

$$|k_i k_{i+2} \dots k_{i+2m}| = |k_{i+1} k_{i+3} \dots k_{i+2m+1}|.$$

Example 8.3.8. The function $\psi = z_2\overline{z}_1 + z_3\overline{z}_2 + z_1\overline{z}_3$ does not satisfy the condition, since it is formed by the radial vector field X and $F = (z_2, z_1, z_3)$. On the other hand, the function $z_2^2\overline{z}_1 + z_3\overline{z}_2 + z_1\overline{z}_3$. Note that the cycle (123) has odd length.

Example 8.3.9. The function $z_2^2\overline{z}_1 + z_4\overline{z}_2 + z_1\overline{z}_3 + z_3\overline{z}_4$ satisfies the Milnor condition since it is formed by the radial vector field and $F = (z_2^2, z_4, z_1, z_3)$.

Example 8.3.10. The function $\psi = z_2\overline{z}_1 + 2z_4\overline{z}_2 + z_1\overline{z}_3 + z_3\overline{z}_4$ satisfies the Milnor condition since the vector field $F = (z_2, 2z_4, z_1, z_3)$ is defined by a cycle with length and the coefficients associated satisfy the condition of the theorem.

The Strong Milnor Condition

Let $F = (k_1 z_{\sigma_1}^{a_1}, \dots, k_n z_{\sigma_n}^{a_n})$ and $X = (t_1 z_1, \dots, t_n z_n)$ be vector fields as above such that the function $\psi_{F,X}$ satisfies the Milnor condition at 0. The function satisfies the *strong Milnor condition* at 0 if for every sufficiently small ϵ the map

$$\phi = \psi/|\psi| : \mathbb{S}_\epsilon^{2n-1} \setminus K_\epsilon \longrightarrow \mathbb{S}^1$$

is the projection map of a smooth fiber bundle. The goal of this section is to establish sufficient conditions on the vector field F for ψ satisfying this stronger condition.

If $v \in \mathbb{R}^2 \simeq \mathbb{C}$, we claim that

$$\mathrm{proj}_{\mathcal{L}_{-\theta}} v = \mathrm{Re}[e^{i\theta}.v].$$

Indeed, the real dot product can be recovered from the Hermitian product:

$$\begin{aligned} \mathrm{proj}_{\mathcal{L}_{-\theta}} v &= \langle v, (\cos\theta, -\sin\theta)\rangle_{\mathbb{R}} \\ &= \mathrm{Re}[v.e^{i\theta}] \end{aligned}$$

Define $\psi_\theta(z) = \mathrm{Re}\langle e^{i\theta}F(z), X(z)\rangle$ and let $M := \psi^{-1}(0)$, $M_\theta := \psi_\theta^{-1}(0)$. It follows that ψ_θ operates as the projection of $\psi(z)$ on the line $\mathcal{L}_{-\theta}$. The following map is a prototype for

the projection map of a fiber bundle associated with the singularities above. Define

$$\begin{aligned}\tilde{\phi} : \mathbb{C}^n \setminus M &\longrightarrow \mathbb{S}^1 \\ z &\longmapsto \arg\left(i\langle F(z), z\rangle\right)\end{aligned}$$

and let $E_\theta := \tilde{\phi}^{-1}(\theta)$. If we restrict $X(z)$ to the normal form $(z_1, \ldots, z_n)$, the map map is $\tilde{\phi}(z) = \arg\left(i\overline{\psi(z)}\right) = -\arg\psi(z) \pm \pi/2$, according to the identity $\tan(\theta \pm \pi/2) = \mp 1/\tan(\theta)$. We have the following remarks:

- The level set M_θ is the set of points for which $e^{i\theta}F$, a complex scalar of F, is orthogonal to the vector field X in the real sense.
- The map $\tilde{\phi}$ has the following meaning. It gives the orthogonal direction of $-\arg\psi(z)$, that is, $\psi_{\tilde{\phi}(z)}(z) = 0$. The motivation for this operation is the following. Seade in [18] considered the vector field X as the gradient of some function g. Therefore, $\tilde{\phi}$ is the angle by which $e^{i\tilde{\phi}}F$ is orthogonal to X and then tangent to the level hypersurface $g^{-1}(z)$. In this case, M is the set of points for which the associated foliations are tangent.

The next result describes the relation between M and M_θ, including a map into $\mathbb{S}^1$.

Lemma 8.3.11. *The following statements hold:*

1. *The space $\mathbb{C}^n$ is the union of all M_θ for $\theta \in [0, \pi]$.*
2. *The level set M is the intersection of all M_θ for $\theta \in [0, \pi]$.*
3. *For any distinct $\theta_1, \theta_2 \in [0, \pi)$, $M_{\theta_1} \cap M_{\theta_2} = M$.*
4. *The level set M_θ has the disjoint decomposition $M_\theta = E_\theta \cup M \cup E_{\theta+\pi}$.*
5. *If ψ satisfies the Milnor condition at 0, the sets $M_\theta^* = M_\theta \setminus \{0\}$ are smooth manifolds.*

Remark 8.3.12. *Note that the fibers $E_\theta, E_{\theta+\pi}$ are glued along the "compactification" boundary M to obtain M_θ. This level set can be described as the set of points for which $e^{i\theta}F$ is tangent, as a real vector field, to the spheres.*

Proof. For each $y \in \mathbb{C}^n$, the vector $\psi(y)$ has zero projection on its orthogonal line, therefore, it belongs into some M_θ. For the second statement, if $\psi(y) = 0$, the element $\psi(y)$ has zero projection on every line $\mathcal{L}_{\pm\theta}$, which shows an inclusion. For the other, if $\psi(y) \neq 0$, it must have nonzero projection on its own direction and the statement follows. Also, the zero projection occurs only on its orthogonal projection, so the two-by-two intersection gives precisely M, which gives the third statement. This observation also implies that, if $\psi_\theta(y) = 0$ and $y \notin M$, then $\psi(y) \in \mathcal{L}_{-\theta\pm\pi/2}$. But $\tilde{\phi}(y) = \theta, \theta + \pi$, because $\arg \psi(y) = -\theta \pm \pi/2$, and we conclude. Conversely, if $\tilde{\phi}(y) = \theta, \theta + \pi$, $\arg \psi(z) = -\theta \pm \pi/2$ and thus the projection of $\psi(z)$ on the line $\mathcal{L}_{-\theta}$ is zero. We also remark that the intersection $M \cap E_\alpha$ is empty if $\alpha \neq 0$ and then the union in item 4 is disjoint. If $\alpha = 0$, the intersection is only the origin. □

The objects described in the lemma will be used to define the desired fiber bundle. The next theorem provides sufficient conditions to guarantee the strong Milnor condition.

Theorem 8.3.13. *Let F and X vector fields as above. Assume the following conditions.*

1. *For each cycle of length 1, the correspondent exponents satisfy $a_i \neq b_i$.*
2. *For each cycle of odd length $r > 1$, the correspondent exponents satisfy $a_i \geq 1$ and $b_i = 1$ and at least one $a_i > 1$.*
3. *For each cycle of even length $r > 1$, the correspondent exponents satisfy $a_i \geq 1$, $b_i = 1$ and at least two a_i, say a_{i_1}, a_{i_2} are strictly bigger than 1, with i_1 being odd and i_2 even in the cycle.*

Then $\psi_{F,X}$ satisfies the strong Milnor condition at 0. More specifically, for every sufficiently small sphere $\mathbb{S}^{2n-1}_\epsilon \subset \mathbb{C}^n$ the function

$$\phi = \psi/|\psi| : \mathbb{S}^{2n-1}_\epsilon \setminus K_\epsilon \longrightarrow \mathbb{S}^1$$

is the projection map of a smooth fiber bundle. Each pair of antipodal fibers $\phi^{-1}(e^{i\theta}$, $\phi^{-1}(e^{i(\theta+\pi)}$ are glued together along K_ϵ forming the smooth $(2n-1)$-dimensional manifold

$$M_\theta \cap \mathbb{S}^{2n-1}_\epsilon = \{z \in \mathbb{S}^{2n-1}_\epsilon : \operatorname{Re} e^{i\theta}\langle e^{i\theta}F, X\rangle = 0\}.$$

The theorem will follow from two lemmas. The first one gives a fiber bundle structure for $\tilde{\phi}$ and the second one is that this structure can be restricted to the sphere minus the link.

Lemma 8.3.14. *There exists a norm-preserving smooth action γ_λ of $\mathbb{S}^1$ on $\mathbb{C}^n$ permuting the level sets M_θ. More specifically, for every $\lambda \in \mathbb{R}$, the element γ_λ sends M_θ to $M_{[\theta+\lambda]}$, where $[\theta+\lambda]$ is the class of $\theta+\pi$ under the identification $\mathbb{S}^1 \simeq \mathbb{R} \mod \pi$. In particular, M is invariant for this action.*

Remark 8.3.15. *The relation modulo is taken modulo π because the level sets M_θ are taken with $\theta \in [0, \pi)$.*

Scheme. We construct an action $\Gamma : \mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that

$$\Gamma(\omega, z_1, \ldots, z_n)) = (e^{i\omega s_1} z_1, \ldots, e^{i\omega s_n} z_n)$$

$$\psi_\theta(\Gamma(\omega, z)) = \psi_{\theta+z}(z),$$

where $s_i \in \mathbb{Q}$. Note that the action is the same as $\mathbb{S}^1 \simeq \mathbb{R} \mod \pi$.

We construct the action by cycles. If $r = 1$, then it is enough to define $s_i = 1/(a_j - b_j)$. If $r > 1$, fix the cycle for which $F = (z_2^{a_1}, z_3^{a_2}, \ldots, z_r^{a_1}, z_1^{a_r})$ and consider the matrix

$$A = \begin{pmatrix} -1 & a_1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & a_2 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & -1 & a_{r-1} \\ a_r & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The determinant is $(-1)^r(1 - a_1 \ldots a_r) \neq 0$. The vector $s = (s_1, \ldots, s_n)$ is the solution of $A.s = (1, \ldots, 1)$. The following equations hold

$$\begin{aligned} -s_1 + a_1 s_2 &= 1 \\ -s_2 + a_2 s_3 &= 1 \\ &\vdots \\ -s_{r-1} + a_{r-1} s_r &= 1 \\ -s_r + a_r s_1 &= 1 \end{aligned}$$

If ψ_r is the part of ψ corresponding to the cycle above, then

$$\begin{aligned} e^{i\theta}\psi_r(\Gamma(\omega, z)) &= e^{i\theta} e^{i\omega(-s_1 + a_1 s_2)} z_2^{a_1} \overline{z}_1 + e^{i\theta} e^{i\omega(-s_2 + a_2 s_3)} z_3^{a_2} \overline{z}_2 + \ldots \\ &\quad + e^{i\theta} e^{i\omega(-s_{r-1} + a_{r-1} s_r)} + e^{i\theta} e^{i\omega(-s_r + a_r s_1)} z_1^{a_r} \overline{z}_r \\ &= e^{i(\theta+\omega)} \psi_r \end{aligned}$$

Taking the real parts and applying for each cycle, we obtain that

$$\psi_\theta \circ \Gamma(\omega, z) = e^{i(\theta+\omega)}\psi = \psi_{\theta+\omega}.$$

□

Lemma 8.3.16. *There exists a flow on $\mathbb{C}^n$ whose orbits are transversal to every sphere around $0 \in \mathbb{C}^n$, except for 0 itself, which is a fixed point, and they converge to zero when the time tends to $-\infty$. This flow leaves invariant each M_θ and thus also M.*

Scheme. We construct an action $\Gamma : \mathbb{R}^+ \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that

$$\Gamma(t, z_1, \ldots, z_n)) = (t^{m_1}, \ldots, t^{m_n})$$

$$\psi(\Gamma(t, z)) = t\psi(z),$$

where $m_i \in \mathbb{Q}$ and $m_i > 0$. The action also works as a flow and the condition above shows that ψ is weighted homogeneous. It implies the statement because for each $z_0 \in \mathbb{C}^n$ fixed, the integral curve satisfies:

$$\langle \Gamma'(t, z_0), z_0 \rangle \neq 0$$

and for negative time, the flow goes to zero. This is important to set a fiber bundle for every small sphere.

The action is constructed for each cycle of σ in the following way. If $r = 1$, then it is enough choose $m_i = 1/(a_i + b_i)$. If $r > 1$, suppose the cycle of the form

$$(z_r^{a_1}, z_1^{a_2}, \ldots, z_{r-1}^{a_r}).$$

Consider also the matrix:

$$B = \begin{pmatrix} a_2 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & a_3 & 1 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & a_r & 1 \\ 1 & 0 & 0 & 0 & 0 & a_1 \end{pmatrix}$$

The determinant is $a_1 \dots a_r + (-1)^{r+1} \neq 0$ since $a_i > 0$ and at least one of these exponents is bigger than one. Take the solution $m = (m_1, \dots, m_n)$ such that $B.m = (1, \dots, 1)$. Then

$$\begin{aligned} a_2 m_1 + m_2 &= 1 \\ a_3 m_2 + m_3 &= 1 \\ &\vdots \\ a_r m_{r-1} + m_r &= 1 \\ m_1 + a_1 m_r &= 1 \end{aligned}$$

If ψ_r is the part of ψ correspondent to the cycle, then:

$$\begin{aligned} \psi_r(\Gamma(t,z)) &= t^{(a_1 m_r + m_1)}.z_r^{a_1}\overline{z}_1 + t^{(a_2 m_1 + m_2)} z_1^{a_2}\overline{z}_2 + \cdots + t^{(a_r m_{r-1} + m_r)} z_{r-1}^{a_r}\overline{z}_r \\ &= t\psi_r(z) \end{aligned}$$

The rest of the proof consists of showing that $m_i > 0$ for all i. First, by writing the expressions and considering the cases r even and r odd. Here, the conditions are used to conclude. This is important because the expression depends on a factor of the form $1 + (-1)^{r-1} a_1 \dots a_r$. □

Corollary 8.3.17. *If the function $\psi_{F,x}$ is formed by the vector field F and X above under the conditions on the exponents a_i as in the previous theorem, then ψ is weighted homogeneous.*

Proof of Theorem 8.3.13. The first lemma shows that the level sets M_θ are all diffeomorphic, in particular, M_θ^*. Moreover, we claim that $\tilde{\phi} : \mathbb{C} \setminus M \longrightarrow \mathbb{S}^1$ is the projection map of a fiber bundle. Recall that

$$M_\theta = E_\theta \cup M \cup E_{\theta+\pi}$$

and the action leaves M invariant. We also have that

$$\begin{aligned} \gamma_\lambda(M_\theta) &= \gamma(E_\theta) \cup M \cup \gamma(E_{\theta+\pi}) \\ &= E_{\theta+\lambda} \cup M \cup E_{\theta+\lambda+\pi} \end{aligned}$$

Since $M_{\theta_1} \cap M_{\theta_2} = M$, the action also translates the fibers E_α of $\tilde{\phi}$. If $W \subset \mathbb{S}^1$ is an arc with length less than π, then $\tilde{\phi}^{-1}(W)$ can be identified with $W \times E_\alpha$. Since $\tilde{\phi}$ is obtained from ϕ through the diffeomorphism $z \mapsto i\overline{z}$, it induces a fiber bundle with total space $\mathbb{C}^n \setminus M$ and projection map $\psi/|\psi|$.

We note that the condition on the flow with the negative time implies that M_θ has a nonempty intersection with every sphere of positive radius. Whence M_θ is globally embedded as a cone in $\mathbb{C}^n$. It remains to see that the restriction to the sphere is also a fiber bundle. We claim that this is a consequence of the second lemma. Indeed, it is enough to verify that the intersection of M_θ with the spheres is transversal and so the same holds for the fibers E_θ. But if it does not occur, there exists an integral curve with a tangent vector not transversal to the sphere, contradicting the second lemma. Finally, the open book structure is a consequence of Lemma 8.3.11, item 4, restricted to the sphere minus the link. Recall that M_θ has the decomposition into disjoint sets. □

Remark 8.3.18. *The open book structure here remounts to the Milnor Fibration Theorem to holomorphic functions with isolated singularity.*

Example 8.3.19. The function $\psi = z_1^2\overline{z}_1 + \cdots + z_n^2\overline{z_n}$ satisfies the strong Milnor condition since it is formed from $F = (z_1^2, \ldots, z_n^2)$ and the radial vector field.

Example 8.3.20. The function $\psi = z_2^2\overline{z}_1 + z_3\overline{z}_2 + z_1^2\overline{z}_3$ satisfies the strong Milnor condition since the permutation (123) is odd and there is at least one exponent $a_i > 1$.

Theorem 8.3.21. *The monodromy of the fiber bundle is the first return map of the action* $\mathbb{S}^1$ *on* $\mathbb{C}^n$ *in Lemma 8.3.14.*

Proof. The monodromy is constructed from a diffeomorphism of the fiber according to the winding number around $\mathbb{S}^1$. Therefore, we choose $\gamma_{2\pi}$, which leaves invariant M_θ and thus E_θ. □

During the previous seminars, we discussed the existence of real analytic mappings from $\mathbb{R}^6$ to $\mathbb{R}^3$ with the origin as an isolated critical point admitting a non-trivial Milnor fibration. A result due to Church and Lamotke REF might be useful. It states that the Milnor fibration is not trivial if and only if at the origin the mapping does not have the topological type of the canonical projection. Therefore, we have tried some examples with mixed functions, however, they do not have an isolated critical point:

$$(z_1|z_1|^2 + z_2|z_2|^2, |z_3|^2)$$
$$(z_1|z_2|^2 + z_2|z_1|^2, |z_3|^2)$$
$$(z_1|z_2|^2 + z_2|z_1|^2 + z_3^2, |z_3|^2)$$
$$(|z_1|^2 z_1 z_2 + |z_2|^2 z_1 z_2, |z_3|^2)$$

Example 8.3.22. Cidinha proposed the following map:

$$f(x, y, z, w_1, w_2, w_3) = (x, y, z^3 + 3(x^2 + y^2)z + w_1^2 + w_2^2 + w_3^2).$$

The mapping f has isolated critical point at 0, however, its fibers do not indicate non-triviality. We set the following change:

$$f(x, y, z, w_1, w_2, w_3) = (x, y, z^3 + 3(x^2 + y^2)z + w_1^2 + w_2^2 - w_3^2).$$

And so apply the aforementioned result to $f^{-1}(0)$ noting that the surface $\sqrt[3]{u^2 - v^2}$ has a singularity at the origin but it is topologically an Euclidean ball in a neighborhod there. Therefore, this example has a trivial fibration associated.

Remark 8.3.23 (Direções Restantes)**.**

1. Estudar a seção de Robustez, resumindo e digitando para assimilar.

2. Estudar a seção de topologia, se baseando ema artigos anteriores do Pepe, resumindo e digitando para assimilar.

Chapter 9

Digression on Singularity Theory

We review basic results of the Singularity Theory, including finite determinacy and its geometric characterization in the holomorphic case for the Mather groups $\mathcal{R}$ and $\mathcal{K}$. We briefly discuss the C^∞-case.

9.1 Mather Groups

Let $f : (\mathbb{F}^n, 0) \longrightarrow (\mathbb{F}^p, 0) \in \mathcal{E}_n$ or $\mathcal{O}_n$, that is, smooth mappings for $\mathbb{F} = \mathbb{R}$ and analytic mappings for $\mathbb{F} = \mathbb{C}$. The following groups are of interest: the right-equivalence group $\mathcal{R}$ and the contact group $\mathcal{K}$. We recall their action. Each $h \in \mathcal{R}$ acts on the source:

$$h.f \mapsto f \circ h^{-1}$$

This group is largely studied when $p = 1$. The contact group acts on the graphic and is described as the set of pairs of diffeomorphisms (h, H), where $h \in \mathcal{R}$ and $H : (\mathbb{F}^n \times \mathbb{F}^p, 0) \longrightarrow (\mathbb{F}^n, \times \mathbb{F}^p, 0)$ such that:

$$\pi_1 \circ H(x, y) = h(x)$$
$$\pi_2 \circ H(x, 0) = 0$$

That is, H maps vertical fibers into vertical fibers and leaves $\mathbb{F}^n \times 0$ invariant. It follows that

$$H(x, y) = (h(x), \theta(x, y)),$$

where $\theta(x, 0) = 0$. Therefore:

$$(h, H).f \mapsto (h(x), \theta(x, f(x)))$$

and two maps are $\mathcal{K}$-equivalents if $\theta(x, f(x)) = \theta(g(h(x)))$. In other words, the graphics are diffeomorphic up to a change of coordinates by h.

The $\mathcal{R}$-equivalence implies $\mathcal{K}$-equivalence. Moreover, the zero fibers of $\mathcal{K}$-equivalent mapping germs are diffeomorphic. Indeed, let $x_0 \in f^{-1}(0)$, then

$$g \circ h(x_0) = \theta(x_0, f(x_0)) = \theta(x_0, 0) = 0.$$

Conversely, let $y_0 \in g^{-1}(0)$. There exists $\theta'(x, y)$ such that $(h, H)^{-1}$ is formed by h^{-1} and $H' = (h^{-1}, \theta')$. We obtain that:

$$f \circ h^{-1}(y_0) = \theta'(y_0, g(y_0)) = \theta'(y_0, 0) = 0.$$

9.2 Geometric Characterization of Finite Determinacy

Recall that a mapping f is k-$\mathcal{G}$-finitely determined if its $\mathcal{G}$-orbit is determined by $j^k(f)(0)$. Moreover, $\mathcal{R}$-finitely determined mapping germs have the origin as an isolated singularity, and in the holomorphic case, the converse is also true. To verify this claim, if $a = (a_i)$ is a critical point, we may suppose the Jacobian matrix as formed by an identity block and zeros. On the other hand, equations of the form

$$\sum_i f_{ji} a_j = x^j$$

must hold, which leads to a contradiction if we fix some of the a_i as nonzero. In other words, the unstable set of the group $\mathcal{R}$ is the set of critical points of f, that is, those for which f is not surjective.

Recall that a mapping germ with $\mathcal{G}_e$-codimension zero is called $\mathcal{G}$-stable. Therefore, the example suggests that stability around the origin is equivalent to $\mathcal{G}$-finite determinacy. The general geometric condition is the following.

Theorem 9.2.1 (Theorem 2.1, [19]). *A germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ is $\mathcal{G}$-finitely determined if and only if there exist neighborhoods $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^p$ of the respective origins such that $V \subset f(U)$ and for each $z \in V$ and every finite subset $S = \{x_1, \ldots, x_n\} \subset f^{-1}(z)$ the multigerm*

$$f : (\mathbb{C}^n, S) \longrightarrow \mathbb{C}^p$$

is $\mathcal{G}$-stable.

An immediate consequence is that $\mathcal{R}$ is a *small* group to study mappings:

Corollary 9.2.2 (Proposition 2.3, [19])**.** *If $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ has a singularity at the origin and is $\mathcal{R}$-finite, then $p = 1$.*

Proof. The previous theorem asserts that the critical set Σ_f is zero-dimensional. By openness of stability, we may suppose that $j^1 f \pitchfork \Sigma^1$, the first Thom-Boardman set. Therefore, $j^1 f^{-1}(\Sigma^1) = \Sigma^1_f$ has codimension $p - n + 1$, and then $p - n + 1 = n$. But $n \geq p$ since otherwise all points were critical. Hence, $n \geq p = 2n - 1$ and so $p = 1$. □

For the contact group $\mathcal{K}$, the unstable set is $\Sigma(f) \cap f^{-1}(0)$ [19, p.492]. We have then the following corollary.

Corollary 9.2.3. *A germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ is $\mathcal{K}$-finitely determined if and only*

$$\Sigma(f) \cap f^{-1}(0) = \{0\}.$$

This result gives rise to the notion of *complete isolated singularity*. It is interesting to observe that even for real singularities this kind of behavior may occur. Then, although we do not have an ICIS, the singularity has similarities with this case.

More generally, we are interested in algebraic conditions for $\mathcal{K}$-finite determinacy of mapping germs with $p \geq 2$. Furthermore, we should expect a complete characterization for the complex case. To obtain this, we define the *Jacobian ideal* $J(f)$ determined as the ideal generated by the minors of the Jacobian matrix $D(f)$. Since the vanishing of $J(f)$ detects the critical points of f, we may rewrite the condition above as

$$\mathcal{V}\left(J(f) + \langle f_1, \ldots, f_p \rangle\right) = \{0\}.$$

Corollary 9.2.4. *An equivalent condition for the k-$\mathcal{K}$-finite determinacy is that*

$$J(f) + \langle f_1, \ldots, f_p \rangle \supset \mathfrak{m}_n^k.$$

Proof. This is a consequence of Ruckert's Nullstellensatz. □

Example 9.2.5. The germ $f(x, y, z) = (x^2 + y^2 + z^3, xy + yz)$ is 4-$\mathcal{K}$-finitely determined. The Jacobian matrix is

$$\begin{pmatrix} 2x & 2y & 3z^2 \\ y & x + z & y \end{pmatrix}$$

and $J(f) + \langle f_1, f_2 \rangle$ is

$$\langle 2x^2 + 2xz - 2y^2, 2xy - 3yz^2, 2y^2 - 3z^2, 2y^2 - 3z^2 - 3z^3 \rangle + \langle x^2 + y^2 + z^3, xy + yz \rangle.$$

We shall prove that $\mathfrak{m}_3^4 \subset J(f) + \langle f_1, f_2 \rangle + \mathfrak{m}_3^5$ and conclude by Nakayama and the previous result.

1. Powers of y^2 are obtained by the third generator.

2. Powers of x^2 are obtained from the fourth generator and from item 1.

3. From the first generator and from items 1 and 2, we get xz^3.

4. Multiplying the fifth and the second generators by y and noting that y^2z^2 belongs in the sum, we obtain that the same holds for xy^2 and y^2z.

5. Looking at the the third generator multiplied by z and considering previous item for y^2z, it follows that z^4 also belongs in the sum.

6. The fifth generator multiplied by xz and then by z^2 shows that xyz^2 and yz^2, respectively, belong in the sum. We conclude the argument.

9.3 On the C^∞-case

We shall study the characterizations of finite determinacy for the real case, that is, the ring $\mathcal{E}_n$. An ideal $I \subset \mathcal{E}_n$ is called elliptic if $\bigcap_{i=1}^{\infty} \mathfrak{m}_i^n \subset I$. The next example shows that this intersection is not empty.

Example 9.3.1. The function defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

belongs in the intersection above and is analytic except at the origin.

Theorem 9.3.2. *Let $I = \langle g_1, \ldots, g_s \rangle \subset \mathcal{E}_n$ be an ideal. The following conditions are equivalent.*

1. I is elliptic.

2. *There exist* $g \in I$ *and constants* $c > 0$ *and* $\alpha > 1$ *such that*

$$|g(x)| \geq c|x|^{\alpha}.$$

3. *There exist* $c', \beta > 0$ *such that*

$$\sum_{i=1}^{s} g_i^2 \geq c|x|^{\beta}.$$

The conditions above also imply that

$$\mathcal{V}(I) = \{0\}.$$

If $g_1, \ldots, g_n$ *are analytical functions, this last statement implies the others.*

A typical example of the theorem is the ideal $I \subset \mathcal{E}_2$ generated by $f = x^2 + y^2$, whose variety is precisely the origin. Moreover, it follows that this ideal is elliptic. In order to state a corollary, we recall a lemma from analysis.

Lemma 9.3.3 (Lema, p.158, [5])**.** *Let* $U \subset \mathbb{R}^n$ *be a neighborhood of the origin and* $r : U \longrightarrow \mathbb{R}$ *a p-times differentiable function. If* r *and its partial derivatives with order less or equal to* p *vanish at the origin, then* $\lim_{x \to 0} r(x)/|x|^p = 0$.

The proof follows by induction considering the statement for r and its partial derivatives and also the Mean Value Theorem.

Corollary 9.3.4. *Let* $f \in \mathfrak{m}_n^{\infty}$ *and* g *satisfy condition 2 of the previous theorem. The function*

$$\phi(x) = \begin{cases} \frac{f(x)}{g(x)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is smooth.

Proof. We must verify that the function is continuous, the partial derivatives of ϕ at the origin exist, and that they are continuous. By hypothesis, we have that

$$|\phi(x)| \leq \frac{1}{c} \frac{|f(x)|}{|x|^{\alpha}}.$$

We also may suppose α integer for small x and so f satisfies the hypothesis of the previous lemma. Thus, taking the limit $x \to 0$, we prove continuity since $\phi(0) = 0$.

Recall that the derivative of ϕ at 0 in the direction x is:

$$\frac{\partial \phi}{\partial x}(0) = \lim_{t \to 0} \frac{\phi(tx)}{t}.$$

By the condition on g, we have that:

$$\lim_{t \to 0} \left| \frac{\phi(tx)}{t} \right| \leq \lim_{t \to 0} \left| \frac{f}{ct|tx|^{\alpha}} \right|.$$

By the condition on f, we can choose $k > \alpha + 1$ such that $f = \sum_i h_i x^{\lambda_i} \in \mathfrak{m}_n^k$, where λ_i is a finite sequence of integers with $|\lambda_i| = k$. It follows that:

$$\lim_{t \to 0} \frac{\phi(tx)}{t} = \lim_{t \to 0} \frac{\sum_i h_i t^k x^{\lambda_i}}{ct|tx|^{\alpha}} = \lim_{t \to 0} t^{k-\alpha-1} \frac{\sum_i h_i x^{\lambda_i}}{c|x|^{\alpha}} = 0.$$

We conclude that the limit exists and ϕ has partial derivatives at 0. For continuity, note that

$$\frac{\partial \phi}{\partial x_i} = \frac{f_{x_i} g - g_{x_i} f}{g^2}$$

at points away from the origin and zero there. Applying the same inequality for g, we have that:

$$\left| \frac{f_{x_i} g - g_{x_i} f}{g^2} \right| \leq \left| \frac{f_{x_i} g - g_{x_i} f}{c^2 |x|^{2\alpha}} \right| \leq \left| \frac{f_{x_i} g}{c^2 |x|^{2\alpha}} \right| + \left| \frac{g_{x_i} f}{c^2 |x|^{2\alpha}} \right|.$$

Considering that the partial derivatives of f satisfy the hypothesis of the lemma, taking limits we obtain the continuity at the origin. The higher derivatives will involve powers of g and partial derivatives of f, which also lie in $\mathfrak{m}_n^{\infty}$ so that the same reasoning applies. We conclude the proof. □

The function g above is usually called a *control function*.

Remark 9.3.5. *Condition 4 does not imply condition 3 of Theorem 9.3.2 in $\mathcal{E}_n$ and hence any other.*

Proof. The function $f(x)$ in 9.3.1 defines a variety consisting only of the origin. If condition 3 is verified, there exists β, c' such that

$$\frac{e^{-\frac{2}{x^2}}}{x^{2\beta}} \geq c'$$

for all x in a small neighborhood of the origin. Taking the limit $x \to 0$, the left-hand side goes to zero, while $c' > 0$. This gives a contradiction. □

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