Mixed Functions and a Geometric Viewpoint

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Contents

Introduction

Mixed Functions

Lipschitz Geometry

Geometry of Mixed Functions and Varieties

Milnor Fibration Theorem

Theorem

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ *be a holomorphic function with singularity at the origin. For sufficiently small* $r > 0$*, the map*

$$\frac{f}{\|f\|} : \mathbb{S}_r^{2n-1} \setminus K_r \longrightarrow \mathbb{S}^1$$

is a locally trivial fibration, where $K_r = \mathbb{S}_r^{2n-1} \cap f^{-1}(0)$ *is the link.*

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is a locally trivial fibration, where $K_r = \mathbb{S}_r^{2n-1} \cap f^{-1}(0)$ *is the link.*

- For polynomials, it was proved in his 1968's classical book.
- After the works of Hironaka and others, it was extended to the general case.
- Roughly speaking, it describes the topology of $V = f^{-1}(0)$ near the critical points.

Real Case I

Theorem (Tube)

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ *be a real analytic mapping such that* $f(0) = 0$ *and the origin is an isolated singularity. For a sufficiently small* $\epsilon > 0$*, there exists* $\delta > 0$ *such that the restriction*

$$f : \mathcal{N}(\epsilon, \delta) \longrightarrow \mathbb{S}_{\delta}^{p-1}$$

is a fiber bundle, where $\mathcal{N}(\epsilon, \delta) = \mathbb{B}_{\epsilon} \cap f^{-1}(\mathbb{S}_{\delta}^{p-1})$ *is the Milnor tube.*

Real Case II

Theorem (Spherical)

The Milnor tube $\mathcal{N}(\epsilon, \delta)$ is diffeomorphic to:

$$\mathbb{S}_{\epsilon}^{n-1} \setminus f^{-1}(\mathbb{D}_{\delta}),$$

where $\mathbb{D}_{\delta} \subset \mathbb{R}^p$ is the open disc centered at the origin with radius δ. The induced projection on this complement can be extended to a fiber bundle

$$\phi : \mathbb{S}_{\epsilon}^{n-1} \setminus K_{\epsilon} \longrightarrow \mathbb{S}^{p-1}$$

which has the expression $f/\|f\|$ over an open tubular neighborhood of K_{ϵ}.

Real Case III

- Milnor remarked that the isolated singularity condition was too strong and asked for non-trivial examples.

- Non-triviality is defined as the fiber in the spherical fibration not being diffeomorphic to a disk.

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- Non-triviality is defined as the fiber in the spherical fibration not being diffeomorphic to a disk.
- Explicit examples are known for $p = 2$.
- In general, there are pairs (n, p) for which examples do not exist and others for which only the existence is proved.

Milnor Conditions

Definition

Let $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ *be a smooth map, where* $n > p$.

1. *We say that* f *satisfies the Milnor condition at* 0 *if* 0 *is an isolated singularity.*

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1. *We say that* f *satisfies the Milnor condition at* 0 *if* 0 *is an isolated singularity.*

2. *We say that* f *satisfies the strong Milnor condition if it satisfies the Milnor condition and the map*

$$\frac{f}{\|f\|} : \mathbb{S}_r^{n-1} \setminus K_r \longrightarrow \mathbb{S}^{p-1}$$

is a locally trivial fibration for every $r > 0$ *sufficient small.*

Contents

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Hopf Fibration

- The Hopf fibration describes the sphere $\mathbb{S}^3$ as a union of circles over points of $\mathbb{S}^2$.

- The explicit map is:

$$\begin{aligned} f : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{C} \times \mathbb{R} \\ (z_1, z_2) &\longmapsto (2z_1\bar{z}_2, \|z_2\|^2 - \|z_1\|^2) \end{aligned}$$

Hopf Fibration

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- It holds that $f(\mathbb{S}^3) = \mathbb{S}^2$.

- f admits an isolated singularity at the origin and so this fibration can be recovered from the fibration theorem for real singularities.

Holomorphic Vector Fields

- Consider the following holomorphic vector fields:

$$F(z) = (z_{\sigma(1)}^{a_1}, \ldots, z_{\sigma(n)}^{a_n}) \;\; ; \;\; X(z) = (z_1^{b_1}, \ldots, z_n^{b_n})$$

  where σ is a permutation of $\{1, \ldots, n\}$.

- Define the map $(\mathbb{R}^{2n}, 0) \longrightarrow (\mathbb{R}^2, 0)$:

$$\psi(z, \bar{z}) = \sum_{i=1}^{n} F_i(z)\overline{X}_i(z).$$

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Theorem (Ruas-Seade-Verjovsky)

Under certain conditions on σ and the exponents a_i, b_i, $\psi(z, \bar{z})$ satisfies the strong Milnor condition at 0: More specifically, for every sufficiently small sphere $\mathbb{S}_\epsilon^{2n-1} \subset \mathbb{C}^n$ the function

$$\phi = \psi/\|\psi\| : \mathbb{S}_\epsilon^{2n-1} \setminus K_\epsilon \longrightarrow \mathbb{S}^1$$

is the projection map of a smooth fiber bundle.

In General

Definition (M. Oka)

A real analytic map $f : (\mathbb{R}^{2n}, 0) \longrightarrow (\mathbb{R}^2, 0)$ *is called a mixed function if there exists a complex analytic function* $F(z_1, \ldots, z_n, w_1, \ldots, w_n)$ *such that* $f = F(z, \bar{z})$.

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Theorem (Spherical Milnor Fibration)

Let $f(z, \bar{z})$ *be a polar weighted homogeneous mixed polynomial. If the radial weight vector is strictly positive and* V *has an isolated singularity at the origin, then for any* $r > 0$, *the map*

$$\phi : \frac{f}{\|f\|} : \mathbb{S}_r^{2n-1} \setminus K_r \longrightarrow \mathbb{S}^1$$

is a locally trivial fibration, where $K_r = V \cap \mathbb{S}_r^{2n-1}$. *Moreover, the diffeomorphism class does not depend on* r.

Mixed Pham-Brieskorn Polynomials

- $F = (z_1^{a_1+b_1}, \ldots, z_n^{a_n+b_n})$ and $X = (z_1^{b_1}, \ldots, z_n^{b_n})$.
- Let us denote:

$$\psi(z, \bar{z}) := f_{\mathbf{a},\mathbf{b}} = z_1^{a_1+b_1}\bar{z}_1^{b_1} + \cdots + z_n^{a_n+b_n}\bar{z}_n^{b_n}$$

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Theorem

The variety $f_{\boldsymbol{a},\boldsymbol{b}}^{-1}(0)$ is topologically equivalent to a Brieskorn variety and the Milnor fibrations are topologically equivalent. More specifically, there exists a homeomorphism $\phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that:

$$f_{\boldsymbol{a},\boldsymbol{b}} \circ \phi := g_{\boldsymbol{a}} = z_1^{a_1} + \cdots + z_n^{a_n}.$$

Fibrations Smoothly Equivalent

Theorem (Oka)

Fix $r > 0$ sufficiently small. Then there exists a diffeomorphism

$$\phi : (\mathbb{S}_r^{2n-1}, K_{f_{\boldsymbol{a},\boldsymbol{b}}}) \longrightarrow (\mathbb{S}_r^{2n-1}, K_{g_{\boldsymbol{a}}})$$

such that $f_{\boldsymbol{a},\boldsymbol{b}} \circ \phi(z) = g_{\boldsymbol{a}}(z)$ for all $z \in \mathbb{S}_r^{2n-1}$ with $\|g_{\boldsymbol{a}}(z)\|$ sufficiently small, where $K_{f_{\boldsymbol{a},\boldsymbol{b}}}$ and $K_{g_{\boldsymbol{a}}}$ are the associated links.

Contents

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A Motivation

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- $V = f^{-1}(0)$ and $K_\epsilon = V \cap \mathbb{S}^{2n-1}$ the link.

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For a sufficient small ϵ, the following pairs are homeomorphic:

$$(B_\epsilon, V \cap B_\epsilon) \simeq (C(\mathbb{S}_\epsilon), C(K_\epsilon)).$$

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Theorem

For a sufficient small ϵ, the following pairs are homeomorphic:

$$(B_\epsilon, V \cap B_\epsilon) \simeq (C(\mathbb{S}_\epsilon), C(K_\epsilon)).$$

- The topology of V is encoded by its link.
- How about the **geometry** of V?
- How does the link K_ϵ evolve to V?

Definitions

Let $(X, 0) \subset \mathbb{R}^n$ be the germ of an analytic set.

Metrics

1. The *outer metric* is the induced metric: $d_o(x, y) = \|x - y\|$ for all $x, y \in X$.

2. The *inner metric* is the length metric: $d_i(x, y) = \inf \ell(\gamma)$ among all the rectifiable arcs $\gamma \subset X$ connecting $x, y \in X$, where $\ell(\gamma)$ denotes its length.

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Equivalences

1. *Inner*: if there exists a homeomorphism $f : (X, 0) \longrightarrow (Y, 0)$ which is bi-Lipschitz with respect to the inner metric.

2. *Outer*: if there exists a homeomorphism $f : (X, 0) \longrightarrow (Y, 0)$ which is bi-Lipschitz with respect to the outer metric.

The Cusp I

- Let $C \subset \mathbb{R}^2$ be the cusp $\{x^3 - y^2 = 0\} = \{(t^2, t^3)\}$.
- $p_1 = (t_0^2, t_0^3)$ and $p_2 = (t_0^2, -t_0^3)$ for fixed $t_0 > 0$.

The Cusp I

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- $p_1 = (t_0^2, t_0^3)$ and $p_2 = (t_0^2, -t_0^3)$ for fixed $t_0 > 0$.
- It holds that:

$$\lim_{t_0 \to 0} \frac{d_o(p_1, p_2)}{d_i(p_1, p_2)} = \lim_{t_0 \to 0} \Theta(t_0^{1/2}) = 0$$

- However, $\lim_{t_0 \to 0} \frac{d_i(p_1,p_2)}{d_o(p_1,p_2)} = \lim_{t_0 \to 0} \Theta(t_0^{-1/2})$ diverges.
- **The inner and outer Lipschitz geometry are not equivalent.**

The Cusp II

- Let C' be the cone over the link of C. If we take the radius equal to 1, then C' is the join of two lines:

$$C' = \begin{cases} (t, t) & \text{if } t \geq 0 \\ (t, -t) & \text{if } t < 0 \end{cases}$$

The Cusp II

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- Define the map $f : C' \longrightarrow C$ given by $f(t, \pm t) = (t^2, \pm t^3)$. We obtain that:

$$d_i(f(p_1), f(p_2)) = \Theta(\|t_1 - t_2\|) = d_i(p_1, p_2)$$

for any points $p_1, p_2 \in C'$.

- **C is inner equivalent to the cone over its link.**

β-Horns

Definition

The β-horn is the semialgebraic set defined by

$$H_\beta = \left\{(x, y, z) \in \mathbb{R}^3 : \left(x^2 + y^2\right)^q = z^{2p}, z \geq 0\right\},$$

where $\beta = p/q \geq 1$ and p, q are natural numbers.

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where $\beta = p/q \geq 1$ and p, q are natural numbers.

Remark

1. $H_\beta \simeq_{\text{inner}} H_{\beta'}$ if and only if $\beta = \beta'$.

2. There exists a large class of examples for which the cone over the link is not bi-Lipschitz homeomorphic to the analytic set.

Theorem (L. Birbrair)

Let $(X, 0) \subset \mathbb{R}^n$ be a semialgebraic surface with an isolated singularity at the origin. Then X is inner bi-Lipschitz equivalent to a β-horn.

Tangent Cone

Theorem (E. Sampaio)

Let $X, Y \subset \mathbb{R}^n$ be subanalytic sets. If the germs (X, x_0) and (Y, y_0) are bi-Lipschitz equivalent, then the tangent cones $(C(X, x_0), x_0)$ and $(C(Y, y_0), y_0)$ are also bi-Lipschitz equivalent.

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Example

The following pair is topologically equivalent:

$$f(z, \bar{z}) = \|z_1\|^2 - z_1^2 \bar{z}_1$$
$$g(z, \bar{z}) = \|z_1\|^2 - z_1$$

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The tangent cones are:

$$\big(C(f^{-1}(0)), 0\big) = \{(0, 0, z) : z \leq 0\}$$
$$\big(C(g^{-1}(0)), 0\big) = \{(x, y, 0)\}$$

The hypersurfaces are not bi-Lipschitz equivalent

Contents

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Questions

Lipschitz Geometry and mixed functions stand between the topological and analytic realms.

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Lipschitz Geometry and mixed functions stand between the topological and analytic realms.

- Line 1: Lipschitz Geometry of Mixed Singularities and Varieties.
- Line 2: Comparison of the (Contact) Geometry of Mixed and Holomorphic Varieties.

Singularities and their Varieties

Theorem (King, Perron)

Let $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be two holomorphic function germs with an isolated singularity at the origin. Then $f^{-1}(0)$ and $g^{-1}(0)$ have the same topological type if and only if f and g or f and $\bar{g}$ are right topologically equivalent.

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Theorem (Yoshinaga-Suzuki)

Let $g_{\boldsymbol{a}}, g_{\boldsymbol{b}}$ be Pham-Brieskorn polynomials of the forms:

$$g_{\boldsymbol{a}} = z_1^{a_1} + \cdots + z_n^{a_n} \ ; \ 2 \leq a_1 \cdots \leq a_n$$
$$g_{\boldsymbol{c}} = z_1^{c_1} + \cdots + z_n^{c_n} \ ; \ 2 \leq c_1 \cdots \leq c_n$$

If the hypersurface germs of $g_{\boldsymbol{a}}$ and $g_{\boldsymbol{c}}$ have the same topological type, then $\boldsymbol{a} = \boldsymbol{c}$.

Mixed Brieskorn Polynomials

Question

In the family of mixed Pham-Brieskorn polynomials, is the bi-Lipschitz type of $f_{\boldsymbol{a},\boldsymbol{b}}$ determined by ***a and b****?*

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In the family of mixed Pham-Brieskorn polynomials, is the bi-Lipschitz type of $f^{-1}_{\boldsymbol{a},\boldsymbol{b}}(0)$ determined by ***a and b****?*

Remark

It would give an infinite family with the same topological type and distinct Lipschitz types.

Contact Structures on Mixed Links

- $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ with isolated singularity at the origin.
- $\mathbb{S}^{2n-1}$ has a canonical contact structure.
- This restricts to a structure on $K = \mathbb{S}^{2n-1}_r \cap f^{-1}(0)$.

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- This restricts to a structure on $K = \mathbb{S}_r^{2n-1} \cap f^{-1}(0)$.
- Some mixed links are related to holomorphic ones:

$$\phi : (\mathbb{S}_r^{2n-1}, K_{f_{\mathbf{a},\mathbf{b}}}) \longrightarrow (\mathbb{S}_r^{2n-1}, K_{g_{\mathbf{a}}})$$

- There exists an "induced" structure.

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Question

How does the canonical structure on $\mathbb{S}^{2n-1}$ restrict to mixed links? If so, what is the relation with the induced structure?

Gracias!



