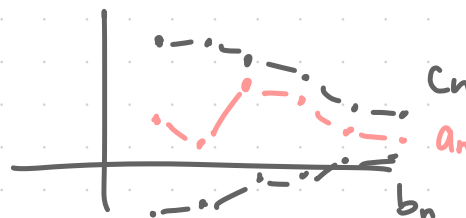


The Squeeze Theorem for Sequences

If we want to compute $\lim_{n \rightarrow \infty} a_n$ and we can pick sequences b_n and c_n where:

- ① $b_n \leq a_n \leq c_n$ exist and MATCH
- ② $\lim_{n \rightarrow \infty} b_n = L$ and $\lim_{n \rightarrow \infty} c_n = L$

Then $\lim_{n \rightarrow \infty} a_n = L$ too.



ex: $\left\{ \frac{\cos(3n)}{n^2} \right\}_{n=1}^{\infty}$ converges or diverges?

since $-1 \leq \cos(3n) \leq 1$

$$-\frac{1}{n^2} \leq \frac{\cos(3n)}{n^2} \leq \frac{1}{n^2}$$

Choose $b_n = -\frac{1}{n^2}$ and $c_n = \frac{1}{n^2}$.

Then $\lim_{n \rightarrow \infty} b_n = 0$ and $\lim_{n \rightarrow \infty} c_n = 0$

match

$\Rightarrow \lim_{n \rightarrow \infty} \frac{\cos(3n)}{n^2} = 0$ by the Squeeze Theorem.

Definitions

- A sequence is **increasing** if $a_n < a_{n+1}$ for all n
- A sequence is **decreasing** if $a_n > a_{n+1}$ for all n
- A sequence that is either increasing or decreasing is called **monotonic**
- A sequence is **bounded above** if there exists a number M such that $a_n \leq M$ for all n
- A sequence is **bounded below** if there exists a number m such that $a_n \geq m$ for all n
- A sequence is said to be **bounded** if it is bounded both above and below

Examples:

$$\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

$$\text{decreasing : } \frac{1}{n} > \frac{1}{n+1}$$

$$\text{bounded? } \frac{1}{n} > 0 \text{ for all } n$$

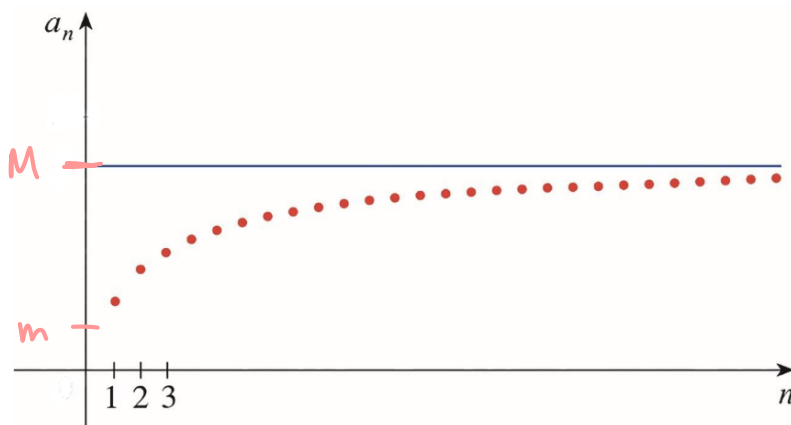
$$\text{and } \frac{1}{n} \leq 1$$

} yes, bounded
above by 1
and bounded
below by 0
 $\Rightarrow$ bounded

$\{(-1)^n n\}$ is not monotonic and not bounded
(check yourself)

The Monotone Convergence Theorem

Every bounded, monotonic sequence is convergent.



← sequence has to keep
increasing, so it gets
trapped and
must converge.

example: $\{ (1 + \frac{2}{n})^n \}_{n=1}^{\infty}$ converges or diverges

"plug in ∞ " : $(1+0)^{\infty} = 1^{\infty}$ $\leftarrow$ indeterminate form

Trick (b/c you have n in the base and exponent) is to use log laws to bring exponent down.

$$L = \lim_{n \rightarrow \infty} \ln(a_n) = \ln(\lim_{n \rightarrow \infty} a_n)$$

$$L = \ln(\lim_{n \rightarrow \infty} a_n)$$

$$e^L = \lim_{n \rightarrow \infty} a_n$$

log law

$$\lim_{n \rightarrow \infty} \ln\left(\left(1 + \frac{2}{n}\right)^n\right) = \lim_{n \rightarrow \infty} n \cdot \ln\left(1 + \frac{2}{n}\right)$$

$$\infty \cdot \ln(1) = \infty \cdot 0$$

$$= \lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{2}{n}\right)}{1/n}$$

$$\leftarrow \frac{0}{0}$$

so can use
L'Hospital's
Rule

$$= \lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{2}{x}\right)}{x^{-1}}$$

$$\stackrel{LH}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\left(1 + \frac{2}{x}\right)} \cdot (2 \cdot (-x^{-2}))}{(-x^{-2})}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{1 + \frac{2}{x}} = 2$$

Answer: $e^1 = e^2$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

← one of many equivalent definitions of Euler's number e

8.2 - Series

First, a quick review of sigma notation:

$$\sum_{n=1}^N a_n$$

← stopping index
← terms (i.e. what you sum)
the index is n
starting index

ex: $\sum_{n=1}^5 \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} = \frac{5269}{3600} \approx 1.46$

a number

We often use n, k, i, j or m as index variables

$$\sum_{n=1}^5 \frac{1}{n^2} = \sum_{k=1}^5 \frac{1}{k^2}$$

A series is a sum with infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots = (?)$$

keeps going forever

ex: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = (?)$

vs. the sequence

$$\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

To make of an infinite series, look at the "partial sums"

for example: $\sum_{n=1}^{\infty} \frac{1}{n}$

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

3rd partial sum $\rightarrow S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$\vdots$

nth partial sum $\rightarrow S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} \leftarrow$

Notice $\{s_n\}$ is a sequence (each n corresponds to a number in the sequence)

We define the value of the series to be the limit of the sequence of partial sums

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \lim_{n \rightarrow \infty} s_n \\ &= \lim_{n \rightarrow \infty} (a_1 + a_2 + \dots + a_n)\end{aligned}$$

This means:

- if $\{s_n\}$ converges to a real number L , then we say that the series converges and its value (or sum) of the series is L
- if $\{s_n\}$ diverges, we say the series diverges.

* Issue: it can be hard to find a formula for s_n . *

But in some special cases, we can find a formula for s_n .

ex: $\sum_{n=0}^{\infty} (-1)^n$

$$a_n = (-1)^n$$

$$S_n = ?$$

$$S_1 = a_0 = 1$$

$$S_2 = a_0 + a_1 = (-1)^0 + (-1)^1 = 1 - 1 = 0$$

$$S_3 = a_0 + a_1 + a_2 = 1 - 1 + 1 = 1$$

$$S_4 = 0$$

$$S_5 = 1$$

$\vdots$

$$\{S_n\}_{n=1}^{\infty} = \{1, 0, 1, 0, 1, 0, \dots\}$$

$$\lim_{n \rightarrow \infty} S_n \text{ DNE } \left(\begin{array}{l} \text{bounces between} \\ 0 \text{ and } 1 \end{array} \right)$$

$\Rightarrow$ series diverges.

example: $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

want: $S_n = ?$

telescoping series

$$S_1 = a_1 = 1 - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = a_1 + a_2 + a_3 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = 1 - \frac{1}{4}$$

...

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1$$

So the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$ converges and its value is 1.

A necessary condition for a series to converge:

Fact: If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.



The Divergence Test: If $\lim_{n \rightarrow \infty} a_n \neq 0$, then
the series $\sum a_n$ diverges.

Means if the limit of the seq. of terms is anything but 0 (including $\pm\infty$ or DNE), then the series diverges.

* Important: You can ONLY conclude DIVERGENCE using the Divergence Test.

If $\lim_{n \rightarrow \infty} a_n = 0$, you know NOTHING!

examples:

① $\sum_{n=1}^{\infty} \frac{n^2-2}{5n^2+1}$

Div. Test: $\lim_{n \rightarrow \infty} \frac{n^2-2}{5n^2+1} = \frac{1}{5} \neq 0$

include in your solution
↓ ↓

series **diverges**
by Div. Test.

For better solutions: avoid using the word "it"
e.g. "the **series** diverges"
vs. "**it** diverges"

② $\sum_{n=0}^{\infty} \frac{1}{2^n}$

Div. Test: $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$

we know nothing
• •
^

The Divergence Test fails.

We'll be able to deal with this series soon
(requires a different test)

Important Points:

- A sequence is a list of numbers.
- A series is a sum of numbers - either equals a number or diverges
- Each series $\sum a_n$ has TWO associated sequences:

① the sequence of terms, a_n

② the sequence of partial sums, s_n

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$$

example: $\sum_{n=0}^{\infty} (-1)^n$

① sequence of terms: $\{a_n\} = \{(-1)^0, (-1)^1, (-1)^2, (-1)^3, \dots\}$
 $= \{1, -1, 1, -1, \dots\}$

② sequence of partial sums:

$$\begin{aligned}\{s_n\} &= \{1, 1+(-1), 1+(-1)+1, 1+(-1)+1+(-1), \dots\} \\ &= \{1, 0, 1, 0, \dots\}\end{aligned}$$

example: $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

① sequence of terms: $a_n = \frac{1}{n} - \frac{1}{n+1}$

$$\begin{aligned} \{a_n\}_{n=1}^{\infty} &= \left\{ 1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \dots \right\} \\ &= \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \dots \right\} \end{aligned}$$

② sequence of partial sums:

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$\begin{aligned} &= \left(1 - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

telescoping

$$\begin{aligned} \{S_n\}_{n=1}^{\infty} &= \left\{ 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, 1 - \frac{1}{5}, \dots \right\} \\ &= \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\} \end{aligned}$$