

## Warm-up Problems

① Explain why  $\int \frac{1}{e^x} dx \neq \ln|e^x| + C$ .

$$u = e^x \\ du = e^x dx$$

$$\int \frac{1}{e^x} dx = \int \underbrace{\frac{1}{e^x} \cdot \frac{e^x}{e^x} dx}_{(e^x)^2 = u^2} du$$

$$\left( \int \frac{1}{x} dx = \ln|x| + C \right. \\ \Rightarrow \int \frac{1}{e^x} dx = \ln|e^x| + C = \underline{x + C} \left. \right)$$

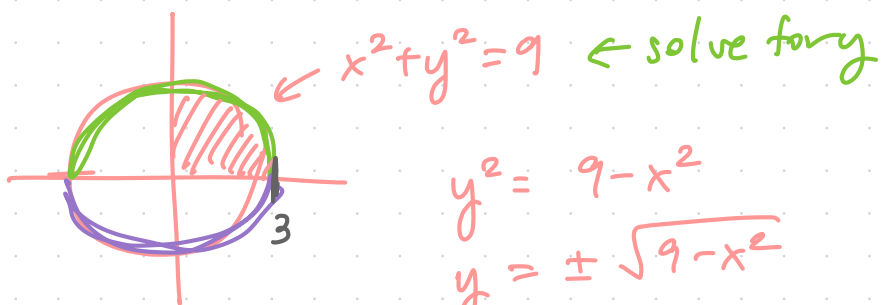
$$\int e^{-x} dx = \int e^u (-1) du = - \int e^u du \\ \boxed{u = -x} \\ \boxed{du = -1 dx} \Rightarrow \underbrace{-1 \cdot du = dx}_{-1 \cdot du = dx} = -e^u + C = -e^{-x} + C$$

②  $\int \underline{x} \sin(x^2) \underline{dx} = \int \sin(u) \cdot \frac{1}{2} du =$

$$u = x^2 \\ du = 2x dx \Rightarrow \frac{1}{2} du = x dx$$
$$\hookrightarrow \frac{1}{2} (-\cos u) + C \\ = \underline{-\frac{1}{2} \cos(x^2) + C}$$

③ Evaluate  $\int_0^3 \underline{\sqrt{9 - x^2}} dx$  using geometry.

↑ top half of a circle



$$y^2 = 9 - x^2$$

$$y = \pm \sqrt{9 - x^2}$$

$$y = \sqrt{9 - x^2}$$

top half

$$y = -\sqrt{9 - x^2}$$

bottom half

$$\int_0^3 \sqrt{9 - x^2} \, dx = \frac{1}{4} \cdot \pi (3)^2 = \frac{9\pi}{4}.$$

examples:

(1) Show  $\int_0^2 3x \sqrt{9-x^2} dx = \int_5^9 \frac{3}{2} \sqrt{u} du$

Use  $u = 9 - x^2$

bounds:  $x=0 \Rightarrow u=9$   
 $x=2 \Rightarrow u=5$

$du = -2x dx \Rightarrow -\frac{1}{2} du = x dx$

$\int_0^2 3x \sqrt{9-x^2} dx = 3 \int_0^2 \underline{x} \sqrt{9-x^2} \underline{dx}$   $\leftarrow -\frac{1}{2} du$

$= 3 \int_9^5 \sqrt{u} \cdot (-\frac{1}{2}) du$

$= -\frac{3}{2} \int_9^5 \sqrt{u} du$

$= \frac{3}{2} \int_5^9 \sqrt{u} du \checkmark$

(2)  $\int x \sqrt{4-x} \underline{dx}$  (trick: "back substitution")

$u = 4 - x$

$du = -1 \cdot dx \Rightarrow -1 \cdot du = dx$

$= \int \underline{x} \sqrt{u} \cdot (-1) \cdot du$

$\leftarrow$  bad! this sub didn't work ;)

If  $u = 4 - x$  then  $x = 4 - u$ .

$$= \int (4-u) \sqrt{u} (-1) du \quad u\text{-sub worked!}$$

$$= \int (u-4) \sqrt{u} du$$

$$= \int u^{3/2} - 4u^{1/2} du$$

$$= \frac{2}{5} u^{5/2} - 4 \cdot \frac{2}{3} u^{3/2} + C$$

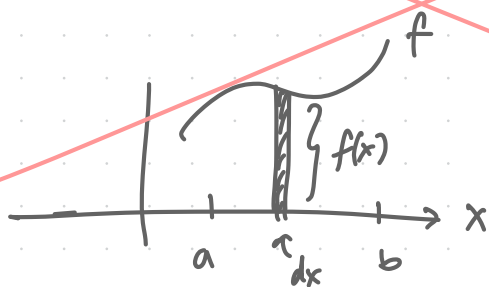
$$= \frac{2}{5} (4-x)^{5/2} - \frac{8}{3} (4-x)^{3/2} + C$$

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~~$$\int u^{3/2} - 4u^{1/2} du$$~~

~~$$= \int u^{3/2} du - 4 \int u^{1/2} du$$~~

$$\int f(x) dx \quad \leftarrow \text{differential}$$



area

$$dA = f(x) dx$$

## Integration by Parts

An integration technique for dealing with products of functions

$$\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Integrate both sides:

$$\int \frac{d}{dx} (f(x)g(x)) dx = \int f'(x)g(x) + f(x)g'(x) dx$$

but  
FTC  
↪

$$f(x)g(x) = \int f'(x)g(x) dx + \int f(x)g'(x) dx$$

$$f(x)g(x) - \int f'(x)g(x) dx = \int f(x)g'(x) dx$$

IBP Formula:  $\int f(x) g'(x) dx = f(x)g(x) - \int \underbrace{f'(x)g(x)}_{\text{new integral, hopefully easier!}} dx$

want to compute

Let  $u = f(x)$  and  $v = g(x)$  written HW  
 $\underline{du} = f'(x)dx$   $dv = g'(x)dx$

IBP Formula with  $u$ 's and  $v$ 's:  $\int \underline{u} dv = uv - \int v du$

example:  $\int x \sin x dx$

Pick  $u = x$  and  $dv = \sin x dx$   
 $du = dx$   $v = -\cos x$

$$\int u dv = uv - \int v du$$

$$\begin{aligned} \int x \sin x dx &= x \cdot (-\cos x) - \int (-\cos x) \cdot dx \\ &= -x \cos x + \int \cos x dx \end{aligned}$$

$$= -x \cos x + \sin x + C \quad \checkmark$$

What if we picked differently?

$$\begin{aligned} \text{Pick } u &= \sin x & \text{and } dv &= x \, dx \\ du &= \cos x \, dx & v &= \frac{1}{2} x^2 \end{aligned}$$

$$\begin{aligned} \int x \sin x \, dx &= uv - \int v \, du \\ &= \sin x \left( \frac{1}{2} x^2 \right) - \int \frac{1}{2} x^2 \cdot \cos x \, dx \end{aligned}$$

worse!

example:  $\int x^2 \cos(2x) \, dx$

|                              |                                                     |
|------------------------------|-----------------------------------------------------|
| $u = x^2$<br>$du = 2x \, dx$ | $dv = \cos(2x) \, dx$<br>$v = \frac{1}{2} \sin(2x)$ |
|------------------------------|-----------------------------------------------------|

$$= uv - \int v \, du$$

$$\begin{aligned} &\int \cos(2x) \, dx \\ &\quad (u=2x) \\ &= \frac{1}{2} \sin(2x) + C \end{aligned}$$

$$\begin{aligned} &= (x^2) \cdot \left( \frac{1}{2} \sin(2x) \right) - \int \frac{1}{2} \sin(2x) \cdot \cancel{2} x \, dx \\ &= \frac{1}{2} x^2 \sin(2x) - \left( \int \sin(2x) x \, dx \right) \end{aligned}$$

When using IBP  
multiple times,  
be consistent with  
choice of  $u$  and  $dv$

$$\begin{cases} u = x \\ du = dx \end{cases} \quad \begin{cases} dv = \sin(2x) dx \\ v = -\frac{1}{2} \cos(2x) \end{cases}$$

$$= \frac{1}{2} x^2 \sin(2x) - \left( uv - \int v du \right)$$

$$= \frac{1}{2} x^2 \sin(2x) - \left( x \left( -\frac{1}{2} \cos(2x) \right) - \int -\frac{1}{2} \cos(2x) dx \right)$$

$$= \frac{1}{2} x^2 \sin(2x) - \left( -\frac{1}{2} x \cos(2x) + \frac{1}{2} \int \cos(2x) dx \right)$$

$$= \frac{1}{2} x^2 \sin(2x) + \frac{1}{2} x \cos(2x) - \frac{1}{2} \cdot \frac{1}{2} \sin(2x) + C$$

$$\int x^3 \cos x dx \quad \text{use IBP } x^3$$

$$\int x^4 \cos x dx \quad \text{use IBP } x^4$$

⋮

For choosing  $u$ , is du nicer?

$$x^n \rightarrow nx^{n-1}$$

$$\ln x \rightarrow \frac{1}{x} \quad \underline{\text{alot}} \text{ nicer}$$

$$\arctan x \rightarrow \frac{1}{1+x^2}$$

