8.1 - Sequences

A sequence is a list of numbers, in order. ← important

We'll be studying infinite sequences, so these will be infinitely long lists of numbers.

ex: $\{1, 2, 4, 8, 16, 32, 64, \ldots\}$ ← assume the pattern continues

terminology

- Numbers in the sequence are called terms
- We use the notation a_n (read "a sub n") for the nth term in the sequence.
- n is the index of the term a_n (tells you the position of the term in the sequence).

In example above,

doubles with each step

$a_1 = 1$
$a_2 = 2$
$a_3 = 4 = 2^{2}$ ←compare w/ index→ $n = 3$
$a_4 = 8 = 2^{3}$ $n = 4$
$a_5 = 16 = 2^{4}$ $n = 5$
$\vdots$
$a_n = ?$

The exponent on 2 is one less than the index so $a_n = 2^{n-1}$.

Another way we can write this seq: $\{2^{n-1}\}_{n=1}^{\infty}$

Notation: $\{a_n\}_{n=1}^{\infty}$ ← "stopping point" (∞), ← starting point ($n=1$)

↑ curly braces indicate its a sequence

Note: the index does not have to start at 1.
It can start at any integer.

example: $\left\{\frac{n+1}{n+2}\right\}_{n=2}^{\infty} = \left\{\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$
$\qquad\qquad\qquad\qquad\qquad\uparrow\ a_2 \quad \uparrow\ a_3 \quad \uparrow\ a_4$

example: the Fibonacci sequence
Start with 0 and 1. Sum the previous two terms to get the next term

$$\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$$

We say this sequence is defined recursively because the value of a_n depends on the terms that came before it.

recursive definition: $\boxed{a_0 = 0,\ a_1 = 1 \text{ and } a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2}$

example: $\{(-1)^n\}$ ← if no starting index is given assume it starts at 1

$$\{(-1)^n\} = \{(-1)^1, (-1)^2, (-1)^3, (-1)^4, \ldots\}$$
$$= \{-1, 1, -1, 1, \ldots\}$$

example: $\left\{(-1)^{n+1} \cdot \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{(-1)^2 \cdot \frac{1}{1}, (-1)^3 \cdot \frac{1}{2}, (-1)^4 \cdot \frac{1}{3}, (-1)^5 \cdot \frac{1}{4}, \ldots\right.$
$$= \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right\}$$

Defn: A seq. whose terms alternate positive to negative is called alternating.
(Note: $\{1, 1, -2, -2, 3, 3, -4, -4, \ldots\}$ is not alternating

Fact: If $\{a_n\}$ is alternating, then a_n can be written as either

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n+1} b_n$$

where $b_n \geq 0$.

depends on whether the 1st term is positive or neg.

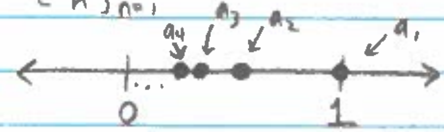
exercise: Find a formula for a_n (start w/ $n=1$)

- $\left\{\frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \frac{5}{243}, \ldots\right\}$ $\qquad a_n = \frac{n}{3^n}$
- $\left\{\frac{1}{3}, -\frac{2}{9}, \frac{3}{27}, -\frac{4}{81}, \frac{5}{243}, \ldots\right\}$ $\qquad a_n = (-1)^{n+1}\frac{n}{3^n}$
- $\left\{\frac{1}{5}, \frac{5}{8}, \frac{25}{11}, \frac{125}{14}, \frac{625}{17}, \ldots\right\}$ $\qquad a_n = \frac{5^{n-1}}{3n+2}$
- $\left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots\right\}$ $\qquad a_n = \frac{1}{n!}$

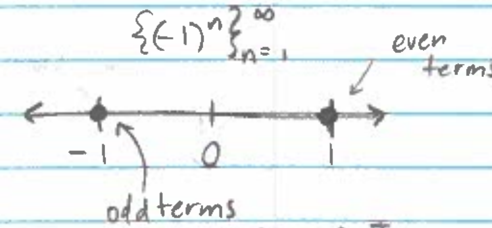
Two ways to visualize sequences:

① as points on a number line

e.g. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

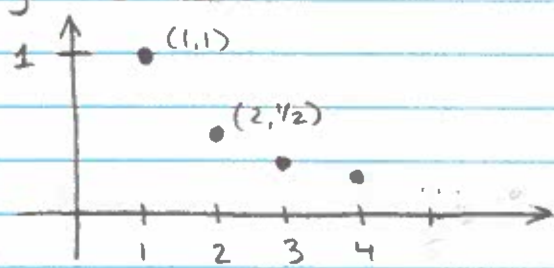


or $\left\{(-1)^n\right\}_{n=1}^{\infty}$



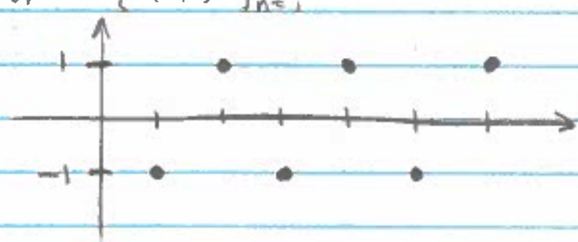
② as functions whose domain is $\{1, 2, 3, \dots\} = \mathbb{N}$ (the natural numbers)

e.g. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$



plot (n, a_n)

or $\left\{(-1)^n\right\}_{n=1}^{\infty}$



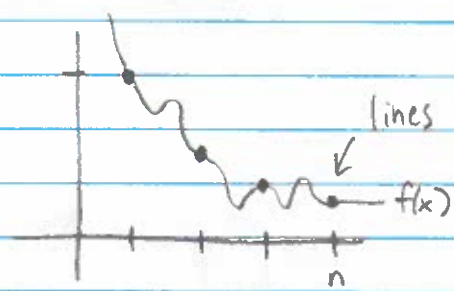
Limits of Sequences (as $n \to \infty$)

$\lim_{n\to\infty} a_n = L$ if we can make a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists and is finite, we say that the sequence **converges**. Otherwise, we say $\{a_n\}$ **diverges**.

ex: $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0 since $\lim_{n\to\infty} \frac{1}{n} = 0$.

Fact: If $f(x)$ is a function on $\mathbb{R}$ such that $f(n) = a_n$ for each natural number n, then $\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n$.



Limit examples:

① $\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{x\to\infty} \frac{2^x}{x^2}$ ← $\frac{\infty}{\infty}$, can use L'Hospital's Rule

$\overset{LH}{=} \lim_{x\to\infty} \frac{\ln 2 \cdot 2^x}{2x}$ ← still $\frac{\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to\infty} \frac{\ln 2(\ln 2 \cdot 2^x)}{2}$

$= \lim_{x\to\infty} \frac{(\ln 2)^2 2^x}{2}$

$= \frac{(\ln 2)^2}{2} \lim_{x\to\infty} 2^x = \infty$

the sequence diverges to ∞

(2) $\lim_{n\to\infty} \frac{n^2+2n}{5n^2+3}$ ← there are several ways to compute this

one way is to delete lower order terms

Idea: as $n\to\infty$, the behavior of a polynomial is completely determined by the highest order term.

$$\lim_{n\to\infty} \frac{n^2+\cancel{2n}}{5n^2+\cancel{3}} = \lim_{n\to\infty} \frac{n^2}{5n^2} = \lim_{n\to\infty} \frac{1}{5} = \frac{1}{5}$$

(3) $$\lim_{n\to\infty} \frac{2n^2+\cancel{n}}{\sqrt{\cancel{3n^2}+7n^4}} = \lim_{n\to\infty} \frac{2n^2}{\sqrt{7n^4}} = \lim_{n\to\infty} \frac{2\cancel{n^2}}{\sqrt{7}\,(\cancel{n^2})} = \frac{2}{\sqrt{7}}$$

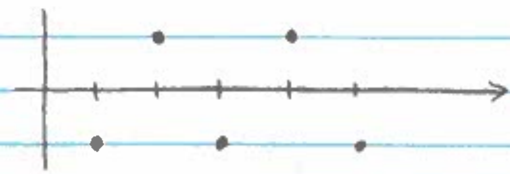
Smaller degree than n^4 ⇒ lower order term

(4) $$\lim_{n\to\infty} \frac{n^4}{\sqrt{n^3+\cancel{2n-2}}} = \lim_{n\to\infty} \frac{n^4}{\sqrt{n^3}} = \lim_{n\to\infty} n^{4-3/2}$$

$$= \lim_{n\to\infty} n^{5/2} = \infty$$

examples where $\lim\limits_{n\to\infty} a_n$ does not exist

- $\{(-1)^n\}_{n=1}^{\infty}$



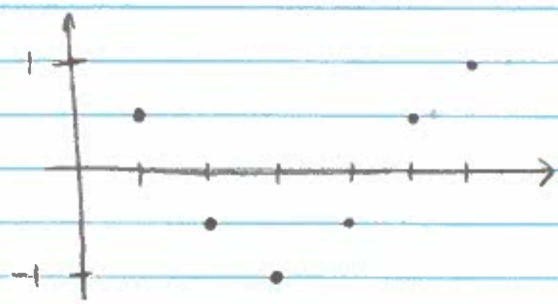
the values bounce between -1 and 1, never converge to a single value

$\lim\limits_{n\to\infty} (-1)^n$ does not exist

the sequence $\{(-1)^n\}$ diverges

- $\left\{\cos\left(\frac{n\pi}{3}\right)\right\} = \left\{\cos\left(\frac{\pi}{3}\right), \cos\left(\frac{2\pi}{3}\right), \cos(\pi), \cos\left(\frac{4\pi}{3}\right), \ldots\right\}$

$= \left\{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \underbrace{\frac{1}{2}, -\frac{1}{2}, \ldots}_{\text{pattern repeats}}\right\}$



does not converge to a single value

$\Rightarrow \lim\limits_{n\to\infty} \cos\left(\frac{n\pi}{3}\right)$ does not exist

You might say that this makes sense since $\lim\limits_{x\to\infty} \cos\left(\frac{\pi x}{3}\right)$ does not exist

BUT be CAREFUL with using the previous fact!

ex: $\left\{\cos\left(\frac{(2n+1)\pi}{2}\right)\right\}_{n=0}^{\infty} = \left\{\cos\left(\frac{\pi}{2}\right), \cos\left(\frac{3\pi}{2}\right), \cos\left(\frac{5\pi}{2}\right), \ldots\right\}$

$= \{0, 0, 0, 0, \ldots\}$

limit is 0 since every term is 0 even though $\lim\limits_{x\to\infty} \cos\left(\frac{(2x+1)}{2}\right.$

The Squeeze Theorem for Sequences

match

If $b_n \le a_n \le c_n$ for all n and $\lim_{n\to\infty} b_n = L$ & $\lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} a_n = L$.

Using the Sq. Thm: If we want to compute $\lim_{n\to\infty} a_n$,

① we pick sequences b_n and c_n so that a_n is between.

② Compute $\lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c_n$ and show that they match.

③ Conclude that a_n converges to the same thing.

example: $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$ ← n! is only defined for integers, can't rewrite this as a limit of a function

Notice $a_n > 0$ for all $n \ge 1$ so choose $b_n = 0$.

Also $\underbrace{a_n \le \frac{1}{n}}_{\text{why?}}$ for all $n \ge 1$ so choose $c_n = \frac{1}{n}$.

$$a_n = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdot \ldots \cdot 3\cdot 2\cdot 1}{n\cdot n\cdot n \cdot \ldots \cdot n\cdot n\cdot n}$$

$$= \underbrace{\left(\frac{n}{n}\right)}_{=1}\cdot\underbrace{\left(\frac{n-1}{n}\right)}_{\le 1}\cdot\underbrace{\left(\frac{n-2}{n}\right)}_{\le 1}\cdot \ldots \cdot\underbrace{\left(\frac{3}{n}\right)}_{\le 1}\underbrace{\left(\frac{2}{n}\right)}_{\le 1}\left(\frac{1}{n}\right)$$

$$\le 1\cdot 1\cdot 1\cdot \ldots \cdot 1\cdot 1\cdot \left(\frac{1}{n}\right) = \frac{1}{n}$$

$\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

example: $\left\{ \left(-\frac{1}{2}\right)^n \right\}_{n=0}^{\infty} = \left\{ 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots \right\}$

Notice that $\left(-\frac{1}{2}\right)^n \le \left(\frac{1}{2}\right)^n$ for each $n \ge 0$

n	$\left(-\frac{1}{2}\right)^n$	$\left(\frac{1}{2}\right)^n$
0	1	1
1	$-1/2$	$1/2$
2	$1/4$	$1/4$
3	$-1/8$	$1/8$
4	$1/16$	$1/16$

they match when n is even, when n is odd the negative one is smaller

Also $-\left(\frac{1}{2}\right)^n \le \left(-\frac{1}{2}\right)^n$ for each $n \ge 0$.

always negative

negative for odd n

they have the same magnitude or absolute value

Choose $b_n = -\left(\frac{1}{2}\right)^n$ and $c_n = \left(\frac{1}{2}\right)^n$.
Then $b_n \le a_n \le c_n$ for all $n \ge 0$.

Also $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = \lim\limits_{n\to\infty} \frac{1}{2^n} = 0$

and $\lim\limits_{n\to\infty} -\left(\frac{1}{2}\right)^n = -\lim\limits_{n\to\infty} \frac{1}{2^n} = 0$

match

$\therefore \lim\limits_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$ by the Squeeze Thm.

(Notice that $c_n = |a_n|$ and $b_n = -|a_n|$.)

$$\boxed{\text{Fact: If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0.}$$

Makes sense b/c $\lim_{n\to\infty} |a_n| = 0$ means that a_n is getting small in absolute value so the sequence must go to 0.

To prove this use the idea from the previous example:

$$-|a_n| \leq a_n \leq |a_n| \quad \text{for all } n$$

Use the Squeeze Theorem with $b_n = -|a_n|$, $c_n = |a_n|$.
Since $\lim_{n\to\infty} |a_n| = 0$, we get $\lim_{n\to\infty} c_n = 0$ match

and

$$\lim_{n\to\infty} b_n = -\underbrace{\lim_{n\to\infty} |a_n|}_{0} = 0$$

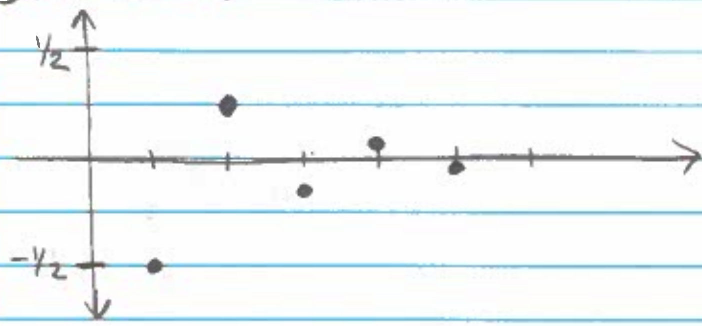
$\therefore \lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

Note: If $\lim_{n\to\infty} |a_n| \neq 0$ then you won't get that the limits of b_n and c_n match!

Means this fact only applies if $\lim_{n\to\infty} |a_n|$ equals zero, no other value!

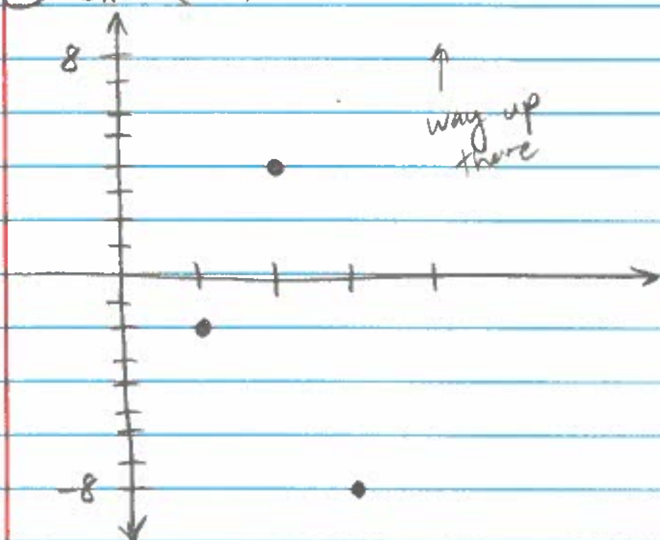
examples:

1. $a_n = \left(-\frac{1}{2}\right)^n$



converges to 0

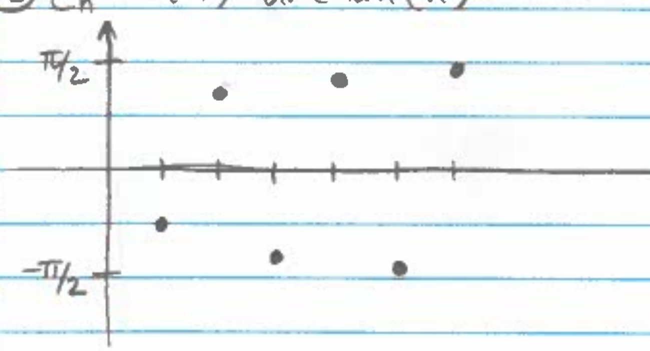
2. $b_n = (-2)^n$



as $n \to \infty$, the points get farther and farther away from each other
$\Rightarrow$ limit DNE, the sequence diverges

3. $c_n = (-1)^n \arctan(n)$

recall that $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$



even terms approach $\pi/2$
odd " " $-\pi/2$

limit does not exist
(does not approach a single value)