

example: (telescoping series)

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

Need to use PFD to
see that this is
telescoping

PFD: $\frac{2}{n^2-1} = \frac{2}{(n+1)(n-1)}$

$$\frac{2}{n^2-1} = \frac{A}{n+1} + \frac{B}{n-1}$$

$$2 = A(n-1) + B(n+1)$$

$$n=1: 2 = 0 + 2B \Rightarrow B=1$$

$$n=-1: 2 = -2A + 0 \Rightarrow A=-1$$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$S_1 = a_2 = 1 - \frac{1}{3}$$

$$S_2 = a_2 + a_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$$

$$S_3 = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}$$

$$= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

$$\left\{ \begin{aligned} S_4 &= 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} \\ &= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \end{aligned} \right.$$

$$\left\{ S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{3}{2} \end{aligned}$$

$\Rightarrow$ series converges to $\frac{3}{2}$

Properties:

$$\textcircled{1} \quad \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

$$\textcircled{2} \quad \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

↑ ↑
if both converge

Careful! $\sum (a_n \cdot b_n) \neq \sum a_n \cdot \sum b_n$

$$(a_1 b_1 + a_2 b_2) \neq (a_1 + a_2)(b_1 + b_2)$$

Shifting the index (or re-indexing)

ex: $\sum_{n=\textcircled{3}}^{\infty} \frac{n+1}{2^{n+2}}$

⁺²
₋₂

← say we want to start at $n=1$ instead.

$$= \sum_{n=1}^{\infty} \frac{(n+2)+1}{2^{n+2}} = \sum_{n=1}^{\infty} \frac{n+3}{2^{n+2}}$$

These have the same terms:

$$\sum_{n=3}^{\infty} \frac{n+1}{2^n} = \frac{4}{2^3} + \frac{5}{2^4} + \frac{6}{2^5} + \dots$$

$$\sum_{n=1}^{\infty} \frac{n+3}{2^{n+2}} = \frac{4}{2^3} + \frac{5}{2^4} + \dots$$

ex: $\sum_{n=0}^{\infty} \frac{3-n}{2n+1}$ $\xleftarrow{+4} (-4)$ $\leftarrow$ re-index to start at $n=4$

$$= \sum_{n=4}^{\infty} \frac{3-(n-4)}{2(n-4)+1}$$

$$= \sum_{n=4}^{\infty} \frac{7-n}{2n-7}$$

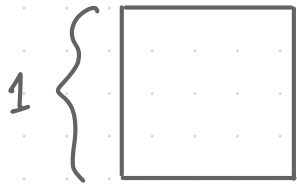
Pulling off terms:

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3}_{\text{?}} + \underbrace{\sum_{n=4}^{\infty} a_n}_{\text{know this value}}$$

Geometric Series

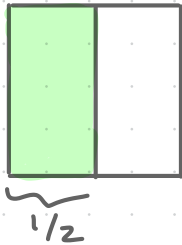
Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Geometric Interpretation:



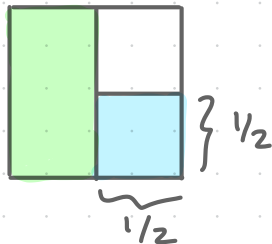
Start with a square with side lengths 1 (so Area = 1)

Step 1: Shade half.



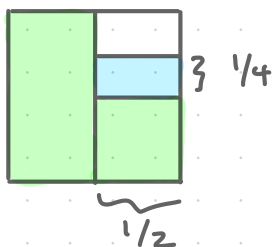
$$\text{shaded area} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Step 2: Shade half of remaining part.



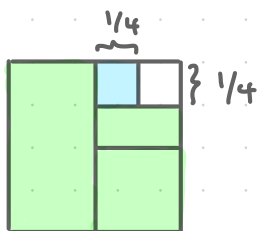
$$\text{shaded area} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Step 3: Shade half of remaining part.



$$\text{shaded area} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Step 4: Shade half of remaining part.



$$\begin{aligned}\text{shaded area} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \\ &= \frac{15}{16}\end{aligned}$$

⋮

Step n : shaded area = $\frac{2^n - 1}{2^n}$

↖
a formula for S_n !

$$\begin{aligned}\lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{2^n - 1}{2^n} = \lim_{n \rightarrow \infty} \left(\frac{2^n}{2^n} - \frac{1}{2^n} \right) \\ &= 1 - 0\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges to } 1$$

A geometric series is any series that can
be written as

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

↑ standard form

ex: $\sum_{n=0}^{\infty} 3 \left(\frac{4}{3}\right)^n$

$$a=3, r=\frac{4}{3}$$

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n$$

$$a=\frac{1}{2}, r=\frac{1}{2}$$

The value r is called the common ratio because any 2 consecutive terms is r the ratio of

$$\sum_{n=0}^{\infty} ar^n$$

$$\text{so } a_n = ar^n$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{a}r^{n+1}}{\cancel{a}r^n} = r$$

ex: $3 + 4 + \frac{16}{3} + \frac{64}{9} + \frac{256}{27} + \dots$
 $n=1$ $n=2$

is geometric

$$r = \frac{a_2}{a_1} = \frac{4}{3}$$

and $\frac{a_4}{a_2} = \frac{64/9}{16/3} = \frac{\cancel{64}^4}{\cancel{9}_3} \cdot \frac{\cancel{3}^3}{\cancel{16}_4} = \frac{4}{3}$

When is a geometric series convergent?

$$\sum_{n=0}^{\infty} ar^n = a \sum_{n=0}^{\infty} r^n$$

* can pull out "a"

Convergence depends on r , not a

Two cases: $r = 1$ and $r \neq 1$

Case 1: If $r = 1$,

$$\sum_{n=0}^{\infty} a(1)^n = \sum_{n=0}^{\infty} a$$

$$= (a + a + a + a + \dots)$$

diverge, adds up to ∞

Case 2: Assume $r \neq 1$.

We want a formula for S_n .

$$S_n = \overset{k=0}{a} + \overset{k=1}{ar} + ar^2 + \dots + \overset{k=n-1}{ar^{n-1}}$$

$$rS_n = r(a + ar + ar^2 + \dots + ar^{n-1})$$

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$$

$$\begin{array}{r} S_n = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} \\ -rS_n = \quad \quad \quad \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-1}} + ar^n \\ \hline S_n - rS_n = a - ar^n \end{array}$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

← this is why we need $r \neq 1$

Now take the limit as $n \rightarrow \infty$ of S_n

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{a - ar^n}{1-r} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} \lim_{n \rightarrow \infty} (1-r^n)$$

Recall: if $|r| < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$

if $r > 1$ or $r \leq -1$,

$\lim_{n \rightarrow \infty} r^n$ diverges

If $|r| < 1$, then $\lim_{n \rightarrow \infty} S_n = \frac{a}{1-r} (1)$

$$= \frac{a}{1-r}$$

series converges $\rightarrow$

If $r > 1$ or $r \leq -1$,

$\lim_{n \rightarrow \infty} S_n$ diverges

$\Rightarrow$ series diverges if $r > 1$ or $r \leq -1$

Also, if $r = 1$, the series diverges.

Geometric Series Test:

A geometric series converges if $|r| < 1$
and diverges if $|r| \geq 1$.

Furthermore, if the series converges,
we can compute the value

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\begin{aligned} \text{ex: } \sum_{n=1}^{\infty} \frac{2^{n+1}}{(-3)^n} &= \sum_{n=1}^{\infty} \frac{2^n \cdot 2}{(-3)^n} \\ &= \sum_{n=1}^{\infty} 2 \left(-\frac{2}{3}\right)^n \end{aligned}$$

$$r = -\frac{2}{3}$$

converges b/c $|r| < 1$

compute the value:

$$\sum_{n=0}^{\infty} 2 \left(-\frac{2}{3}\right)^{n+1} = \sum_{n=0}^{\infty} 2 \left(-\frac{2}{3}\right) \left(-\frac{2}{3}\right)^n$$

$$a = 2 \left(-\frac{2}{3}\right) = -\frac{4}{3}$$

$$\frac{a}{1-r} = \frac{-4/3}{1-(-2/3)} = -\frac{4}{5}$$

Easier way:

$$\begin{aligned} \text{value of a} & \quad \downarrow \\ \text{convergent} & \quad \text{first term in series} \\ \text{geom. series} & = \frac{1-r}{} \end{aligned}$$

→ don't need standard form

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{(-3)^n}$$

$$\text{first term: } \frac{2^2}{(-3)^1} = -\frac{4}{3}$$

$$\frac{r}{1-r} = \frac{-4/3}{1-(-2/3)} = -\frac{4}{5}$$

$$a_n = \frac{2^{n+2}}{7^n} = \frac{4 \cdot 2^n}{7^n} = 4 \underbrace{\left(\frac{2}{7}\right)^n}_{\downarrow 0}$$