

Convergence of random walks to Wiener process

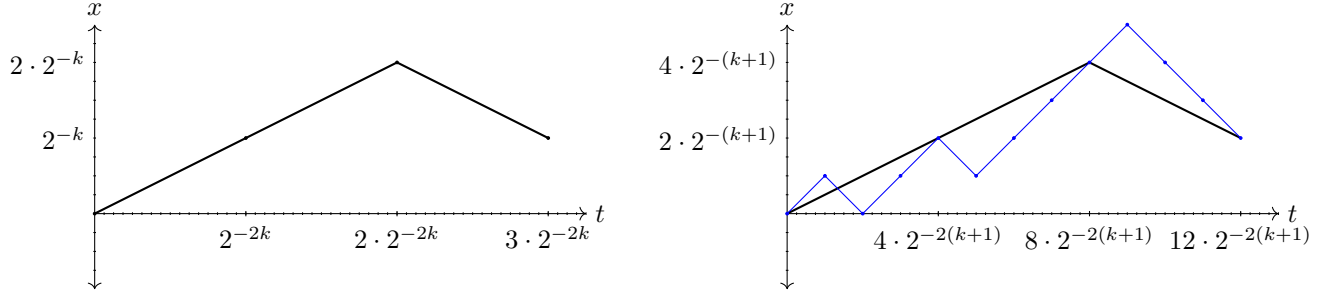
We seek to model the Wiener process, $W_t, t \in [0, \infty)$, with the goal of having a.s. uniform convergence on compact intervals of time. We follow the work of [Kni62]. For a.s. convergence, we need a single probability space for which the probability of the limiting object, a scaled random walk

$$\begin{aligned} R_k(t) &= R_k(m2^{-2k}), \quad \frac{m}{2^{2k}} \leq t < \frac{m+1}{2^{2k}} \\ &= \frac{1}{2^k} \sum_{i=1}^{m2^{-2k}} X_i, \quad X_i \sim \text{Uniform}\{-1, +1\}, \end{aligned}$$

is consistent for all refinement levels k . We will build each probability space, $(\Omega_k, \mathcal{F}_k, P_k)$, and then create a map, $M_k : \Omega_k \rightarrow \Omega_{k-1}$ for which

$$P_{k-1}(E) = P_k(M_k^{-1}(E)), \quad \forall E \in \mathcal{F}_{k-1}, \quad \forall k \geq 0.$$

The resulting family of probability spaces $\{(\Omega_k, \mathcal{F}_k, P_k)\}$, $k \geq 0$, together with the maps $\{M_k\}$, $k \geq 0$, is a stochastic process [Boc55].



With the consistency of the probability measures handled, we can consider the goal of uniform convergence on compact intervals of time. This analysis requires consideration of the non-uniqueness of the map M_k^{-1} . In particular, we track the so called time lag,

$$L_{k,r}(m2^{-2k}) = E_{r-1} \cdots E_{k+1} E_k(m) 2^{-2r} - m 2^{-2k}, \quad r > k,$$

where $E_{k-1}(m)$ is the least integer such that for $M_k \omega_k = \omega_{k-1}$, the coarser (ordered) set is contained in the finer (ordered) set:

$$\{\omega_{k-1}(2^{-2(k-1)}), \dots, \omega_{k-1}(m2^{-2(k-1)})\} \subset \{\omega_k(2^{-2k}), \dots, \omega_k(E_{k-1}(m)2^{-2k})\}.$$

It suffices to study $E_k(1)$ since the time lag at refinement level k for one step of the random walk is

$$L_{k,k+1}(2^{-2k}) = E_k(1) 2^{-2(k+1)} - 2^{-2k}.$$

By noting that $E_k(1)$ follows a distribution $2(X+1)$, $X \sim \text{Geo}(1/2)$, we can get an a.s. and uniform bound on the time lag on intervals of time $t \in [0, m2^{-2k}]$:

$$\text{For all } \delta > 0, \quad P \left[\max_{0 \leq t \leq m2^{-2k}} |L_{k,r}(t)| > \delta \right] < \frac{m2^{-4k+1}}{3\delta^2}.$$

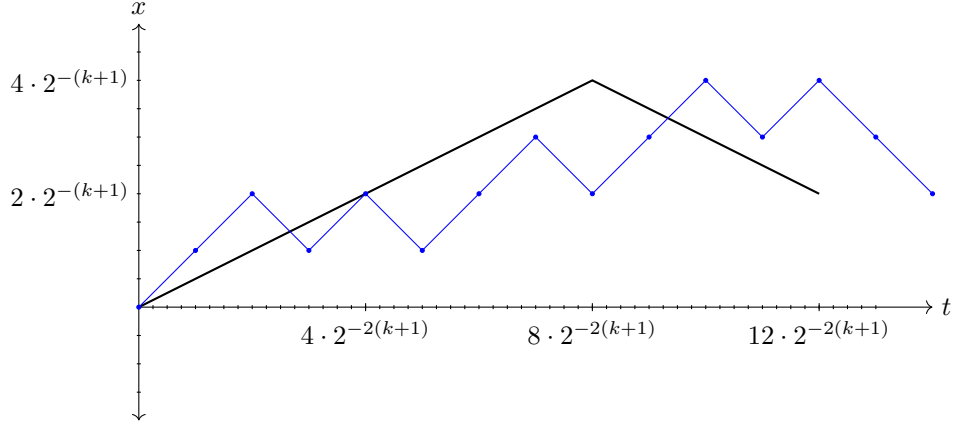


Figure 1: $L_{k,k+1}(3) = (E_k(3)/4 - 3)2^{-2k} = (\frac{2+8+4}{4} - 3)2^{-2k} = \frac{1}{2} \cdot 2^{-2k}$.

In fact, we can get a bound independent of the number of refinement levels between each time lag, that is, we can get uniformity w.r.t. r for our bound on the time lag:

$$\text{For fixed } k \text{ and } m, \text{ and } \delta = 2^{-2k+1}, P \left[\sup_{0 \leq t \leq m2^{-2k}, k \leq r} |L_{k,r}(t)| > \delta \right] < C$$

where C depends on k and m . Hence, the limit of time lag, $\lim_{r \rightarrow \infty} L_{k,r}(m2^{-2k})$ exists a.s., and thus the times $m2^{-2k} + \lim_{r \rightarrow \infty} L_{k,r}(m2^{-2k})$ are a.s. dense in $(0, \infty)$. It is on these times that we define the limiting object, R_∞ , that is to be our separable Brownian motion, and it is on these times that R_∞ is a.s. uniformly continuous in finite time intervals. We set $R_\infty(t) = \lim_{k \rightarrow \infty} R_k(t)$ when the time lag limit exists, and we take the value 0 when it does not (that is, 'sawtooth' paths with infinite time lag are assigned measure 0). The final result is that $R_\infty(t)$ is a separable Brownian motion, and, for all $\epsilon, \delta > 0$, and k large,

$$P \left[\max_{0 \leq t \leq T, r \geq k} |R_\infty(t) - R_r(t)| > \delta \right] < D$$

where D depends on T, k, δ .

Though it is not efficient for modeling, this method can be useful for understanding path dependent functionals of Brownian motion, in particular the local time of Brownian motion at a point can be modeled using the ideas here. The main result of a.s. uniform convergence on compact intervals can be expanded to any diffusion process which has a speed measure that is absolutely continuous w.r.t. Lebesgue measure, and with the addition of a time change, the result can be further expanded to sticky Brownian motions [Ami91].

References

- [Ami91] Madjid Amir. "Sticky Brownian motion as the strong limit of a sequence of random walks". In: *Stochastic Processes and their Applications* 39.2 (Dec. 1991), pp. 221–237. URL: <https://ideas.repec.org/a/eee/spapps/v39y1991i2p221-237.html>.
- [Boc55] Salomon Bochner. *Harmonic Analysis and the Theory of Probability*. California Monographs in Mathematical Sciences. Berkeley and Los Angeles: University of California Press, 1955.
- [Kni62] F. B. Knight. "Random walks and a sojourn density process of Brownian motion". In: *Transactions of the American Mathematical Society* 103.2 (1962), pp. 218–228. DOI: 10.1090/S0002-9947-1962-0132134-0.