The goal of this project is to develop "function sense" about the decay rate of functions. This skill is important for determining convergence of improper integrals, and it will become important again when we study convergence of series.

Problems 1-4 will help develop your *numerical* "function sense."

1. Consider the improper integral $\int_3^\infty \frac{1}{x^2 \ln(x)} dx$. Use technology to find:

   (a) $\int_3^{10} \frac{1}{x^2 \ln(x)} dx =$

   (b) $\int_3^{100} \frac{1}{x^2 \ln(x)} dx =$

   (c) $\int_3^{1000} \frac{1}{x^2 \ln(x)} dx =$

2. Which of the following statements is closest to what you can conclude regarding the convergence/divergence of the above improper integral?

   (a) Based on numerical evidence, the above integral converges.
   (b) Based on numerical evidence, the above integral diverges.
   (c) Based on numerical evidence, it appears that the above integral converges.
   (d) Based on numerical evidence, it appears that the above integral diverges.

3. Now, consider the improper integral $\int_3^\infty \frac{\ln(x)}{\sqrt{x}} dx$. Again, use technology to find

   (a) $\int_3^{10} \frac{\ln(x)}{\sqrt{x}} dx =$

   (b) $\int_3^{100} \frac{\ln(x)}{\sqrt{x}} dx =$

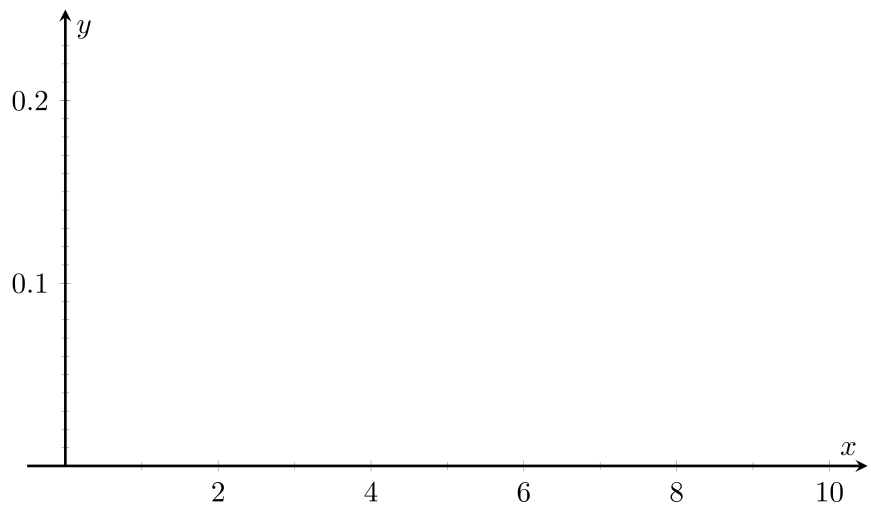
   (c) $\int_3^{1000} \frac{\ln(x)}{\sqrt{x}} dx =$

4. Which of the following statements is closest to what you can conclude regarding the convergence/divergence of the above improper integral?

   (a) Based on numerical evidence, the above integral converges.
   (b) Based on numerical evidence, the above integral diverges.
   (c) Based on numerical evidence, it appears that the above integral converges.
   (d) Based on numerical evidence, it appears that the above integral diverges.

Problems 5-8 will help develop your *graphical* "function sense."

5. Use technology to help you draw the graphs of $f(x) = \dfrac{1}{x^2 \ln(x)}$ and $g(x) = \dfrac{1}{x^2}$. Pay attention to any intersection points, and pay attention to which function has higher values.



6. Using the graph you just made:

   (a) Do you think $\displaystyle\int_3^\infty \frac{1}{x^2 \ln(x)}\,dx \quad \leq \quad \int_3^\infty \frac{1}{x^2}\,dx$ is true? Why or why not?

   (b) What does this suggest about the convergence/divergence of the two integrals and why?

7. Use technology to help you draw the graphs of $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$ and $g(x) = \dfrac{1}{\sqrt{x}}$. Do these functions both have horizontal asymptotes?

8. Using the graph you just made

   (a) Do you think $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx \quad \geq \quad \int_3^\infty \frac{1}{\sqrt{x}}\,dx$ is true? Why or why not?

   (b) What does this suggest about the convergence/divergence of the two integrals and why?

The punchline:

Comparison Theorem for Integrals
If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If ________________ is convergent, then ________________ is convergent.

(b) If ________________ is divergent, then ________________ is divergent.

If we know $\int_a^\infty f(x)dx$ diverges that what can we conclude about $\int_a^\infty g(x)dx$?

And if we know $\int_a^\infty g(x)dx$ converges, what can we conclude about $\int_a^\infty f(x)dx$?

Putting it into practice:

9. Consider the integral $\displaystyle\int_1^\infty \frac{1+\sin^4(2x)}{\sqrt{x}}dx$.

   (a) Do you think this improper integral converges or diverges?

   (b) A good comparison function is:

   (c) Write down the inequality that compares $\dfrac{1+\sin^4(2x)}{\sqrt{x}}$ to your answer to (b).

   (d) How will this inequality help you prove that your guess is correct?

10. Consider the integral $\int_2^\infty \frac{1}{x+e^x}dx$.

    (a) Do you think this improper integral converges or diverges?

    (b) A good comparison function is:

    (c) Write down the inequality that compares $\frac{1}{x+e^x}$ to your answer to (b).

    (d) How will this inequality help you prove that your guess is correct?

The limits and limitations of your "function sense."

11. Consider the integral $\int_1^\infty \frac{1}{\sqrt{x^2+1}}dx$

    (a) Do you think this improper integral converges or diverges?

    (b) A good comparison function is:

    (c) Write down an inequality that compares $\frac{1}{\sqrt{x^2+1}}$ to your answer to (b).

    (d) How will this inequality help you prove your guess? (or will it?)

12. If your attempt at comparison above failed, then make a guess about the convergence/divergence of this integral by investigating the above integral numerically.

13. Even though $\frac{1}{x}$ did not produce the desired inequality, it may still provide a useful comparison.

    (a) Show that $\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2+1}}}{\frac{1}{x}} = 1$

    (b) What does this say about the relationship between $\frac{1}{\sqrt{x^2+1}}$ and $\frac{1}{x}$?

14. Try to write your own theorem that involves the limit idea above. I will get you started.

> **The Limit Comparison Test:** If $f(x)$ and $g(x)$ are positive continuous functions, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ __________, then:

More practice:

15. Consider the integral $\int_2^\infty \frac{\cos^2(x)}{x^2} dx$.

    (a) Do you think this improper integral converges or diverges?

    (b) A good comparison function is:

    (c) Write down the inequality that compares $\frac{\cos^2(x)}{x^2}$ to your answer to (b).

    (d) How will this inequality help you prove that your guess is correct?

16. Consider the integral $\int_3^\infty \frac{1}{\sqrt{x^2-1}} dx$.

    (a) Do you think this improper integral converges or diverges?

    (b) A good comparison function is:

    (c) Write down the inequality that compares $\frac{1}{\sqrt{x^2-1}}$ to your answer to (b).

    (d) How will this inequality help you prove that your guess is correct?

17. Now determine the convergence/divergence of the following integrals by comparing them to another improper integral whose convergence/divergence you already know, or can more easily calculate.

    (a) $\int_1^\infty e^{-x^2}\,dx$

    (b) $\int_1^\infty \frac{3}{\sqrt{x^3+x}}\,dx$

    (c) $\int_1^\infty \frac{5+2\sin x}{x^2+2}\,dx$

    (d) $\int_3^\infty \frac{5+2\sin x}{x-2}\,dx$