

PFD example

$$\int \frac{x^2+3x+9}{x^2(x^2+9)} dx$$

$$x^2(x^2+9) \cdot \left(\frac{x^2+3x+9}{x^2(x^2+9)} \right) = \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} \right) x^2(x^2+9)$$

$$x^2+3x+9 = Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2$$

$$= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2$$

$$\underline{0}x^3 + \underline{1}x^2 + \underline{3}x + \underline{9} = (\underline{A+C})x^3 + (\underline{B+D})x^2 + \underline{9A}x + \underline{9B}$$

Equate coefficients: $A+C=0$

$$B+D=1$$

$$9A=3 \Rightarrow A=3$$

$$9B=9 \Rightarrow B=1$$

$$C=-A=-3$$

$$D=1-B=1-1=0$$

$$\frac{x^2+3x+9}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}$$

$$= \frac{3}{x} + \frac{1}{x^2} + \frac{-3x+0}{x^2+9}$$

$$\int \frac{x^2+3x+9}{x^2(x^2+9)} dx = \int \frac{3}{x} + \frac{1}{x^2} - \frac{3x}{x^2+9} dx$$

$$= 3 \int \frac{1}{x} dx + \int \frac{1}{x^2} dx - 3 \int \frac{x}{x^2+9} dx$$

$$u=x^2+9$$

$$du=2x dx$$

$$= 3 \ln|x| + \int x^{-2} dx - 3 \int \frac{1}{u} \cdot \frac{1}{2} du$$

$$= 3 \ln|x| - x^{-1} - \frac{3}{2} \ln|x^2+9| + C$$