Arc Length Formula
If a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then its length is

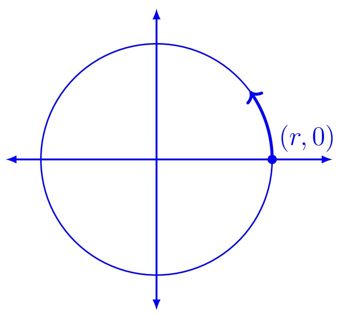
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

1. Follow the steps below to use the arc length formula to compute the circumference of a circle of radius r.

(a) Give a parametrization of the circle of radius r centered at the origin.

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$$
$$0 \le t \le 2\pi$$

(b) Check that your parametrization traverses the circle exactly once. What is the starting point of your parametrization? Which direction does your parametrization go, clockwise or counterclockwise?



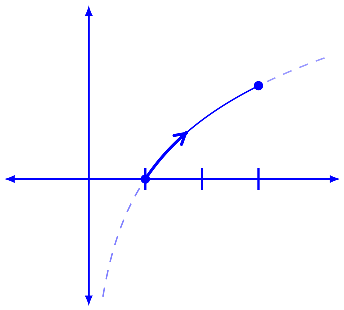
$x(0) = r\cos(0) = r$ and $y(0) = r\sin(0) = 0$ so the starting point is $(r, 0)$.

For a small positive t, $y = r\sin t$ is positive so it goes counterclockwise from the starting point.

(c) Compute the length of the curve, i.e. the circumference of the circle. Is your answer what you expected?

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{2\pi} r\, dt \\ &= (rt)\Big|_0^{2\pi} = r(2\pi) - 0 = 2\pi r \end{aligned}$$

2. (a) Graph the curve $y = \ln x$ where $1 \le x \le 3$. Find a parametrization of the curve.



$$\begin{cases} x = t \\ y = \ln t \end{cases}$$
$1 \le t \le 3$

(b) Does your parametrization traverse this curve exactly once? How do you know?

Yes, because $y = \ln x$ is a function of x, so every input x has exactly one output y.

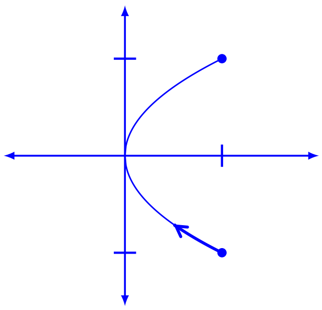
When we let $x = t$, that means each t in the interval $[1, 3]$ corresponds to exactly one point on the curve.

(c) Set up an integral that represents the arc length of this curve. You do not have to evaluate the integral.

First $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = \dfrac{1}{t}$. Using the arc length formula,

$$L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^3 \sqrt{1 + \left(\frac{1}{t}\right)^2}\, dt$$

3. (a) Graph the curve $x = y^2$ where $-1 \leq y \leq 1$. Find a parametrization of the curve.



$$\begin{cases} x = t^2 \\ y = t \end{cases}$$
$$-1 \leq t \leq 1$$

(b) Does your parametrization traverse this curve exactly once? How do you know?

Yes, because $x = y^2$ is a function of y, so every input y has exactly one output x.

When we let $y = t$, that means each t in the interval $[-1, 1]$ corresponds to exactly one point on the curve.

(c) Set up an integral that represents the arc length of this curve. You do not have to evaluate the integral.

First $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 1$. Using the arc length formula,

$$L = \int_{-1}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-1}^{1} \sqrt{(2t)^2 + 1}\, dt$$

4. Think about how you were able to find a parametrization for the curve in problem 2. Can you use that process to find a parametrization for any curve given as $y = f(x)$ where $a \le x \le b$? What would the arc length formula be in that case?

We can parametrize this as: $\begin{cases} x = t \\ y = f(t) \end{cases}$ where $a \le t \le b$.

Then $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = f'(t)$. Using the arc length formula,

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{1 + \left(f'(t)\right)^2}\, dt$$

Switching t back to x, we get

$$\boxed{L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.}$$

Similarly, if a curve is given as $x = g(y)$ for $c \le y \le d$, we can parametrize this as: $\begin{cases} x = g(t) \\ y = t \end{cases}$ where $c \le t \le d$.

Then $\dfrac{dx}{dt} = g'(t)$ and $\dfrac{dy}{dt} = 1$. Using the arc length formula,

$$L = \int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_c^d \sqrt{\left(g'(t)\right)^2 + 1}\, dt$$

Switching t back to y, we get

$$\boxed{L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\, dy.}$$