

Recap: Taylor & Maclaurin series

Given a function f and a base point a , the Taylor series is

$$\sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{\text{Taylor coefficients}} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Maclaurin series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

To find Taylor / Maclaurin series for a given function, we start with taking several derivatives of f and looking for a pattern.

$\leftarrow a=0$

Standard Maclaurin Series

Radius of Convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$R = 1 \quad \leftarrow \text{IOC is } (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$R = \infty$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$R = 1$$

↑ use the standard Maclaurin series to find
Maclaurin series for related functions

$\leftarrow a=0$

ex: $f(x) = x^3 \sin(5x)$, Maclaurin series for f

Instructions: "Use a known
Maclaurin series to find
a Maclaurin series for
 $f(x) = x^3 \sin(5x)$."

$$\left[\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right]$$

$$\sin(5x) = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{(2n+1)!}$$

$$f(x) = x^3 \cdot \sin(5x) = x^3 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1} \cdot x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{5^{2n+1} x^{2n+4}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^{2n+1}}{(2n+1)!} x^{2n+4}$$

ex: Show that the Maclaurin series representation of $\cos(x)$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{using } \sin(x).$$

Idea: $\frac{d}{dx}(\sin x) = \cos x$

$$\begin{aligned} \cos x &= \frac{d}{dx}(\sin x) && \swarrow \text{from table} \\ &= \frac{d}{dx} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \\ &= \frac{d}{dx} \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots \\ &= 1 - \frac{\cancel{3}x^2}{\cancel{3} \cdot 2!} + \frac{\cancel{5}x^4}{\cancel{5} \cdot 4!} - \frac{\cancel{7}x^6}{\cancel{7} \cdot 6!} + \dots \end{aligned}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$n=0$ $n=1$ $n=2$ $n=3$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

example from WebAssign

8. [-/3 Points]

DETAILS

SCALCCC4 8.6.011.

(a) Use differentiation to find a power series representation for the following function.

$$f(x) = \frac{1}{(7+x)^2} = \left(\frac{1}{7+x}\right) \left(\frac{1}{7+x}\right)$$

- ☐ $\sum_{n=0}^{\infty} (-1)^n (n+1) \frac{x^{2n}}{7^{n+3}}$
- ☐ $\sum_{n=1}^{\infty} (-1)^n (n+2) \frac{x^{2n}}{7^{n+1}}$
- ☐ $\sum_{n=0}^{\infty} (-1)^n (n+1) \frac{x^n}{7^{n+2}}$
- ☐ $\sum_{n=2}^{\infty} (-1)^n (n+1) \frac{x^n}{7^{n+2}}$
- ☐ $\sum_{n=0}^{\infty} (-1)^n (n-1) \frac{x^n}{7^{n+1}}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

What is the radius of convergence?

R =

(b) Use part (a) to find a power series for the following function.

$$f(x) = \frac{1}{(7+x)^3}$$

Idea: $\frac{d}{dx} \left(\frac{-1}{7+x} \right) = \frac{d}{dx} \left(-(7+x)^{-1} \right)$

$$= + (7+x)^{-2} \cdot 1 = \frac{1}{(7+x)^2}$$

$$-\left(\frac{1}{7+x}\right) = -\left(\frac{\frac{1}{7}}{1+\frac{x}{7}}\right)$$

$$= -\frac{1}{7} \left(\frac{1}{1+\frac{x}{7}} \right)$$

$$= -\frac{1}{7} \left(\frac{1}{1 - (-\frac{x}{7})} \right) \quad r = -\frac{x}{7}$$

$$= -\frac{1}{7} \sum_{n=0}^{\infty} \left(-\frac{x}{7}\right)^n$$

$$= -\frac{1}{7} \sum_{n=0}^{\infty} \left(-\frac{1}{7}\right)^n x^n$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{7}\right)^{n+1} x^n$$

$$f(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \left(-\frac{1}{7}\right)^{n+1} x^n \right)$$

← const.

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{7}\right)^{n+1} \cdot n x^{n-1}$$

q

Tricks for finding P.S. representations using

$$\frac{1}{1-x} = \sum x^n \quad \text{or another known Maclaurin series.}$$

- substitute for x

e.g. $\sin(\underline{5x})$ using $\sin x$

- multiply by powers of x

e.g. $x^3 \cdot \sin x$ using $\sin x$

- integrating / differentiating

e.g. relating $\ln(1+x)$ to $\frac{1}{1+x}$,

using $\sin x$ for finding
p.s. for $\cos x$.

(extra examples on website)

ex: $f(x) = x \arctan(x^2)$

(a) Find a Maclaurin series for f .

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arctan(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$f(x) = x \cdot \arctan(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{2n+1}$$

(b) Find $f^{(n)}(0)$ using the Maclaurin series for f .

$$\text{Taylor coeff: } C_n = \frac{f^{(n)}(0)}{n!}$$

$$\frac{f^{(11)}(0)}{11!} = C_{11} \leftarrow \text{coeff. for the term involving } x^{11}$$

$$x^{4n+3} = x^{11}$$

$$4n+3 = 11$$

$$4n = 8$$

$$\boxed{n=2}$$

$$\sum_{h=0}^{\infty} \frac{(-1)^h}{2h+1} \cdot x^{4h+3}$$

coefficient for $n=2$ is $\frac{(-1)^2}{2(2)+1} = \boxed{\frac{1}{5}}$

$$\frac{f^{(11)}(0)}{11!} = \frac{1}{5}$$

$$f^{(11)}(0) = \frac{11!}{5}$$

for $f^{(10)}(0) \dots$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+3} = x^3 - \frac{1}{3} x^7 + \frac{1}{5} x^{11} - \dots$$

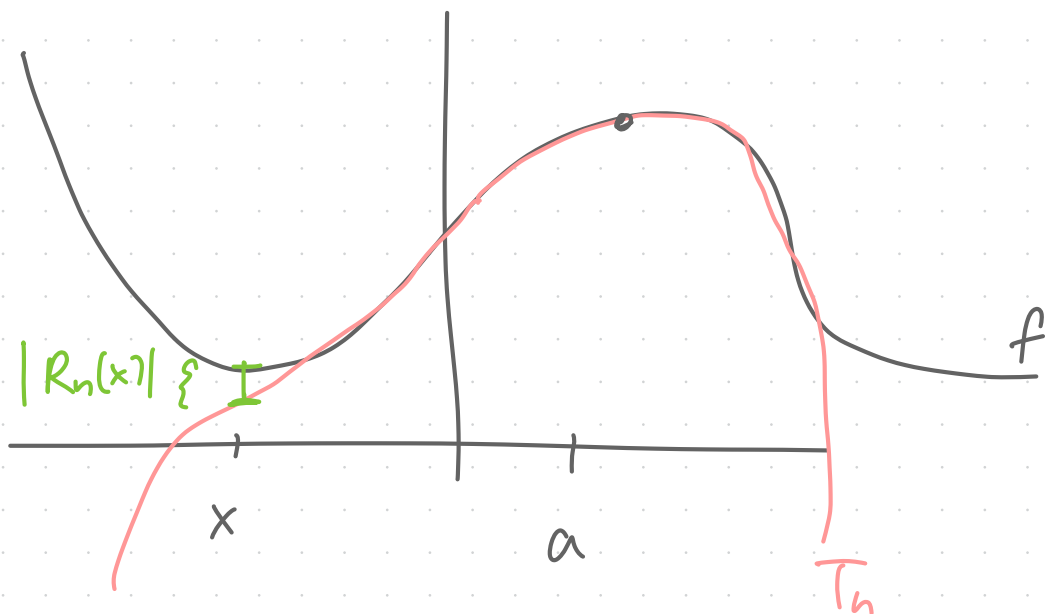
$$C_{10} = 0 \quad \text{so} \quad f^{(10)}(0) = 0$$

* 8.8 - Approximations Using Taylor Polynomials

If $T_n(x)$ is the n th degree Taylor polynomial of $f(x)$ centered at a

$$T_n(x) \approx f(x) \quad \text{for } x \text{ near } a$$

We can tell how good this approximation is using Taylor's Inequality



$$R_n(x) = f(x) - T_n(x)$$

remainder
(error)

↑
true
value
at x

↑
approximation
at x

$$|R_n(x)| = |\text{error in approx. } f \text{ using } T_n|$$

Taylor's Inequality (given on final)

If $|f^{(n+1)}(x)| \leq M$ for all x satisfying

$|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

a - base point

M - a bound for the $(n+1)$ th derivative of f (need to find)

n - "stopping index" (like our other remainder estimates)

d - distance from base point where Taylor's ineq. is valid

x - a point or range of points