

Taylor's Inequality

The remainder of the Taylor series for $f(x)$ centered at a is defined as

$$R_n(x) = f(x) - T_n(x),$$

where $T_n(x)$ denotes the n th degree Taylor polynomial (also centered at a). Taylor's Inequality says that if $|f^{(n+1)}(x)| \leq M$ for all x satisfying $|x - a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } x \text{ satisfying } |x - a| \leq d.$$

1. Use Taylor's inequality to determine the number of terms needed to estimate $\cos(\frac{1}{2})$ using the Taylor series centered at $a = \frac{\pi}{6}$ to within $0.0001 = 10^{-4}$.

2. (a) Find the 2nd degree Taylor polynomial, $T_2(x)$ for $f(x) = x^{-2}$ based at $a = 1$.
- (b) Use Taylor's Inequality to estimate the accuracy of the approximation using $T_2(x)$ to estimate $f(x)$ in the interval $[0.9, 1.1]$.

3. How many terms of the Maclaurin series for $f(x) = \cos(2x)$ are needed to guarantee that the approximation of $f(x)$ on the interval $[-\frac{1}{3}, \frac{1}{3}]$ is within 10^{-6} ?

If you're using a Taylor polynomial to approximate f at a particular value of x instead of a range of x , the following formulation of Taylor's Inequality is sometimes useful.

Alternate Statement of Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for all x between a and x_0 , then

$$|R_n(x_0)| \leq \frac{M}{(n+1)!} |x_0 - a|^{n+1}.$$

4. How large should n be to guarantee that the approximation of $e^{-0.1}$ using an n th degree Taylor polynomial for e^x centered at $a = 0$ is within 0.0001?

5. Estimate the range of values of x for which the approximation

$$\ln(x) \approx \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2$$

is accurate to within 0.01.