

Web Assign ...

Going from polar eq. to Cartesian eq. } $x = r \cos \theta$
OR Cartesian to polar } $y = r \sin \theta$

ex: $r = 4 \sec \theta$ $\leftarrow$ polar

$$r = \frac{4}{\cos \theta}$$

$$r \cos \theta = 4$$

$$x = 4 \leftarrow \text{Cartesian}$$

ex: $r^2 \sin(2\theta) = 2$

$$r^2 (\cancel{2} \sin \theta \cos \theta) = \cancel{2}$$

$$r^2 \sin \theta \cos \theta = 1$$

$$(r \sin \theta) \cdot (r \cos \theta) = 1$$

$$y \cdot x = 1$$

$$xy = 1$$

hyperbola

ex: $r = 2 \cos \theta$ $\leftarrow$ ex. from yesterday

$$\underbrace{r \cdot r}_{x^2 + y^2} = 2 \underbrace{\cos \theta \cdot r}_x$$

↑
using $x^2 + y^2 = r^2$ (comes from $x = r \cos \theta$, $y = r \sin \theta$)

ex: for Cartesian to polar, just plug in
 $x = r \cos \theta$, $y = r \sin \theta$ and solve for r

$$x = -y^2$$

$$r \cos \theta = - (r \sin \theta)^2$$

$$r \cos \theta = -r^2 \sin^2 \theta$$

$$r \cos \theta + r^2 \sin^2 \theta = 0$$

$$r (\cos \theta + r \sin^2 \theta) = 0$$

$$r = 0 \quad \text{or} \quad \cos \theta + r \sin^2 \theta = 0$$

↑
just the
origin

$$r \sin^2 \theta = -\cos \theta$$

$$r = -\frac{\cos \theta}{\sin^2 \theta}$$

answer

Calc with Polar, ctd

Areas enclosed by Polar Curves

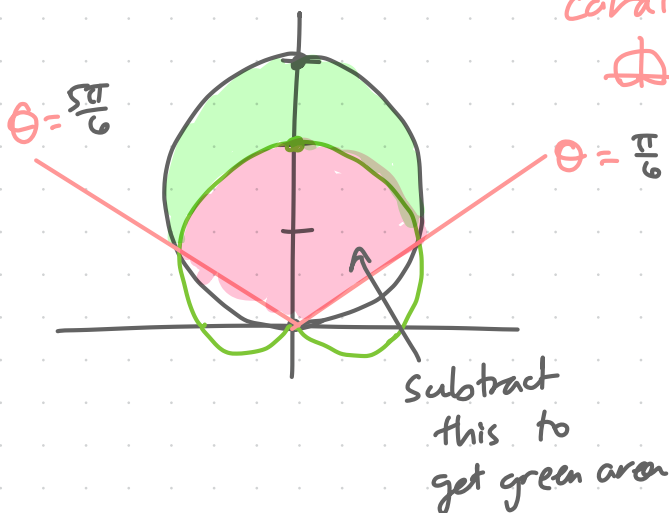
circle
↓

ex: Find the area of region inside $r = 3\sin\theta$
and outside $r = 1 + \sin\theta$

~~cardioid~~

cardioid

~~circle~~



intersection points

$$3\sin\theta = 1 + \sin\theta$$

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2} (\underbrace{3\sin\theta}_{\text{outer curve}})^2 d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2} (\underbrace{1 + \sin\theta}_{\text{inner curve}})^2 d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} \cdot 9 \sin^2 \theta - \frac{1}{2} (1 + \sin \theta)^2 \right) d\theta$$

$$= \frac{1}{2} \int_{\pi/6}^{5\pi/6} 9 \sin^2 \theta - (1 + 2 \sin \theta + \sin^2 \theta) d\theta$$

$$= \frac{1}{2} \int_{\pi/6}^{5\pi/6} \underline{8 \sin^2 \theta} - 2 \sin \theta - 1 d\theta$$

$$= \frac{1}{2} \int_{\pi/6}^{5\pi/6} 8 \left(\frac{1}{2} (1 - \cos(2\theta)) \right) - 2 \sin \theta - 1 d\theta$$

$\downarrow \quad \downarrow$
 $+ 2 \cos \theta + \theta$

$$= \dots = \pi$$

In general, if $f(\theta) \geq g(\theta) \geq 0$, the area inside f but not g on $[a, b]$ is

$$\int_a^b \frac{1}{2} (f(\theta)^2 - g(\theta)^2) d\theta$$

Arc Length for Polar Curves

$r = f(\theta)$. Consider θ as a parameter.

$$\begin{cases} x = r \cos \theta = f(\theta) \cos \theta \\ y = r \sin \theta = f(\theta) \sin \theta \end{cases}$$

for $a \leq \theta \leq b$ $\leftarrow$ depends on the curve

Recall: $L = \int_a^b ds$ where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$



$\uparrow$
arc length
differential
(little piece of length)

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \end{aligned}$$

Why?

$$x = f(\theta) \cos \theta$$

$$y = f(\theta) \sin \theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta) \cos \theta - f(\theta) \sin \theta\right)^2 + \left(f'(\theta) \sin \theta + f(\theta) \cos \theta\right)^2$$

$$= \left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right)^2 + \left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right)^2$$

$$\rightarrow = \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta$$

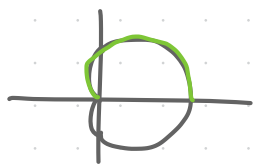
$$= \left(\frac{dr}{d\theta}\right)^2 (\underbrace{\cos^2 \theta + \sin^2 \theta}_1) + r^2 (\underbrace{\sin^2 \theta + \cos^2 \theta}_1)$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Arc Length Formula for $r = f(\theta)$:

$$L = \int_a^b ds \quad \text{where} \quad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

ex: Find the length of $r = 1 + \cos\theta$.



by symmetry,

$$L = \int_0^{2\pi} ds = 2 \cdot \int_0^{\pi} ds.$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2} d\theta$$

$$= \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_1} d\theta$$

$$= \sqrt{2 + 2\cos\theta} d\theta$$

$$L = 2 \int_0^{\pi} ds = 2 \int_0^{\pi} \sqrt{2+2\cos\theta} d\theta$$

$$= 2 \int_0^{\pi} \frac{\sqrt{2+2\cos\theta}}{1} \cdot \frac{\sqrt{2-2\cos\theta}}{\sqrt{2-2\cos\theta}} d\theta$$

$$= 2 \int_0^{\pi} \frac{\sqrt{4-4\cos^2\theta}}{\sqrt{2-2\cos\theta}} d\theta$$

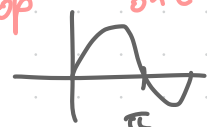
$$= 2 \int_0^{\pi} \frac{\sqrt{4(1-\cos^2\theta)}}{\sqrt{2-2\cos\theta}} d\theta$$

$= \sin^2\theta$

$$= 2 \int_0^{\pi} \frac{\sqrt{4\sin^2\theta}}{\sqrt{2-2\cos\theta}} d\theta$$

$$= 2 \int_0^{\pi} \frac{|2\sin\theta|}{\sqrt{2-2\cos\theta}} d\theta$$

can drop b/c $\sin\theta \geq 0$ on $[0, \pi]$



$$= 2 \int_0^{\pi} \frac{2\sin\theta}{\sqrt{2-2\cos\theta}} d\theta$$

$$u = 2-2\cos\theta$$

$$du = 2\sin\theta d\theta$$

$$= 2 \int_0^4 \frac{1}{\sqrt{u}} du$$

$$\theta=0 \Rightarrow u=0$$

$$\theta=\pi \Rightarrow u=4$$

$$= \dots = 8$$

On desmos, "r = "
 $\uparrow$
 $f(\theta)$