THE ABEL-JACOBI THEOREM *

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Abstract

It is a well-known result that a principal divisor on a compact Riemann surface has degree zero. Except for the Riemann sphere $\mathbb{P}^1$, the converse is not true (see the proof of Corollary 1). For a complex torus, a criterion to a divisor be principal is given based on the Abel map [2, Theorem V.2.8].

Abel's Theorem extends this result by constructing the Abel-Jacobi map to a general compact Riemann surface X. The proof presented here follows [1] and is based on a result of functional analysis. After, Abel-Jacobi theorem describes the quotient of divisors of degree zero modulo principal divisors and a new space associated to X is constructed. In the end, we return to the case of surfaces with genus one together with an application. The references are explicitly given along the text.

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Notation

The table below fixes some notations. All the spaces are defined on X. Other objects are introduced along the work.

X	compact Riemann surface of genus $g > 0$
Ω^k	space of complex k-forms, $k = 0, 1, 2$
$\Omega^{0,1}$	space of (0,1)-forms
$\Omega^{1,0}$	space of (1,0)-forms
$\mathcal{H}^{1,0}$	space of holomorphic 1-forms
$\mathcal{H}^{0,1}$	$\operatorname{coker} \overline{\partial} \colon \Omega^0 \longrightarrow \Omega^{0,1}$
$\mathrm{H}_1(\mathrm{X}, \mathbb{Z})$	homology group
$\mathrm{H}_1(\mathrm{X}, \mathbb{R})$	real homology group
$\mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{R})$	real de Rham cohomology group
$\mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{C})$	complexified de Rham cohomology group
$\mathrm{Div}(\mathrm{X})$	group of divisors
$\mathrm{Div}_0(\mathrm{X})$	subgroup of divisors of degree zero
$\mathrm{PDiv}(\mathrm{X})$	subgroup of principal divisors
$L(D)$	space of meromorphic functions with poles bounded by D
$L^1(D)$	space of meromorphic 1-forms with poles bounded by D

1 The Main Theorem and Some Consequences

In this preliminary section a powerful theorem related to existence of solutions to a special differential equation on Riemann surfaces is introduced. It has important implications relating the analytical and topological structures of X.

Theorem 1 (Main - [1], Theorem 5)**.** *Let ρ be a 2-form on X. There is a solution f to the $\Delta f := 2i\overline{\partial}\partial f = \rho$ if and only if $\int_X \rho = 0$, and the solution is unique up to the addition of a constant.*

Proof. See [1, Chapter 9] or [9]. □

The next two theorems are direct consequences.

Theorem 2 ([1], Theorem 6)**.**

1. *The map*

$$\begin{aligned} \sigma \colon \mathcal{H}^{1,0} &\longrightarrow \overline{\mathcal{H}^{0,1}} \\ \omega &\longmapsto [\omega + \overline{\omega}] \end{aligned} \tag{1}$$

is an isomorphism.

2. The bilinear map

$$B : \mathcal{H}^{1,0} \times \mathcal{H}^{0,1} \longrightarrow \mathbb{C} \qquad (\omega, [\theta]) \longmapsto \int_X \omega \wedge \theta \tag{2}$$

induces an isomorphism from $\mathcal{H}^{0,1}$ *to* $(\mathcal{H}^{1,0})^*$.

3. Consider the inclusion

$$i\colon \mathcal{H}^{1,0} \longrightarrow \mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{C}) \qquad \omega \longmapsto [\omega] \tag{3}$$

Then the map

$$\tilde{i}\colon \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \longrightarrow \mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{C}) \qquad (\omega, \theta) \longmapsto i(\omega) + \overline{i(\sigma^{-1}(\theta))} \tag{4}$$

is an isomorphism.

Remark 1. *For a complex vector space* V*, the conjugate space* $\overline{V}$ *denotes the complex vector space where the scalar multiplication is given by*

$$\lambda.v = \overline{\lambda} v,$$

for $\lambda \in \mathbb{C}$*,* $v \in V$*. This space is naturally isomorphic to* V*.*

We shall describe only the isomorphism of item 2. A complete proof can be found in [9]. If V is a real vector space of finite dimensional equipped with an inner product $\langle ., . \rangle$, then an isomorphism between V and its dual V^* is induced by the map φ below:

$$\varphi\colon V \longrightarrow V^* \qquad v \longmapsto \langle v, . \rangle\colon V \longrightarrow \mathbb{R} \qquad w \longmapsto \langle v, w \rangle$$

If V is a complex vector space of finite dimension equipped with a Hermitian form, a similar map provides an isomorphism between V and the conjugate $\overline{V}$. Now, from item 3, $\mathcal{H}^{1,0}$ and $\mathcal{H}^{0,1}$ are complex vector spaces of finite dimension. Set

$$B_{[\theta]} : \mathcal{H}^{1,0} \longrightarrow \mathbb{C} \qquad \omega \longmapsto \int_X \omega \wedge \theta$$

The composition

$$B_{[\theta]} \circ \sigma\colon \overline{\mathcal{H}^{0,1}} \longrightarrow \mathbb{C} \qquad \overline{\omega} \longmapsto \int_X \overline{\omega} \wedge \theta$$

defines an Hermitian form on $\overline{\mathcal{H}^{0,1}}$. Indeed, it is related to the Dirichlet product (see [1, Subsection 5.3.3]). Define the map

$$\begin{aligned} \varphi\colon\ \mathcal{H}^{0,1} &\longrightarrow (\mathcal{H}^{1,0})^* \\ [\theta] &\longmapsto B_{[\theta]} \end{aligned} \tag{5}$$

If $\theta \in \operatorname{Ker}\varphi$, then $B_\theta \equiv 0$, which implies $B_{[\theta]} \circ \sigma \equiv 0$. Hence $\theta \in \operatorname{Im}\overline{\partial} = 0 \in \mathcal{H}^{0,1}$. Then the kernel of φ is trivial. From item 1, $\dim(\mathcal{H}^{0,1}) = \dim(\mathcal{H}^{1,0})$ and φ is the desired isomorphism. The next Theorem will be used in the proof of Abel's result.

Theorem 3. *Let ω be a smooth (0,1)-form on X. Then, the equation*

$$\overline{\partial} u = \omega$$

has a smooth solution u if and only if

$$\int_X \omega \wedge \theta = 0$$

for all holomorphic 1-forms θ.

Proof. Recall the isomorphism (5). The condition is equivalent to the form B_θ be null. This implies that $\omega \in \operatorname{Im}\overline{\partial} \subset \Omega^{0,1}$, that is, $\omega = \overline{\partial} f$ for some smooth function f which is the desired solution. Conversely, if the equation has a solution, then $\overline{\partial} u = \omega$. Therefore $\omega \in \operatorname{Im}\overline{\partial}$ and $B_\theta \equiv 0 \in (\mathcal{H}^{1,0})^*$. □

2 Abel's Theorem

The proof of Abel's Theorem presented here follows [1, Subsection 12.2.1]. Some work is devoted to show the equivalence of this statement with the most common form of Abel's result. Now, note that every divisor $D \in \mathrm{Div}_0(\mathrm{X})$ can be decomposed as

$$D = \sum_i (-p_i + q_i),$$

where $p_i \neq q_i$ for each i, with possible repetitions of p_i's or q_i's in the sum. The fundamental result of this section is the following.

Theorem 4. *Let $D = \sum_i (-p_i + q_i) \in \mathrm{Div}_0(\mathrm{X})$. Then, D is a principal divisor if and only if there are paths γ_i from p_i to q_i such that*

$$\sum_i \int_{\gamma_i} \omega = 0 \tag{6}$$

for all holomorphic 1-forms ω.

Proof. Suppose the existence of paths as in the statement. We will construct smooth maps $F_i \colon X \longrightarrow \mathbb{P}^1$ with the desired local behavior. Firstly, we choose local coordinate discs (U_{p_i}, z_{p_i}), (U_{q_i}, z_{q_i}) centered at p_i and q_i, respectively. Set

$$F_i = \begin{cases} z_{q_i}, \text{ on } U_{q_i} \\ \frac{1}{z_{p_i}}, \text{ on } U_{p_i} \end{cases}$$

Applying a homotopic transformation on $\gamma_i \cap U_{p_i}$ and $\gamma_i \cap U_{q_i}$, which does not change the integral, we may assume F_i negative on these intersections. Given a cover of γ_i and considering yet the homotopical adjusts, the function F_i is extendended to a neighborhood U_{γ_i} of γ_i in such way that (see Figure 1):

$$F_i^{-1}[-\infty, 0] = \gamma_i.$$

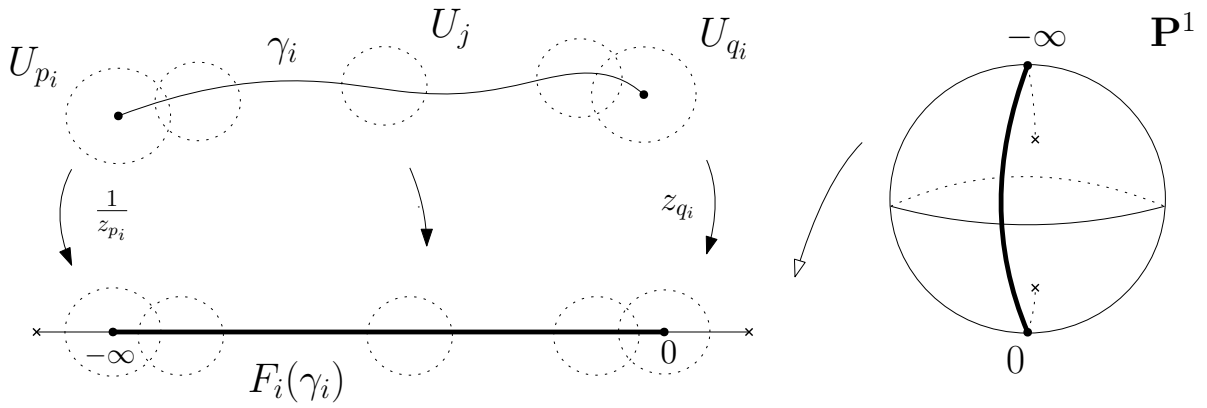


Figure 1: The function F_i on a neighbourhood of γ_i

Now, using bump functions, F_i is smoothly extended to a non-zero constant function on X. Therefore, we have the following properties for F_i:

- A meromorphic function on small discs centered at p_i and q_i;
- has a single and unique pole at p_i and a single and unique zero at q_i;
- real and negative on γ_i;
- a smooth map from X to $\mathbb{P}^1$;
- a non-zero constant outside a neighborhood $\tilde{U_{\gamma_i}}$ of γ_i.

Let $F = \prod_i F_i$. Our desired function will be of the form $e^u F$ for some smooth function u. Away from the poles, the function f shall satisfy

$$\overline{\partial}(f) = 0, \tag{7}$$

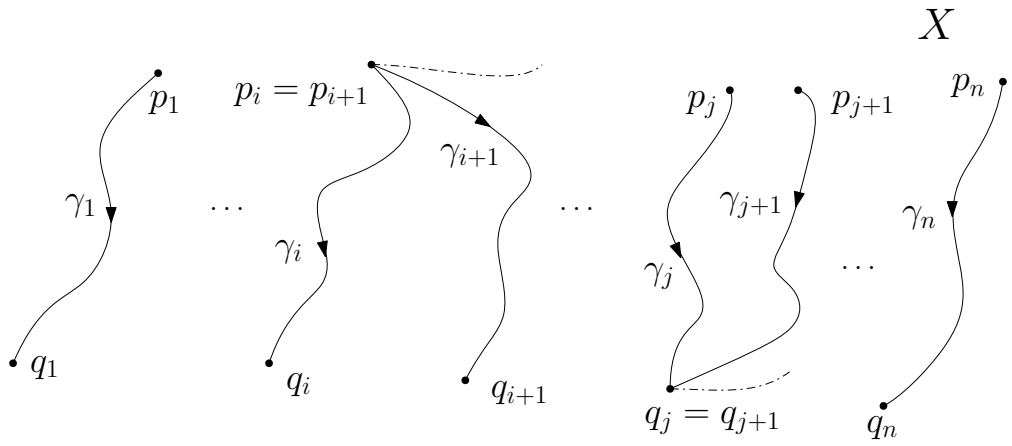


Figure 2: The general context in X for possible multiple poles and zeros

since it must be holomorphic away from these points. On the other hand,

$$\begin{aligned}\overline{\partial}(e^u F) &= e^u \overline{\partial}(u) F + \overline{\partial} F e^u \\ &= e^u \overline{\partial} u F + \sum_i \frac{\overline{\partial} F_i}{F_i} F e^u \\ &= e^u F \left(\overline{\partial} u + \sum_i \frac{\overline{\partial} F_i}{F_i} \right).\end{aligned}$$

Thus, on $X \setminus \cup_i \{p_i, q_i\}$ equation (7) is equivalent to

$$\overline{\partial}(u) = -\sum_i \chi_i, \tag{8}$$

where $\chi_i = F_i^{-1} \overline{\partial} F_i$. Since $\overline{\partial}(F_i)$ vanishes on U_{p_i} and U_{p_i}, χ_i extends to a smooth (0,1)-form on X. Before to apply Theorem 3, we shall relate the functional $\sum_i B_{\chi_i}$ with integrals over γ_i. We claim that

$$\int_X \chi_i \wedge \omega = 2\pi i \int_{\gamma_i} \omega \tag{9}$$

for each i and every holomorphic 1-form ω. For this, cut the surface along γ_i. On this cut surface X^* we can define a branch of log, which is analougous to define this function on the complex plane excluding the real negative axis. Then $\chi_i = \overline{\partial}(\log F_i)$ and

$$\int_{X^*} d(\log F_i \cdot \omega) = \int_{X^*} d(\log F_i) \wedge \omega + \int_{X^*} \log F_i \wedge d\omega.$$

Since $\log F_i$ and ω are holomorphics, $d\omega = 0$ and

$$d(\log F_i) = \overline{\partial}(\log F_i),$$

$$d(\log F_i \cdot \omega) = \overline{\partial}(\log F_i \cdot \omega).$$

By Stoke's Theorem, this implies

$$\int_{\gamma_i = \partial X^*} \log F_i \cdot \omega = \int_{X^*} \overline{\partial}(\log F_i \cdot \omega) = \int_{X^*} \chi_i \wedge \omega.$$

Recall that F_i is real and negative on γ_i, thus the function $\log F_i$ is constant and equal to $2\pi i$ along the this cut. Indeed, this can be seen in the associated Riemann surface, where the logarithm has a discontinuity of $2\pi i$ on each level. Finally,

$$\sum_i B_{\chi_i}(\omega) = \sum_i \int_{\gamma_i} \chi_i \wedge \omega = 2\pi i \sum_i \int_{\gamma_i} \omega = 0, \tag{10}$$

and the first part follows from Theorem 3.

Conversely, let $f\colon X \longrightarrow \mathbb{P}^1$ be a meromorphic function with divisor $D = \sum_i (q_i - p_i)$. Let $y \in \mathbb{P}^1$ a branch point of f. Consider the local Laurent expansion of f in a coordinate neighborhood of y. Multiplying, if necessary, the coordinate function by a constant, we may assume that there are no branch points of f on the real axis. Therefore, $f^{-1}[-\infty, 0]$ is a union of smooth paths γ_i from p_i to q_i, after a possible recombination of pairs (p_i, q_i), which corresponds only to a different decomposition of the divisor D. Consider F as constructed above. The function f/F has no zero or pole, and is smooth. Further, the exponential is surjective, thus there exists a function u satisfying

$$e^u = \frac{f}{F}.$$

For each point in X, we can write locally $u = \log(\frac{f}{F})$ for some branch of log. Hence u is a smooth fuction. Considering equations (8) and (9), the result follows applying the Theorem 3 again. □

The above theorem can be restated in the following way.

Theorem 5. *Let $D = \sum_i (-p_i + q_i) \in \mathrm{Div}_0(\mathrm{X})$. Fix a point $p_0 \in X$ such that $p_0 \neq p_i, q_i$ for all i. Then D is a principal divisor if and only if there are paths α_i connecting p_0 to p_i and paths β_i connecting p_0 to q_i such that*

$$\sum_i \left(\int_{\alpha_i} \omega + \int_{\beta_i} \omega \right) = 0 \tag{11}$$

for all holomorphic 1-form ω.

Proof. Suppose that D is a principal divisor. By Theorem 4 there are paths γ_i from p_i to q_i satisfying (6). For each i, choose paths α_i from p_0 to p_i and β_i from q_i to p_0 of the statement such that

$$\alpha_i + \beta_i + \gamma_i = \partial Y_i,$$

for some triangularizable region Y_i (see Figure 3).

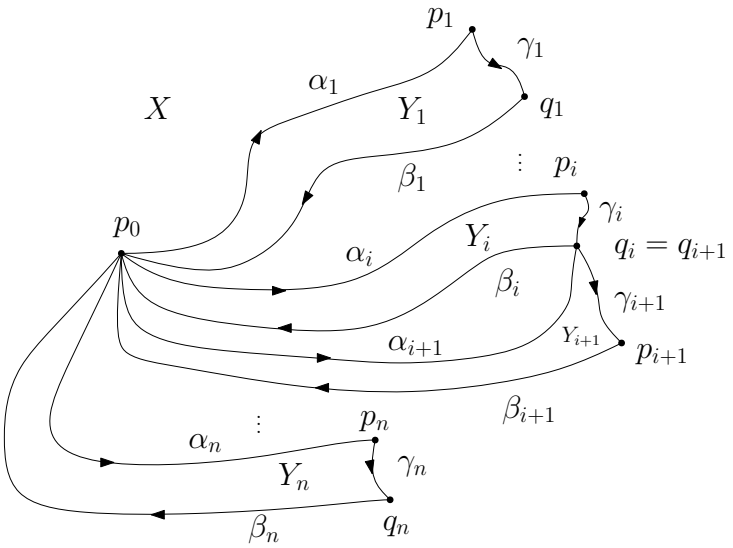


Figure 3: Regions and curves for the respective pairs

Then, for any holomorphic 1-form ω,

$$\int_{\gamma_i} \omega + \int_{\alpha_i} \omega + \int_{\beta_i} \omega = \int_{\partial Y_i} \omega = \int_{Y_i} d\omega = 0,$$

which implies

$$\sum_i \left(\int_{\alpha_i} \omega + \int_{\beta_i} \omega \right) = -\sum_i \int_{\gamma_i} \omega = 0.$$

Conversely, given the paths α_i, β_i, we may choose paths γ_i in a suitable way as above to equation (6) be satisfied. Hence, D is a principal divisor and the result follows. □

We introduce now the Jacobian of X, the Abel-Jacobi map and Abel's Theorem in its most common form. The main reference for this part is [2, Section VIII.1].

Recall that the first homology group $\mathrm{H}^1(\mathrm{X}, \mathbb{Z})$ is the quotient of the group of closed chains modulo the subgroup of boundary chains. Regarding the polygonal model for a compact Riemann surface X of genus $g > 0$, the group $\mathrm{H}^1(\mathrm{X}, \mathbb{Z})$ is freely generated by elements $a_1, \ldots, a_g, b_1, \ldots b_g$ which represent closed paths on X. Therefore

$$\mathrm{H}^1(\mathrm{X}, \mathbb{Z}) \simeq \mathbb{Z}^{2g}.$$

By Stoke's theorem, the integral of a closed 1-form ω over a boundary chain is zero. Thus, this integral only depends on the homology class. In particular, a well-defined map on $\mathcal{H}^{1,0}$ is obtained:

$$\int_{[c]} : \mathcal{H}^{1,0} \longrightarrow \mathbb{C}.$$

These linear functional are called *periods* and form a subgroup Λ in the dual space $(\mathcal{H}^{1,0})^*$.

Definition 1. *The Jacobian of X is the quoetient group*

$$\mathrm{Jac(X)} := \left(\mathcal{H}^{1,0}\right)^* / \Lambda.$$

Recall that $\mathcal{H}^{1,0}$ is a complex vector space of complex dimension g (see Theorem 2). Hereafter we fix the basis

$$\mathcal{B} = \{\omega_1, \ldots, \omega_g\}$$

for $\mathcal{H}^{1,0}$. A natural identification is

$$\begin{aligned} \mathcal{H}^{1,0^*} &\longrightarrow \mathbb{C}^g \\ \lambda &\longmapsto (\lambda(\omega_1), \ldots, \lambda(\omega_g)). \end{aligned}$$

In particular,

$$\begin{aligned} \Lambda &\longrightarrow \mathbb{C}^g \\ \int_{[c]} &\longmapsto \left(\int_{[c]}(\omega_1), \ldots, \int_{[c]}(\omega_g)\right). \end{aligned}$$

The next theorem describes the inclusion $\Lambda \subset \mathbb{C}^g$ and provides a geometric description of $\mathrm{Jac(X)}$.

Theorem 6. *The subgroup of periods Λ is a lattice in $\mathbb{C}^g$. Thefore, the Jacobian* $\mathrm{Jac(X)}$ *is a 2g-real-dimensional complex torus.*

Proof. Consider the isomorphisms in (1) and (4). The composition

$$\tilde{i} \circ \overline{\sigma} \colon\ \mathcal{H}^{1,0} \oplus \mathcal{H}^{1,0} \longrightarrow \mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{C})$$

is also an isomorphism. Taking its real part we get the identification

$$\begin{aligned} \mathcal{H}^{1,0} &\longrightarrow \mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{R}) \\ \omega &\longmapsto [\alpha + \overline{\alpha}] \end{aligned}$$

By Poincaré's duality, $\mathrm{H}^1_{\mathrm{d}}(\mathrm{X}, \mathbb{R}) \simeq \mathrm{H}_1(\mathrm{X}, \mathbb{R})$. Thus $(\mathcal{H}^{1,0})^* \simeq \mathrm{H}_1(\mathrm{X}, \mathbb{R})$. Note that if $T \colon\ \mathcal{H}^{1,0} \longrightarrow \mathbb{C}$ is a linear map, then the real part of T provides a linear map from $\mathcal{H}^{1,0}$ to $\mathbb{R}$. In particular, the action of Λ on holomorphic 1-forms becomes

$$\int_{[c]} \omega \longmapsto \int_{[c]} (\omega + \overline{\omega}) = 2 \int_{[c]} \mathrm{Re}(\omega).$$

This yields an homomorphism $\mathrm{H}_1(\mathrm{X}, \mathbb{Z}) \longrightarrow \mathrm{H}_1(\mathrm{X}, \mathbb{R})$ which is the inclusion induced by $\mathbb{Z} \subset \mathbb{R}$. Then Λ is a lattice. The complex vector space structure on $(\mathcal{H}^{1,0})^*$ induces the complex structure on $\mathrm{Jac(X)}$ and the result follows. □

Fix a base point $p_0 \in X$ and define

$$\begin{aligned} A\colon X &\longrightarrow \left(\mathcal{H}^{1,0}\right)^* \\ p &\longmapsto A(p)\colon \mathcal{H}^{1,0} \longrightarrow \mathbb{C} \\ &\qquad\qquad \omega \longmapsto \int_\gamma \omega \end{aligned}$$

for some path γ_p from p_0 to p. If γ_p' is another path connecting p_0 to p, the functional $A(p)$ changes by an integration of ω over the closed chain $\gamma_p - \gamma_p'$. Thus, we get a well-defined map

$$\begin{aligned} A : X &\longrightarrow \mathrm{Jac(X)} \\ p &\longmapsto A(p) = \Big(\int_{\gamma_p} \omega_1, \ldots, \int_{\gamma_p} \omega_g\Big) \mod \Lambda \end{aligned}$$

This map is called the *Abel-Jacobi map*. An extension to divisors, also denoted by A, is the following:

$$\begin{aligned} A : \mathrm{Div(X)} &\longrightarrow \mathrm{Jac(X)} \\ \sum_p n_p p &\longmapsto \sum_p n_p A(p) \end{aligned}$$

Note that A is a homomorphism of abelian groups and depends on the base point. However, the restriction

$$A_0 : \mathrm{Div_0(X)} \longrightarrow \mathrm{Jac(X)}$$

is independent of this choice.

Lemma 1 ([2], Lema VIII.2.1)**.** *The Abel-Jacobi map A_0 is independent of the chosen base point p_0 on X.*

Proof. For two choices of base points p_0, p_0', the functional $A(p)$ differs by

$$j = \left(\int_\gamma \omega_1, \cdots \int_\gamma \omega_g\right) \mod \Lambda,$$

where γ is a path connecting the points. If $D = \sum_p n_p p$ satisfies $\sum_p n_p = 0$, then $A_0(D)$ changes by $\sum_p n_p j = j \sum_p n_p = 0$, since A_0 is a homomorphism of abelian groups. □

We are now in conditions to prove Abel's theorem in your most common form.

Theorem 7 (Abel - [2], Theorem VIII.2.2)**.** *Let $D = \sum_i (-p_i + q_i) \in \mathrm{Div_0(X)} \in \mathrm{Div_0(X)}$. Then D is a principal divisor if and only if*

$$A_0(D) = 0 \mod \Lambda.$$

Proof. We shall see that the statement is equivalent to Theorems 4 and 5. If D is principal, for each pair (p_i, q_i), we may choose p_i as the base point for $A_0(D_i)$, where $D_i = -p_i + q_i$, and the predicted paths γ_i of Theorem 4. Thus, for any holomorphic 1-form ω, it holds that

$$\begin{aligned} A_0(D)(\omega) &= \sum_i A_0(D_i)(\omega) \\ &= \sum_i \left(-A_0(p_i) + A_0(q_i)\right)(\omega) \\ &= \sum_i \int_{\gamma_i} \omega \\ &= 0. \end{aligned}$$

Conversely, if $A_0(D) = 0 \in \mathrm{Jac(X)}$ with base point p_0, then $A_0(D) = \int_{[c]}$ for some $[c] \in \mathrm{H}^1(\mathrm{X}, \mathbb{Z})$. Choose paths α_i from p_i to p_0 and β_i from p_0 to q_i and α connecting p_1 to a point $q \in c$. Set $\hat{\alpha_1}$ as the path $\alpha_1 + \alpha + c - \alpha$.

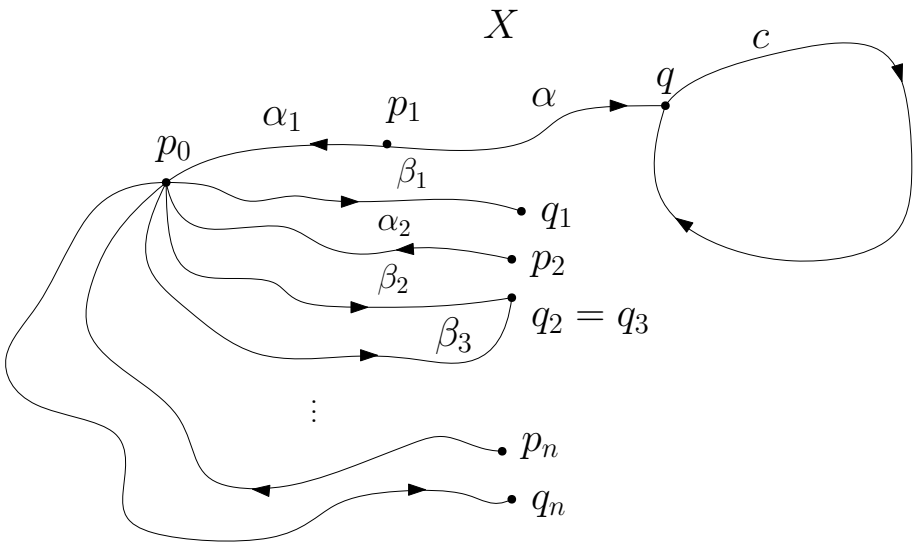


Figure 4: Paths connecting p_0 and the points p_i,q_i

Therefore, considering the independence of paths, for any holomorphic 1-form ω we have

$$\begin{aligned} A_0(D)(\omega) &= \sum_{i\geq 2} \left(\int_{\alpha_i} \omega + \int_{\beta_i} \omega \right) + \int_{\hat{\alpha_1}} \omega \\ &= \sum_i \left(\int_{\alpha_i} \omega + \int_{\beta_i} \omega \right) + \int_c \omega \\ &= \int_c \omega, \end{aligned}$$

which implies $\sum_i \left(\int_{\alpha_i} \omega + \int_{\beta_i} \omega \right) = 0$. This is the condition in Theorem 5 to D be a principal divisor. □

Corollary 1 ([4], Corollary III.6.4.2)**.** *Let X be a compact Riemann surface with genus $g > 1$. Then $A\colon X \longrightarrow \mathrm{Jac(X)}$ is an injective mapping of X onto a proper submanifold $A(X) \subset \mathrm{Jac(X)}$.*

Proof. Let $p, q \in X$ distinct points. If $A(p) = A(q)$, then $A_0(D) = 0$, where $D = -p+q$. By Abel's Theorem, D is a principal divisor for some function f. In particular, f has a single pole at p. Since $g > 1$, this is a contradiction and the map is injective. □

3 The Jacobi Inversion Problem

By Abel's Theorem, the map A_0 can be reduced to

$$\hat{A}_0\colon \mathrm{Pic_0(X)} \longrightarrow \mathrm{Jac(X)}$$

where $\mathrm{Pic_0(X)} := \mathrm{Div_0(X)} \,/\, \mathrm{PDiv(X)}$ is called the *Picard group*. The Jacobi inversion problem asserts that $\hat{A}_0$ is surjective and hence a new characterization of Jac(X) is obtained. The main reference for this section is [3, Section 21].

Lemma 2 ([3], Lemma 21.3)**.** *There exist g distinct points $p_1, \ldots, p_g \in X$ with the following property: Every holomorphic 1-form $\omega \in \mathcal{H}^{1,0}$ which vanishes at all the points $p_1, \ldots, p_g$ is identically zero.*

Proof. For $p \in X$, let

$$\mathcal{H}_p = \{\omega \in \mathcal{H}^{1,0} : \omega(p) = 0\}.$$

Then $\mathcal{H}_p = L^{(1)}(p)$. If K is a canonical divisor, it holds that $L^{(1)}(p) \simeq L(K+p)$. Further, $\mathcal{H}^{1,0} = L^{(1)}(0) \simeq L(K)$. By [2, Lemma 3.15], it follows that $\mathcal{H}_p = \mathcal{H}^{1,0}$ or has codimension one.

Let $(p_n) \subset X$ an infinite and non-constant sequence of points in X. Since X is compact, the set of zeros of a holomorphic 1-form is finite. Thus $\bigcap_i \mathcal{H}_{p_i} = \emptyset$. Moreover, the dimension of $\mathcal{H}^{1,0}$ is g. Therefore, there exist g points $p_1, \ldots, p_g$ (after a possible reindexation) such that $\bigcap_{i=1}^{g} \mathcal{H}_{p_i} = \emptyset$. These points satisfy the desired properties of the statement. □

Theorem 8 (Abel-Jacobi - [3], Theorem 21.7)**.** *The map*

$$\hat{A}_0\colon \mathrm{Pic_0(X)} \longrightarrow \mathrm{Jac(X)}$$

is an isomorphism.

Proof. By Abel's Theorem we only need to check that the map is surjective. Choose points $p_1, \ldots, p_g$ as in Lemma 2 and coordinate neighborhoods (U_j, z_j) centered at p_j, for $j = 1, \ldots, g$. Define a map

$$F\colon U_1 \times \cdots \times U_g \longrightarrow \mathbb{C}^g$$

in the following way. If $z = (z_1, \ldots, z_g) \in U_1 \times \ldots U_g$, then

$$F(z) = (F_1(z), \ldots, F_g(z)),$$

where

$$F_i(z) = \sum_{j=1}^{g} \int_{p_j}^{z_j} \omega_i.$$

Since U_j is simply connected, the integral $\int_{p_j}^{z_j} \omega_i$ is well-defined. Note also that $F(p) = 0$. Let $\omega_i = \varphi_{ij} dz_j$ on U_j. Then F is smooth and has Jacobian matrix

$$J_F(p) = \left(\frac{\partial F_i}{\partial z_j} \right) = (\varphi_{ij}(p_j)).$$

Consider now the linear map

$$\begin{aligned} T\colon \mathcal{H}^{1,0} &\longrightarrow \mathbb{C}^g \\ \omega &\longmapsto \left(\frac{\omega}{dz_1}(p_1), \ldots, \frac{\omega}{dz_g}(p_g) \right) \end{aligned}$$

where $\frac{\omega}{dz_j}$ are the holomorphic functions which define ω on U_j. By Lemma 2, $\mathrm{Ker}(T) = 0$ and T is an isomorphism. Moreover, in the basis $\mathcal{B}$, the matrix of T is

$$\left(\frac{\omega_i}{dz_j}(p_j) \right) = (\varphi_{ij}).$$

Therefore, $J_F(p)$ is invertible and we can apply the Inverse Function Theorem for F: There are $V \subset U_1 \times \cdots \times U_g$ and $W \subset \mathbb{C}^g$ containing the origin such that $F\colon V \longrightarrow W$ is a diffeomorphism.

Let $[\hat{y}] \in \mathrm{Jac(X)}$ represented by $\hat{y} \in \mathbb{C}^g$. For sufficient large $n \in \mathbb{N}$, $y := \frac{\hat{y}}{n} \in W$. Then there exist $q = (q_1, \ldots, q_j) \in V$ such that $F(q) = y$. In particular, if y_k is the k-th coordinate of y, then

$$y_k = \sum_j \int_{p_j}^{q_j} \omega_k. \tag{12}$$

Choose a base point $p_0 \neq p_j, q_j$ in X and paths γ_j, from p_j to q_j and paths α_j, β_j, connecting p_0, p_j and p_0, q_j, respectively. As in Theorem 5, this choice can be made in such way that $\alpha_j + \gamma_j - \beta_j$ is a boundary chain and by Stoke's theorem

$$\int_{\gamma_j} \omega_k = \int_{\beta_j} \omega_k - \int_{\alpha_j} \omega_k.$$

Substituting in (12), we get

$$y_k = \sum_{i=1}^{g} \left(-A(p_j) + A(q_j) \right)(\omega_k).$$

Finally, the class of $n.D = n\sum_j (p_j - q_j)$ in $\mathrm{Pic}_0(\mathrm{X})$ satisfies

$$A_0([n.D]) = n.A_0[D] = [\hat{y}]$$

as required. □

A second space associated to a compact Riemann surface of genus g is the g-direct product

$$X^g := X \times \cdots \times X.$$

Choose g arbitrary points $p_1, \ldots, p_g \in X$ and define a map

$$\psi \colon X^g \longrightarrow \mathrm{Pic}_0(\mathrm{X})$$

in the following way. For $(x_1, \ldots, x_g) \in X^g$, set

$$\psi(x_1, \ldots, x_g) = \sum_{j=1}^{g} (x_j - p_j).$$

Consider the composition $J := A_0 \circ \psi$:

$$(x_1, \ldots, x_g) \longmapsto \sum_{j=1}^{g} (x_j - p_j) \longmapsto \sum_{j=1}^{g} A_0(x_j - p_j) \mod \Lambda.$$

The next theorem contrasts with the Corollary 1.

Theorem 9 ([3], Theorem 21.9)**.** *With the notation as above, the map*

$$J \colon X^g \longrightarrow \mathrm{Jac}(\mathrm{X})$$

is surjective.

Proof. It suffices to show that $\psi \colon X^g \longrightarrow \mathrm{Pic}_0(\mathrm{X})$ is surjective. Given $[D] \in \mathrm{Pic}_0(\mathrm{X})$, set

$$D_1 = D + p_1 + \cdots + p_g.$$

Since $\deg(D_1) = g$, by Riemann-Roch Theorem, it holds that

$$\dim(L(D_1)) - \dim(K - D_1) = 1,$$

which implies $\dim(L(D_1)) \geq 1$. Thus there exists a non-zero meromorphic function f on X such that $\mathrm{div}(f) + D_1 \geq 0$. Let $D_2 = \mathrm{div}(f) + D_1$. We may write

$$D_2 = x_1 + \ldots x_g,$$

for some points x_i with possible repetition. Therefore

$$\sum_i (x_i - p_i) = D + \mathrm{div}(f).$$

That is,

$$\psi(x_1, \ldots, x_g) = [D] \in \mathrm{Pic}_0(\mathrm{X}),$$

as desired. □

4 Curves of Genus One

In this last section we deal with the case $g = 1$, where Abel's Theorem shows that such surfaces are isomorphic to the complex torus. In particular, they have a structure of abelian group which will be described by the Abel-Jacobi map and has an interesting geometric description. This section follows [2, Section VIII.5].

If $g = 1$, the Jacobian Jac(X) is a complex torus and the same argument in the proof of Corollary 1 shows that $A\colon X \longrightarrow \mathrm{Jac(X)}$ is injective. On the other and, locally A is of the form

$$p \longmapsto \int_{p_o}^{p} \omega,$$

where $\omega \in \mathcal{H}^{1,0}$, then A is a holomorphic map between two Riemann surfaces. We have shown

Theorem 10 ([2], Proposition VIII.5.2)**.** *Let X be a compact Riemann surface with genus one. Then* $\mathrm{Jac(X)} \simeq X$*. In other words, X is isomorphic to a complex torus.*

Identifying $\mathrm{Jac(X)} \simeq X$, the map A on divisors is simply

$$\begin{aligned} A\colon \mathrm{Div(X)} &\longrightarrow \mathrm{Jac(X)} \simeq X \\ \sum_p n_p p &\longmapsto \sum_p n_p . p \end{aligned}$$

that is, A sends a formal sum to the sum in the group law of X. The neutral element (also called the origin) of this sum can be arbitrarily chosen, since it is the base point of A.

By Plücker's formula, a smooth projective plane cubic is isomorphic to a complex torus, so is an abelian group. Let p_0 be the origin. This point corresponds to $0 \in \mathrm{Jac(X)}$. We shall see that this choice determines essentially the group law.

Consider $H(X, Y, Z) = aX + bY + cZ$ a homogeneous polynomial. The set $H = 0$ determines a plane in $\mathbb{C}^3$, so is a line in $\mathbb{P}^2$. Recall that, informally, the hyperplane divisor $\mathrm{div}(H)$, which is non-negative, counts the intersections of the line determined by H with X. Since X is cubic, by Bezout's theorem,

$$\deg(\mathrm{div}(H)) = 3.$$

Moreover, we may write

$$\mathrm{div}(H) = p_1 + p_2 + p_3.$$

Definition 2. *Let* $\mathrm{div}(H) = p_1 + p_2 + p_3$ *a hyperplane divisor and L_H the line determined by $H = 0$.*

1. *If the p_i's are distinct, L_H is called transverse to X. Otherwise, this line is tangent to X.*

2. *If all three points are equal, L_H is called a flexed tangent to X and the point is a flex point of X.*

Define yet

$$\xi(H) = A(\mathrm{div}(H)) = p_1 + p_2 + p_3 \in X.$$

The next Lemma justifies the nomenclature.

Lemma 3. *A line L_H determined by the homogeneous polynomial H is tangent to X at p if and only if the intersection divisor satisfies*

$$\mathrm{div}(H) \geq 2p.$$

Proof. This is a particular case of [2, Lemma VII.3.4]. □

Thus the tangent line L_H is tangent to X at the double, or triple, point in the formal sum $p_1 + p_2 + p_3$.

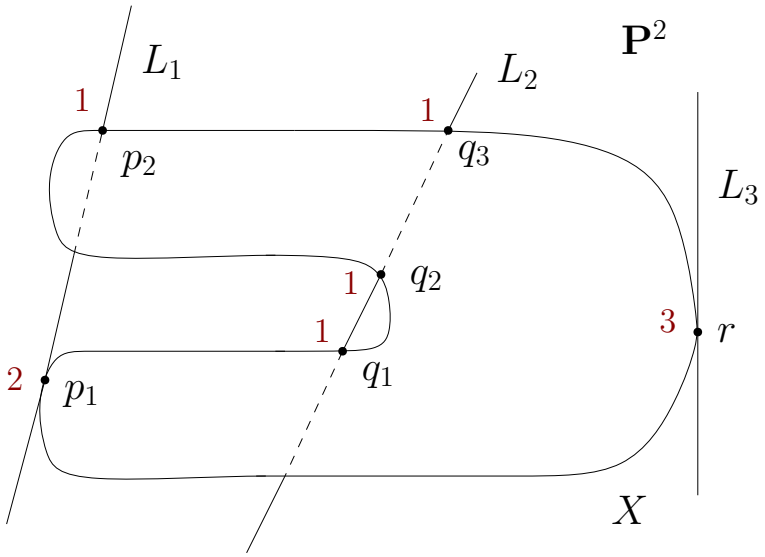


Figure 5: The lines L_1 and L_3 are tangents, L_2 is transverse. Numbers in red denote multiplicity

Lemma 4. *If H' is any other linear homogeneous polynomial, then $\xi(H) = \xi(H')$.*

Proof. By [2, Lemma 2.3, V], $\mathrm{div}(H)$ and $\mathrm{div}(H')$ are linearly equivalent, that is, $\mathrm{div}(H) - \mathrm{div}(H') = \mathrm{div}(f)$ for some meromorphic function f. Then

$$A(\mathrm{div}(H)) - A(\mathrm{div}(H')) = A(\mathrm{div}(f)) = p_0 \in X,$$

by Abel's theorem. Which implies $\xi(H) = \xi(H')$ as points of X. □

This point is denoted only by ξ and it is called the *collinearity point* of the plane cubic X. A characterization of collinarity in terms of divisors is:

Three points in X are collinear if they form the intersection divisor of a line with X.

Flex points are of special interest and they exist in a finite number.

Lemma 5 ([2], Lemma VIII.5.3)**.** *There are exactly 9 flex points on a smooth projective plane cubic curve X, given by the nine solutions to the equation $3p = \xi$ in the group law of X.*

Proof. We may suppose $X \simeq \mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$. If

$$p = \alpha_1 + \beta_1\tau \text{ and } \xi = \alpha_2 + \beta_2\tau$$

the equation can be rewritten as

$$3\alpha_1 - \alpha_2 + (3\beta_1 - \beta_2)\tau \in \Gamma,$$

where $\alpha_i, \beta_i \in [0, 1]$. Hence

$$\alpha_1 = \frac{n + \alpha_2}{3} \text{ and } \beta_1 = \frac{m + \beta_2}{3},$$

for $m, n \in \mathbb{Z}$. Each equality above has 3 solutions, then the equation has 9 solutions. Now, the tangent line L at p has intersection divisor of the form $p + p + q$, for some $q \in X$. Then, if p satisfies the equation,

$$2p + q = \xi = 3p,$$

which implies $p = q$, so p is a flex point. Conversely, every flex point p form a hyperplane divisor $\mathrm{div}(H)$ and by Lemma 4, $3p = \xi(H) = \xi$.

□

Proposition 1. *The collinearity point ξ is the third point of intersection of the tangent line at the origin p_0 with the cubic X.*

Proof. The intersection divisor of the tangent line at p_0 is of the form $p_0 + p_0 + q$ for some $q \in X$. Moreover, $2p_0 + q = \xi$. Since $p_0 = 0$ in the group law, $q = \xi$, as stated. □

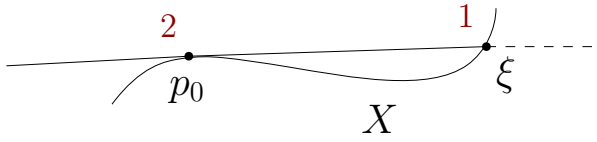


Figure 6: Geometric characterization of the collinearity point

When the origin p_0 is a flex point, we obtain a complete geometric characterization of the group law.

Proposition 2 ([2], Proposition VIII.5.4)**.** *Let X be a smooth projective plane cubic curve whose origin p_0 is a flex point.*

1. *Three points are collinear if and only if they sum to zero in the group law.*
2. *The inverse $-p$ of a point p is the third point of intersection of the line joining p_0 and p with X.*
3. *The sum $p+q$ is the inverse of the third point of intersection of the line joining p and q with X.*
4. *The point $2p$ is the inverse of the third point of intersection of the tangent line to X at p with X.*

Proof.

1. The origin p_0 satisfies $3p_0 = \xi$, then $\xi = p_0$. But three points are collinear if and only if they sum to $\xi = 0$ in the group law.
2. Note that $p + q = 0$ if and only if $p + q + p_0 = 0$, since $p_0 = 0$. Rewritting,
$$p + q + p_0 = \xi$$
which occurs if and only if p, q, and p_0 are collinear.
3. A point r is the inverse of $p + q$ if and only if $p + q + r = \xi = 0$. This is also the condition to the points be collinear.
4. The equation $2p + r = \xi = 0$ implies p and r collinear and the line tangent to p (see Lemma 3), as desired.

□

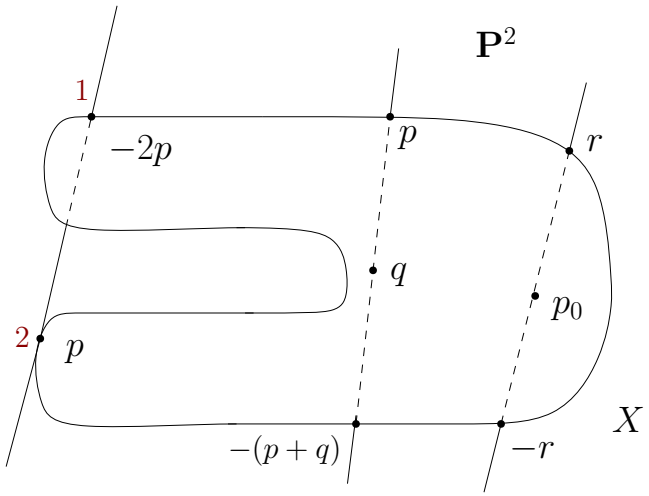


Figure 7: Geometric characterization of the group law

Indeed, one can construct a group structure for a smooth projective plane cubic directly in a more complicated way, whithout the use of Abel-Jacobi map [10].

5 Further Remarks

To conclude, we list briefly some additional remarks on the subject.

- Starting only with an exact sequence of sheaves, one can obtain an exact sequence of cohomology groups from which the statement of Abel's Theorem is obtained. The Jacobian turns out to be isomorphic to the quotient where $\mathcal{O}_X$ is the sheave of holomorphic functions and $\mathbb{Z}$-constant functions [2, p.319].
- Using Abel-Jacobi map, holomorphic line bundles are described in terms of the quotient Div(X) / PDiv(X), which is also isomorphic to Jac(X) $\times\mathbb{Z}$ [2, p.357].
- The theory also develops to consider an induced holomorphic map $\hat{J}$ from $\mathrm{Sym}^d(X)$ to Jac(X), where $\mathrm{Sym}^d(X)$ is the quotient of X^g by the action of the permutation group [1, Subsection 12.2.2].
- Abel's Theorem has a generalization to projective algebraic varieties [6].
- The original problem solved by Abel was related to elliptic integrals. We refer to [8] for a description and development of Abel's work.

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