

Arc Length Formulas

The length of a curve is

$$L = \int_a^b ds.$$

← posted on website

The form of ds depends on how the curve is given.

If the curve is given parametrically:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If the curve is given as $y = f(x)$:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

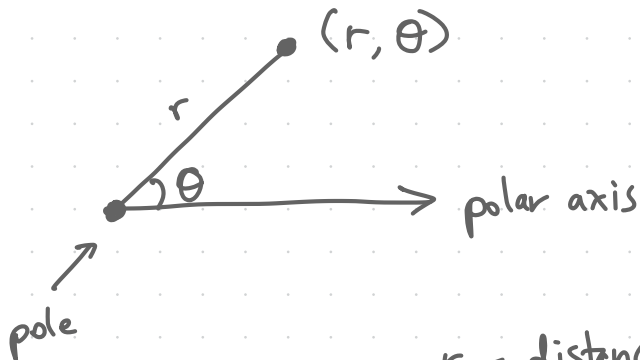
$$\frac{dy}{dx} = f'(x)$$

If the curve is given as $x = g(y)$:

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Appendix H.1 - Polar Coordinates

The location of points in the plane is described in reference to the origin (AKA the pole)



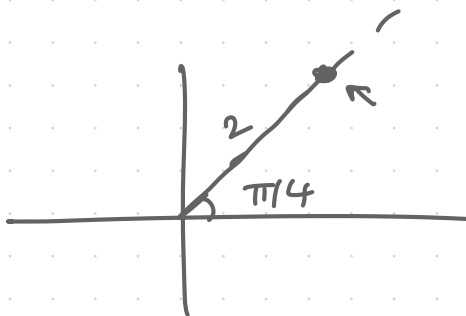
r - distance to origin

θ - angle in radians measured from the polar axis

(standard xy -coordinates are called Cartesian coordinates)

examples:

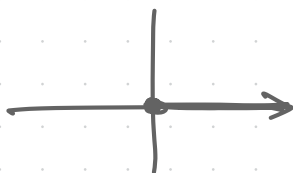
① $(r, \theta) = (2, \pi/4)$



② $(r, \theta) = (2, 9\pi/4)$

same point

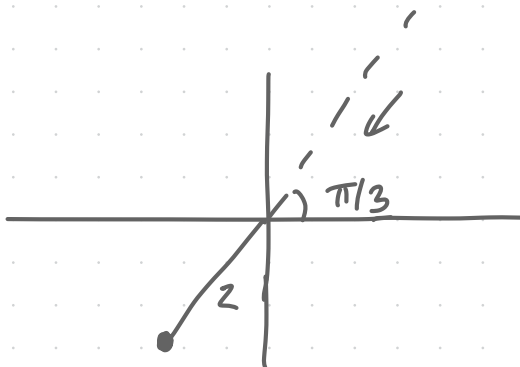
③ $(r, \theta) = (0, 3\pi)$



if $r=0$, θ anything,
get the origin

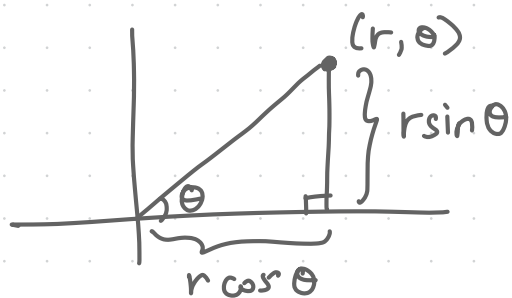
④ $(r, \theta) = (-2, \pi/3)$

r can be negative



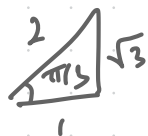
Unlike Cartesian coordinates, each point does not have a unique representation in polar coordinates

Relationship between polar & Cartesian



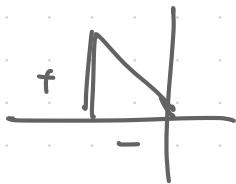
$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

ex: Convert $(r, \theta) = (-4, 2\pi/3)$ into Cartesian



$$x = -4 \cos(2\pi/3) = -4 \left(-\frac{1}{2}\right) = 2$$

$$y = -4 \sin(2\pi/3) = -4 \left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$



$$(x, y) = (2, -2\sqrt{3})$$

To go from Cartesian to polar:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\begin{aligned} \rightarrow x^2 + y^2 &= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\ &= r^2 (\cos^2 \theta + \sin^2 \theta) \end{aligned}$$

$$x^2 + y^2 = r^2$$

$$\rightarrow r = \sqrt{x^2 + y^2}$$

Also $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{r \sin \theta}{r \cos \theta} = \frac{y}{x}$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

BUT $\arctan()$ returns values in

$$-\frac{\pi}{2} \text{ to } \frac{\pi}{2}$$

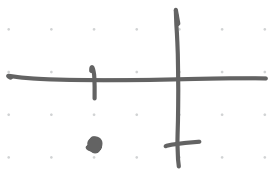


$$\theta = \arctan\left(\frac{y}{x}\right) \text{ or } \arctan\left(\frac{y}{x}\right) + \pi$$

compute $\arctan\left(\frac{y}{x}\right)$, then
check quadrant is correct.

If not, add π .

ex: $(x, y) = (-1, -1)$



$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1 + 1} = \sqrt{2} \end{aligned}$$

$$\arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-1}{-1}\right)$$

$$= \arctan(1) = \frac{\pi}{4}$$

↑
wrong
quadrant!

$$\theta = \frac{\pi}{4} + \pi$$

$$= \frac{5\pi}{4}$$

$$(r, \theta) = (\sqrt{2}, 5\pi/4)$$

A polar equation is an equation of the form $F(r, \theta) = 0$.

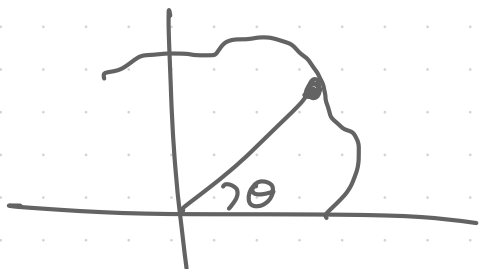
E.g. $r - 2\cos\theta = 0$

$$r^2 \cdot \theta = 0$$

$$r = 1 + \sin\theta \Rightarrow r - 1 - \sin\theta = 0$$

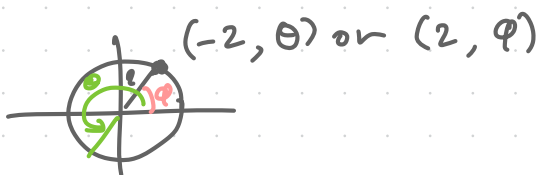
often polar eq's are written

$$r = f(\theta)$$

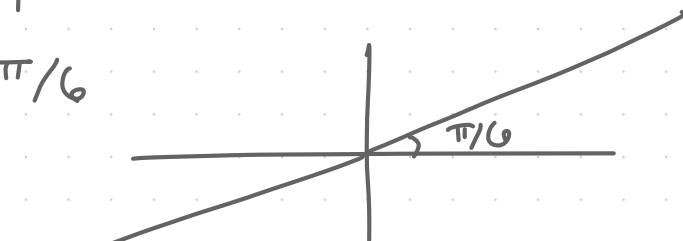


The graph of a polar equation
the set of points that have some
polar representation that satisfies
the equation

ex: $r = 2$ $\rightarrow$ graph



ex: $\theta = \pi/6$

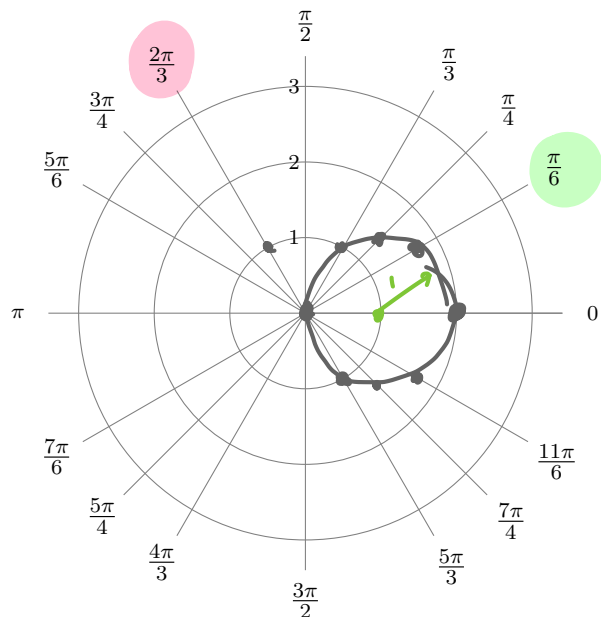


↑ pts here could be represented
 $(-r, \pi/6)$

H.1 Curves in Polar Coordinates - Examples

Example 1: $r = 2 \cos \theta$

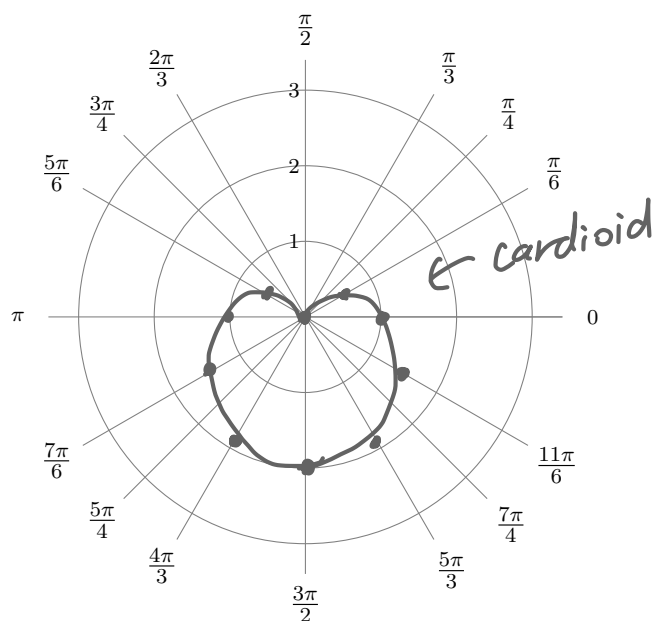
θ	r
0	$2 \cos(0) = 2$
$\frac{\pi}{6}$	$2 \left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \approx 1.7$
$\frac{\pi}{4}$	$2 \left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.4$
$\frac{\pi}{3}$	$2 \left(\frac{1}{2}\right) = 1$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	-1
$\frac{3\pi}{4}$	$-\sqrt{2}$
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	-2



can use $x = r \cos \theta$
 $y = r \sin \theta$ to see that this is a circle

Example 2: $r = 1 - \sin \theta$

θ	r
0	1
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	1
$\frac{7\pi}{6}$	$\frac{3}{2}$
$\frac{4\pi}{3}$	$1 - \left(\frac{-\sqrt{3}}{2}\right) \approx 1.9$
$\frac{3\pi}{2}$	2
$\frac{5\pi}{3}$	$1 + \left(\frac{\sqrt{3}}{2}\right) \approx 1.9$
$\frac{11\pi}{6}$	$\frac{3}{2}$



example 1: is it a circle?

← multiply by r

$$r = 2 \cos \theta$$

$$r^2 = 2(r \cos \theta)$$

$$x^2 + y^2 = 2x$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

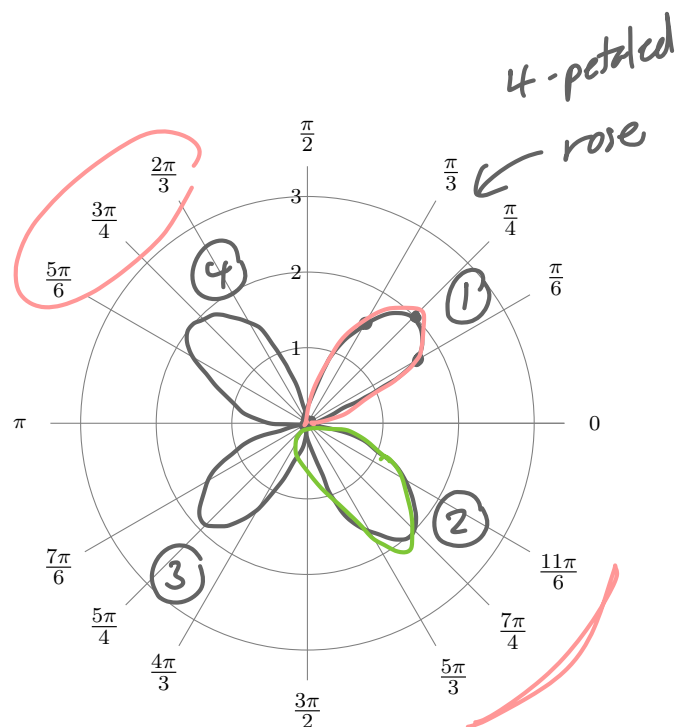
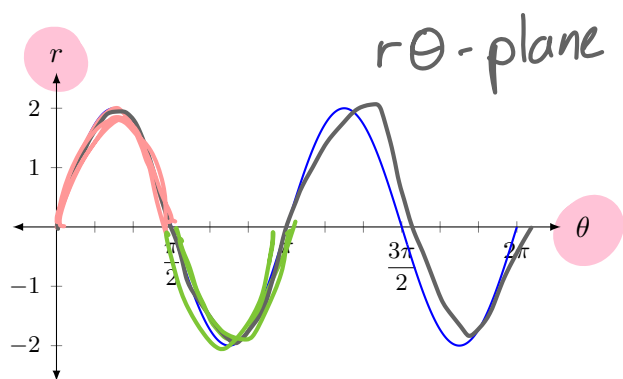
$$x^2 + y^2 = r^2$$

$$x^2 - 2x + 1 + y^2 = 0 + 1$$

$$(x-1)^2 + y^2 = 1$$

← complete the square

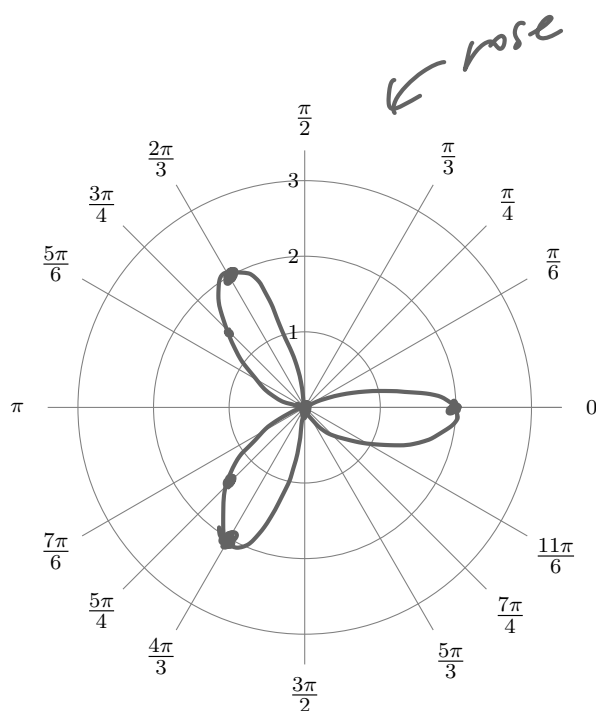
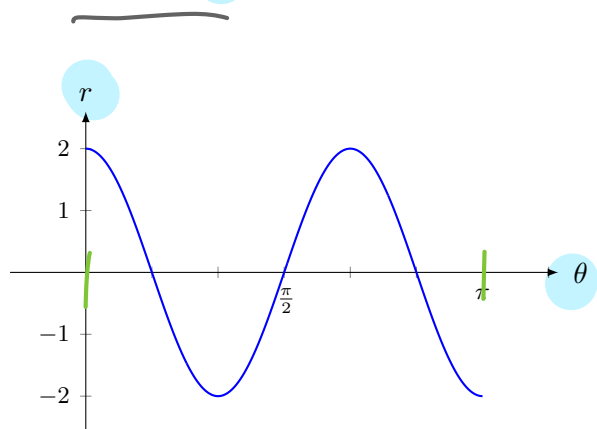
Example 3: $r = 2\sin(2\theta)$



θ	r	θ	r
0	0	π	0
$\frac{\pi}{6}$	$\sqrt{3}$	$\frac{7\pi}{6}$	$-\sqrt{3}$
$\frac{\pi}{4}$	2	$\frac{5\pi}{4}$	-2
$\frac{\pi}{3}$	$\sqrt{3}$	$\frac{4\pi}{3}$	$-\sqrt{3}$
$\frac{\pi}{2}$	0	$\frac{3\pi}{2}$	0
$\frac{2\pi}{3}$	$-\sqrt{3}$	$\frac{5\pi}{3}$	$\sqrt{3}$
$\frac{3\pi}{4}$	-2	$\frac{7\pi}{4}$	2
$\frac{5\pi}{6}$	$-\sqrt{3}$	$\frac{11\pi}{6}$	$\sqrt{3}$

use symmetry
to fill in the table

Example 4: $r = 2 \cos(3\theta)$



θ	r
0	2
$\frac{\pi}{6}$	0
$\frac{\pi}{4}$	$-\sqrt{2} \approx -1.4$
$\frac{\pi}{3}$	-2
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	2
$\frac{3\pi}{4}$	$\sqrt{2}$
$\frac{5\pi}{6}$	0
π	-2

$$2 \cos\left(3 \cdot \frac{\pi}{4}\right)$$

$$\begin{array}{c|c} - & + \\ \hline + & + \end{array}$$

using desmos to plot polar eq's

$r =$ _____