

Stochastic Processes

Change of Measure and Radon-Nikodým Derivative

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1 Introduction

In financial models, a sample space Ω representing all possible future states of the financial markets is first set up. Naturally, the set of possible scenarios for the future is associated with a probability measure $\mathbb{P}$ corresponding to the real world, which means it can be sought by empirical estimation of the model parameters and thus is referred to as the actual probability measure. However, when it comes to pricing derivative securities, the actual probability measure $\mathbb{P}$ has its own disadvantages. Hence, another measure $\tilde{\mathbb{P}}$, referred to as the risk-neutral probability measure, is introduced for its neat summary of the result of solving systems of equations occurring in determining the price of a derivative security.

The importance of these two probability measures in financial models merits our attention. In this report, we will inspect the mathematical foundations of these two probability measures, that is, change of measure and Radon-Nikodým theorem, and apply those theoretical concepts to clarify our understanding of these two relevant probability measures.

2 Change of Measure

When we discuss change of measure, it is natural to ask ourselves first how to change a probability measure to another one. One of the easiest ways to do that is by means of a random variable Z , in which we weight the original probability measure $\mathbb{P}$ in the sample space Ω according to Z . Nonetheless, since the changed measure is required to remain a probability measure, there are some constraints on the random variable Z . The following theorem verifies our consideration.

Theorem 2.1. Let $(\Omega, F, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative random variable with $\mathbb{E}Z = 1$. For $A \in F$, define

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a non-negative random variable, then

$$\tilde{\mathbb{E}}X = \mathbb{E}XZ.$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}Y = \tilde{\mathbb{E}}\frac{Y}{Z}$$

for every non-negative random variable Y

Remark:

1. $Z \geq 0$ *a.s.* guarantees $\tilde{\mathbb{P}} \geq 0$.
2. $\mathbb{E}Z = 1$ guarantees $\tilde{\mathbb{P}}(\Omega) = 1$.
3. $\tilde{\mathbb{P}}$ is countably additive by MCT and thus is a probability measure.
4. Z tells us where we should revise the probability upward ($Z > 1$) and where we should revise the probability downward ($Z < 1$).
5. The weight process through Z is set-by-set rather than ω -by- ω .

3 Radon-Nikodým Theorem

Definition 3.1. Let Ω be a non-empty set and F a σ -algebra of subsets of Ω . Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on (Ω, F) are said to be equivalent if they agree which sets in F have probability zero.

Remark:

1. Under the assumptions of Theorem 2.1 with $Z > 0$ *a.s.* included, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent.
2. Because $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent, there is no need to specify which measure we refer to when we say an event occurs almost surely.
3. It is mandatory that the actual probability measure is equivalent to the risk-neutral probability measure. They may disagree on how probable the possibilities are, but they must agree on what is possible and what is impossible. Thus, we are allowed to set up hedges that work with $\tilde{\mathbb{P}}$ -probability one. These hedges then also work with $\mathbb{P}$ -probability one.

Theorem 3.1 (Radon-Nikodým Theorem). Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be equivalent probability measures defined on (Ω, F) . Then there exists an almost surely positive random variable Z such that $\mathbb{E}Z = 1$ and

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for every } A \in F.$$

The random variable Z is defined as the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ and denoted by

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

Remark:

1. The Radon-Nikodým theorem tells that the relationship between two equivalent probability measures is connected through an almost surely positive random variable Z . The result holds between the actual probability measure and the risk-neutral probability measure.
2. We have known how to obtain an equivalent probability measure through an almost surely positive random variable Z . This theorem conforms that it is the only way to obtain two equivalent probability measures. Hence, it provides the sufficient and necessary condition for two equivalent probability measures.
3. The random variable Z is unique in the sense that if Z_1 and Z_2 satisfy the theorem, then $Z_1 = Z_2$ *a.s.*.

Proof. Let $A = \{Z_1 > Z_2\}$. Then

$$\int_A [Z_1(\omega) - Z_2(\omega)] d\mathbb{P}(\omega) = \tilde{\mathbb{P}}(A) - \tilde{\mathbb{P}}(A) = 0$$

Since the integrand is strictly positive on A , $\mathbb{P}(\{Z_1 > Z_2\}) = 0$ for the equation to hold. Similarly, $\mathbb{P}(\{Z_1 < Z_2\}) = 0$. Hence, $Z_1 = Z_2$ *a.s.*.

References

- [1] Steven Shreve: Stochastic Calculus for Finance I
- [2] Steven Shreve: Stochastic Calculus for Finance II