

p-series

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Recall: $\int_1^{\infty} \frac{1}{x^p} dx$

converges if $p > 1$
diverges if $p \leq 1$

The Int. Test tells us that p-series follow the same rules:

p-series Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \leq 1$

cvgs if $p > 1$
diverges if $p < 1$

If $p = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$
which div.

ex: $\sum_{n=1}^{\infty} \frac{n}{n^{5/2}}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$p = \frac{3}{2} > 1$
converges

ex: $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

$p = \frac{1}{3} < 1$
diverges

$$\left[\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n+5}} \right]$$

} similar to a
p-series
 $\Rightarrow$ compare to a
p-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n}} = \sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

$p = \frac{1}{3}$

The (Direct) Comparison Test (for series)

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ converges and $a_n \leq b_n$ for all n , then $\sum a_n$ also converges.
- If $\sum b_n$ diverges and $a_n \geq b_n$ for all n , then $\sum a_n$ also diverges.

To use: Given $\sum a_n$, you pick $\sum b_n$.

Good series to compare:

- geometric series
- p-series

Note: you can only use Comparison Test on series with positive terms!

Check that $a_n, b_n \geq 0$ for all n .

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(detailed solution)

$$a_n = \frac{1}{2^n + n} \geq 0 \quad \checkmark \quad \text{for all } n$$

Choose $b_n = \frac{1}{2^n}$. Then $b_n \geq 0$ $\checkmark$
for all n .

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{converges,}$$

$r = \frac{1}{2}$ so $|r| < 1$

Want: $a_n \stackrel{?}{\leq} b_n \quad \checkmark$

$$\frac{1}{2^n + n} \leq \frac{1}{2^n}$$

$$\cancel{2^n} \leq \cancel{2^n} + n$$

$$0 \leq n \quad \leftarrow \text{true because } n \text{ is a natural number}$$

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ also $\Rightarrow$ original ineq. was true
converges by Comparison Test (DCT)

BARE MINIMUM SOLUTION (same example)

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ $\leftarrow$ positive terms $\checkmark$

Pick $b_n = \frac{1}{2^n} \geq 0$. $\checkmark$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges since $|r| = \left|\frac{1}{2}\right| < 1$

Want: $\frac{1}{2^n + n} \leq \frac{1}{2^n}$ $\checkmark$

$\uparrow$
larger denominator
so smaller

original series converges by Comparison Test
not "it"

BAD SOLUTION (will lose points for write-up)

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

$$b_n = \frac{1}{2^n}$$

unclear what converges
converges, $|r| = \left|\frac{1}{2}\right| < 1$

← did you check or not?
 $a_n \leq b_n$ and $\sum b_n$ converges

so $\sum a_n$ converges by
Comparison Test.

Standard of Proof: can another Calc 2 student read and understand your solution?

* Note: compare the terms not the series

$$a_n \leq b_n \quad \text{NOT} \quad \sum a_n \leq \sum b_n$$

ex: $\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+n}}$ ← positive terms ✓

Compare to $b_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \geq 0$ ✓

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad p = \frac{3}{2} > 1$
 $\Rightarrow$ converges

Want: $a_n \leq b_n$

$\frac{\cos^2(n)}{\sqrt{n^3+n}} \stackrel{!}{\leq} \frac{1}{n^{3/2}}$ ✓

↑ true b/c $\cos^2 n \leq 1$
 and $\sqrt{n^3+n} \geq n^{3/2}$

Tip: compare the numerator & denominator separately

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3+n}}$

also converges by Comp. Test.

ex: $\sum_{n=2}^{\infty} \frac{\sqrt{n^4+1}}{n^3-2}$ $\leftarrow$ positive terms ($n \geq 2$)

Choose $b_n = \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n} \geq 0$ ✓

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

Want: $b_n \leq a_n$

~~$\frac{1}{n} \leq \frac{\sqrt{n^4+1}}{n^3-2}$~~ (?)

Tip: use form of b_n from before you simplified

$\frac{\sqrt{n^4}}{n^3} \leq \frac{\sqrt{n^4+1}}{n^3-2}$ ✓

numerator is larger and denominator is smaller, so bigger

$\sum_{n=2}^{\infty} \frac{\sqrt{n^4+1}}{n^3-2}$

also diverges by DCT.

Limit Comparison Test:

If $\sum a_n$ and $\sum b_n$ are both series with positive terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \quad \text{where} \quad 0 < c < \infty$$

strict ineq.!

then either both series converge or both diverge.

Intuition: Can think of $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$

as a race.

Assume $a_n, b_n \rightarrow 0$.

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, $a_n \rightarrow 0$ faster.

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, $b_n \rightarrow 0$ faster.

★ If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ for $0 < c < \infty$,
 a_n and b_n
grow at the
same rate.

ex: $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ $\leftarrow$ positive terms $\checkmark$

Pick $b_n = \frac{1}{n} \geq 0$ $\checkmark$

$\sum b_n = \sum \frac{1}{n}$ diverges (harmonic)

If using DCT, we would want

$\frac{1}{n} \leq \frac{1}{n+\sqrt{n}}$ $\leftarrow$ not true $\therefore$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} = 1 \end{aligned}$$

Since $0 < 1 < \infty$, $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ also

diverges by the Limit Comparison Test (LCT).