

1. Consider the transformation T given by $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} x_1 - x_3 \\ 3x_2 \end{bmatrix}$.

(a) Compute $T \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right)$ and $T \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) + T \left(\begin{bmatrix} r \\ s \\ t \end{bmatrix} \right)$.

$$T \begin{pmatrix} a+r \\ b+s \\ c+t \end{pmatrix} = \begin{bmatrix} (a+r) - (c+t) \\ 3(b+s) \end{bmatrix} = \begin{bmatrix} a-c \\ 3b \end{bmatrix} + \begin{bmatrix} r-t \\ 3s \end{bmatrix} = \begin{bmatrix} a-c+r-t \\ 3b+3s \end{bmatrix}$$

(b) Compute $T \left(c \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right)$ and $cT \left(\begin{bmatrix} r \\ s \\ t \end{bmatrix} \right)$.

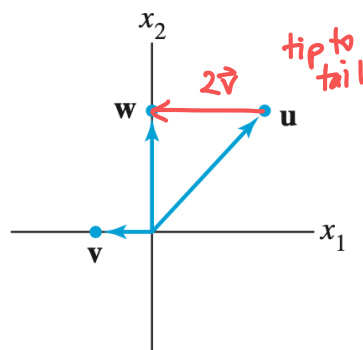
$$T \begin{pmatrix} cr \\ cs \\ ct \end{pmatrix} = \begin{bmatrix} cr-ct \\ 3cs \end{bmatrix} = c \begin{bmatrix} r-t \\ 3s \end{bmatrix} = cT \begin{pmatrix} r \\ s \\ t \end{pmatrix}$$

- (c) Is T linear?

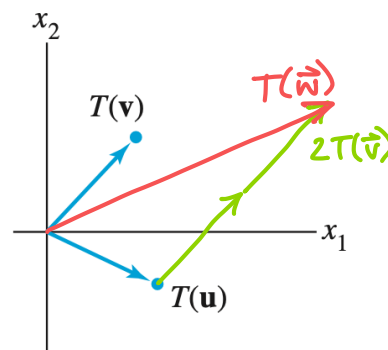
Yes because $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(c\vec{x}) = cT(\vec{x})$ by part (a) and (b).

2. The figure shows vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Draw the image $T(\mathbf{w})$ as accurately as possible.

(Hint: first write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.)



$$\vec{w} = \vec{u} + 2\vec{v}$$

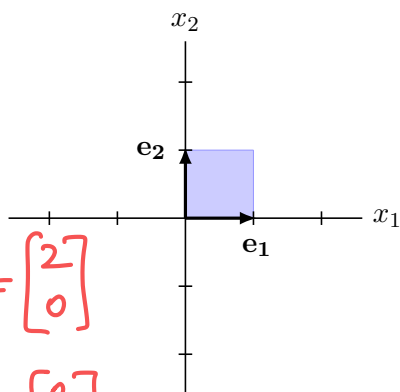


$$T(\vec{w}) = T(\vec{u} + 2\vec{v}) = T(\vec{u}) + 2T(\vec{v})$$

Since T is linear

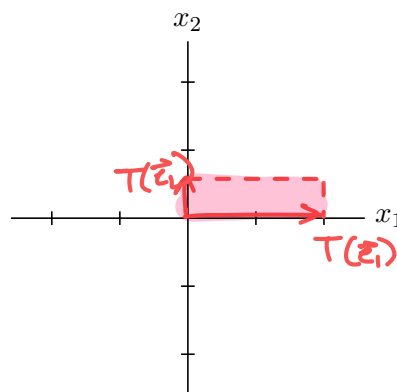
3. Consider the matrix transformation $T_A(\mathbf{x}) = A\mathbf{x}$ for each matrix A . Draw the image of $\mathbf{e}_1$, $\mathbf{e}_2$ and the unit square under T_A . Describe the transformation geometrically.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

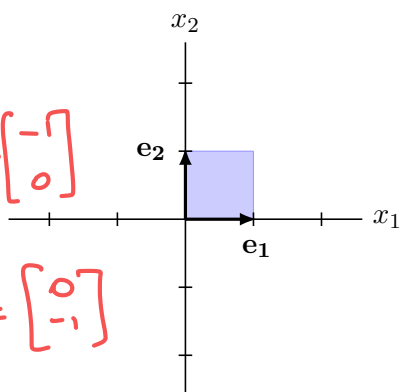


$$A\vec{e}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A\vec{e}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

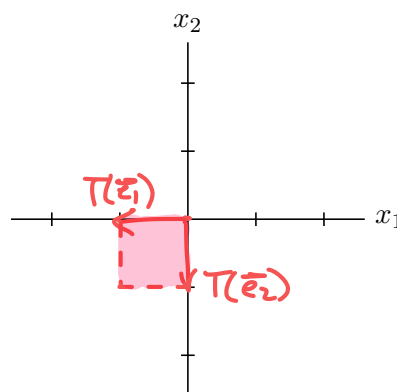


$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

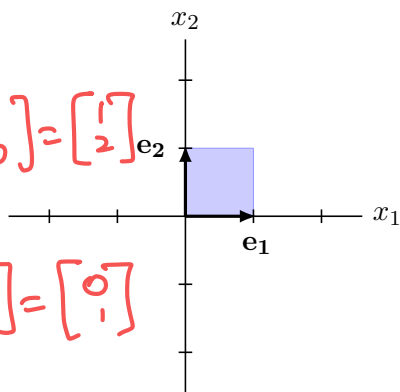


$$A\vec{e}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$A\vec{e}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

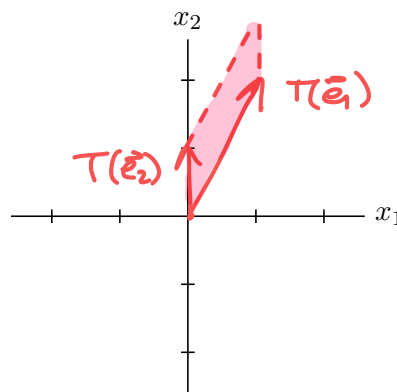


$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$



$$A\vec{e}_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A\vec{e}_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



4. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation that maps $\mathbf{e}_1$ to $\mathbf{y}_1$ and $\mathbf{e}_2$ to $\mathbf{y}_2$ where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard unit vectors in $\mathbb{R}^2$ and

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

Find the images of $\mathbf{v} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ under T .

Given: $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

Notice $\vec{v} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\vec{e}_1 - 3\vec{e}_2$

$$S_0 \quad T(\vec{v}) = T(5\vec{e}_1 - 3\vec{e}_2) = 5T(\vec{e}_1) - 3T(\vec{e}_2) = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$$

$$\text{Also } T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) = x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

$$\text{OR } \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

5. (a) Justify the following statement:

If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

In other words, T maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

$0 \vec{x} = \vec{0}$ for any vector $\vec{x}$ in $\mathbb{R}^n$

the scalar zero the zero vector linear

Since T is linear, $T(0\vec{x}) = 0T(\vec{x}) = \vec{0}$.
 $\parallel$
 $T(\vec{0})$

- (b) Is the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = mx + b$ a linear transformation?

No if $b \neq 0$

E.g. $f(x+y) = m(x+y) + \underline{b}$

$$f(x) + f(y) = (mx+b) + (my+b) = m(x+y) + \underline{2b}$$

(This is called an affine transformation.)