

Recap from yesterday's activity

Taylor polynomials centered at $a = 0$

1st degree Taylor polynomial: $T_1(x) = f(0) + f'(0)x$

2nd degree Taylor polynomial: $T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

3rd degree Taylor polynomial: $T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

4th degree Taylor polynomial: $T_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

add 1 more term

n th degree Taylor polynomial, $T_n(x)$

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$\frac{f(0)}{0!}x^0$

$\frac{f'(0)}{1!}x^1$

0 derivatives of f
 $f^{(0)}(0) = f(0)$

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

What about Taylor polynomials centered at $a \neq 0$?

Think back to tangent lines

$$y = f(a) + f'(a)(x-a)$$

vs

$$y = f(0) + f'(0)x$$

change 0 to a
and x to $x-a$

Taylor polynomial $T_n(x)$ at $a \neq 0$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

The coefficients in the Taylor poly. are called Taylor coefficients

$$C_k = \frac{f^{(k)}(a)}{k!}$$

ex: Find the 3rd deg. Taylor polynomial of

$$f(x) = \frac{1}{x} \text{ at } a = 2.$$

$$T_3(x) = C_0 + C_1(x-2) + C_2(x-2)^2 + C_3(x-2)^3$$

$f(x) = x^{-1}$	$f(2) = \frac{1}{2}$	$C_0 = \frac{f(2)}{0!} = \frac{1}{2}$
$f'(x) = -x^{-2}$	$f'(2) = -\frac{1}{4}$	$C_1 = \frac{-1/4}{1!} = -\frac{1}{4}$
$f''(x) = 2x^{-3}$	$f''(2) = \frac{2}{8} = \frac{1}{4}$	$C_2 = \frac{1/4}{2!} = \frac{1}{8}$
$f'''(x) = -6x^{-4}$	$f'''(2) = -\frac{6}{16} = -\frac{3}{8}$	$C_3 = \frac{-3/8}{3!} = -\frac{3}{8} \cdot \frac{1}{6}$ $= -\frac{1}{16}$

$$T_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

ex: Find the 6th degree Taylor poly of $f(x) = \sin x$ at $a=0$.

↖ MacLaurin poly.

$f(x) = \sin x$	$f(0) = \sin(0) = 0$	$C_0 = \frac{0}{0!} = 0$
$f'(x) = \cos x$	$f'(0) = \cos(0) = 1$	$C_1 = \frac{1}{1!} = 1$
$f''(x) = -\sin x$	$f''(0) = 0$	$C_2 = \frac{0}{2!} = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$	$C_3 = \frac{-1}{3!} = -\frac{1}{6}$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$	$C_4 = \frac{0}{4!} = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$	$C_5 = \frac{1}{5!} = \frac{1}{120}$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$	$C_6 = \frac{0}{6!} = 0$

$$T_6(x) = \cancel{C_0} + C_1 x + \cancel{C_2 x^2} + C_3 x^3 + \cancel{C_4 x^4} + C_5 x^5 + \cancel{C_6 x^6}$$
$$= x - \frac{1}{6} x^3 + \frac{1}{120} x^5$$

Notice $T_6(x)$ is actually degree 5.

"nth degree Taylor poly" can have degree less than n.

Also, $T_6(x) = T_5(x) + \underbrace{\text{next term}}_0$

$T_6(x) = T_5(x) \leftarrow$ for this example

ex: The 4th deg. Taylor polynomial of some function f is

$$T_4(x) = \underline{6} + \underline{1}(x-1) - 2(x-1)^2 + \frac{1}{2}(x-1)^3 - \frac{5}{12}(x-1)^4$$

Find $f^{(4)}(\underline{1})$, $f'''(1)$, $f'(1)$ and $f(1)$.

base point $a = \underline{1}$

Use $C_k = \frac{f^{(k)}(1)}{k!}$

For $f^{(4)}(1)$, use $C_4 = \frac{f^{(4)}(1)}{4!}$

Also from T_4 , $C_4 = -\frac{5}{12}$ ↗ equal

$$\frac{f^{(4)}(1)}{4!} = -\frac{5}{12} \Rightarrow f^{(4)}(1) = -\frac{5 \cdot 24}{12} = -10$$

For $f'(1)$, use $C_1 = \frac{f'(1)}{1!} = f'(1)$

From T_4 , $C_1 = 1$

$f'(1) = 1$

★ Important! can't use $T_4(x)$ based at $a=1$ to find $f^{(k)}(0)$ or any other point. ONLY at $a=1$.

8.5 Power Series

Up to now, we've been studying the convergence or divergence of series like

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

↑ ↑ ↑
numbers

← if this converges,
it will be a
number

With power series, we introduce a variable x .

Definition: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

Here $\{c_n\}$ is the sequence of coefficients and a is the **center** or **base point** of the power series.

When we plug a value for x into a power series, we get a "normal" series.

Example: $\sum_{n=1}^{\infty} \frac{x^n}{n3^n} = \sum \frac{1}{n3^n} \cdot x^n \leftarrow C_n = \frac{1}{n3^n}, a=0$

If $x=1$, $\sum_{n=1}^{\infty} \frac{(1)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n3^n} \leftarrow \text{cugs, compare to } \sum \frac{1}{3^n}$

a number

If $x=3$, $\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \leftarrow \text{diverges (harmonic)}$

A power series is a FUNCTION of x , but
we have to be careful about the domain.

Definition: The **interval of convergence (IOC)** of a power series is the set of all points, x , for which the power series converges.

Example: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

$\uparrow$ $c_n = 1, a = 0$

Where does this converge?

$\sum_{n=0}^{\infty} x^n$ is geometric with $r = x$

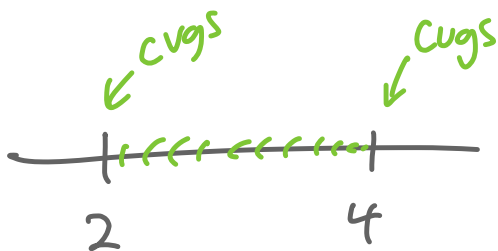
converges if $|x| < 1$
diverges if $|x| \geq 1$

geometric series test
 $\downarrow$

Facts about the Interval of Convergence

IOC is $(-1, 1)$

- If a power series converges at two separate points, it also converges at all points between them. In other words, the interval of convergence will always be an interval (unless it's a single point).



Assume our power series converges at $x=2$ and $x=4$.
P.S. also converges for all x in $(2, 4)$

- The base point is always in the interval of convergence. Another way of saying this is that a power series always converges at its base point.

Why? $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$

plug in $x = a$

$c_0 + 0 + 0 + 0 + \dots$
which definitely converges.

- The interval of convergence is always symmetric about the base point.

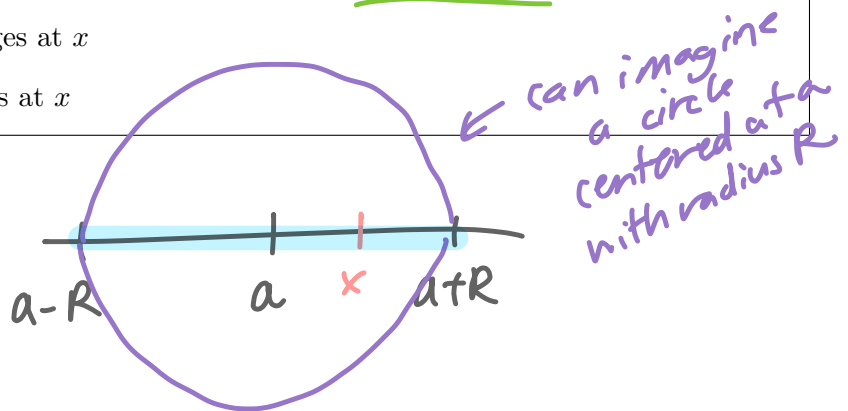
e.g. $\sum x^n$ has $(-1, 1)$ as its IOC



Definition: The **radius of convergence** of a power series $\sum c_n(x-a)^n$ is a number $R \geq 0$ such that

- (i) if $|x-a| < R$, the series converges at x
- (ii) if $|x-a| > R$, the series diverges at x

$|x-a| < R$
 distance between x and a
 radius



If radius of convergence is R ,
 then IOC goes from $a-R$
 to $a+R$

Facts about the Radius of Convergence

- The radius of convergence does not tell you what happens when $|x-a| = R$.

If you know the radius of convergence
 of a P.S., the IOC can be

true if
 $x = a+R$
 or $x = a-R$

$(a-R, a+R)$, $[a-R, a+R]$, $(a-R, a+R]$,
 or $[a-R, a+R)$

- We allow $R = \infty$, which means the power series converges at every real number.

- $R = 0$ means the power series converges ONLY at the base point a .

- We use the Ratio Test to find the radius of convergence.

We'll see in examples