

# Eilenberg-MacLane Spaces \*

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## Abstract

The Eilenberg-MacLane spaces are studied. We show the existence and uniqueness, up to homotopy, for CW complexes. The remarkable relation with Cohomology theory is introduced and several examples are presented. The references are explicitly given throughout the text.

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# 1 Introduction

The Eilenberg-MacLane spaces were introduced in the 1950s by Samuel Eilenberg and Saunders MacLane. As one of the first applications, the notion allows us to show the existence of CW complexes with all homotopy groups prescribed. Their fundamental role, however, is related to Cohomology theory and Fibrations. In particular, they relate Cohomology groups and homomorphisms of fundamental groups with sets of free homotopy classes. We will give a glimpse of these two results.

The text is organized as follows. This first section starts with the basic definition and it is dedicated to review and state some results applied throughout the work. In the second section, we show the existence and uniqueness up to homotopy (in the set of CW complexes), we distinguish the case  $n = 1$  from the general. A miscellaneous of examples is given in Section 3, for instance, closed surfaces and infinite spaces, as well as a review about some constructions and descriptions. Section 4 intends to introduce the deeper theory and we start with a preliminary discussion on Cohomology theory. We apply these results to some of the previous examples. In the direction of a more advanced study, we finish with some remarks.

**Definition 1.1.** *Let  $n \geq 1$  and a group  $G$  (abelian if  $n > 1$ ). A path-connected topological space  $X$  is an Eilenberg-MacLane space, or simply a space of type  $K(G, n)$ , if the only nontrivial homotopy group of  $X$  is  $\pi_n$  and it is equal to  $G$ .*

**Proposition 1.1.** *If  $X$  and  $Y$  are spaces of type  $K(G, n)$  and  $K(H, n)$  respectively, then  $X \times Y$  is a space of type  $K(G \times H, n)$ .*

*Proof.* The result follows from the fact that  $\pi_i(X \times Y) \simeq \pi_i(X) \times \pi_i(Y)$  for all  $i \geq 1$ , see [7, Proposição 6.40].  $\square$

**Example 1.1.** Consider the circle  $\mathbb{S}^1$ . We have that  $\pi_1(\mathbb{S}^1) = \mathbb{Z}$  and the universal covering is  $\mathbb{R}$ , which is a contractible space. On the other hand, the covering map induces an isomorphism between the higher homotopy groups [7, Proposição 8.10]. We conclude that  $\mathbb{S}^1$  is a space of type  $K(\mathbb{Z}, 1)$ . By Proposition 1.1, we have that the  $n$ -torus  $\mathbb{S}^1 \times \dots \times \mathbb{S}^1 = \mathbb{T}^n$  is a space of type  $K(\mathbb{Z}^n, 1)$ .

## 1.1 Presentation of Groups

We review briefly the concepts of free group and free abelian group. These groups will appear in the construction of Eilenberg-MacLane spaces, namely, as homotopy groups of bouquet of spheres.

We based on [5, Sections 1.9 and 2.1]. From an arbitrary set  $S$  we may construct a group  $F(S)$  which contains  $S$  and satisfies a particular property. If  $S^{-1}$  is an another set in bijection with  $S$ ,  $F(S)$  is the set of sequences of arbitrary length formed by the elements of  $S$  and  $S^{-1}$  and the product is given by concatenation. The group  $F(S)$  is called the *free group on  $S$* . The following property is called *universal*:

**Theorem 1.1.** *Let  $G$  be group,  $\varphi : S \longrightarrow G$  a set map, and  $F(S)$  the free group on  $S$ . Then there is a unique homomorphism  $\phi : F(S) \longrightarrow G$  such that  $\phi \circ i = \varphi$ , where  $i$  denotes the inclusion.*

The group  $F(S)$  is unique up to an isomorphism. The case  $S = G$ , where  $G$  is itself a group is of special interest. Let  $\text{Id} : G \longrightarrow G$  the identity map on  $G$ . By Theorem 1.1, there exists a surjective homomorphism  $\phi : F(G) \longrightarrow G$  such that  $\phi \circ i = \text{Id}$ . In particular,  $F(G)/\text{Ker}(\phi) \simeq G$ . We will represent a group  $G$  by a set of generators which satisfy some relations.

**Definition 1.2.**

1. *Let  $G$  be a group,  $S \subset G$  and  $R \subset F(G)$  subsets. The group  $G$  is said to be defined by the generators  $s \in S$  and the relations  $r \in R$  if  $G \simeq F(S)/T$ , where  $T$  is the smallest normal subgroup containing the subgroup generated by  $R$ . Then we say that  $(S, R)$  is a presentation for  $G$ .*
2.  *$G$  is finitely generated if there exists a presentation  $(S, R)$  with  $S$  finite.*
3.  *$G$  is finitely presented if it is finitely generated and  $R$  is also finite.*

We may choose a subset  $S$  of generators for  $G$  and a subset  $R$  of generators for  $\text{Ker}(\phi)$ , then  $F(S) = F(G)$  and  $(S, R)$  is a presentation for  $G$ . For two groups  $G$  and  $H$  we may define a new group  $G * H$  with product defined as the alternating concatenation of elements in  $G$  and in  $H$ . On the other hand, each element of an arbitrary set  $S$  generates a subgroup in  $F(S)$  consisting of its powers. This group is isomorphic to  $\mathbb{Z}$ . We conclude that

$$F(S) = *_{\alpha \in S} \mathbb{Z}_{\alpha}.$$

Now, our goal is to construct an abelian group  $A(S)$  from an arbitrary set  $S$  which also satisfies similar properties as above.

**Definition 1.3.** *Let  $A$  be an abelian group.*

1. *For a non-empty subset  $X \subset A$  the subgroup  $\langle X \rangle$  consists of all possible sums  $\sum_{k=1}^m n_k x_k$ , where  $m \in \mathbb{N}$ , and  $n_k \in \mathbb{Z}$  and  $x_k \in X$  for all  $k$ .*
2. *A basis for  $A$  is a set  $X \subset A$  with  $A = \langle X \rangle$  such that if for distincts  $x_1, \dots, x_m \in X$  and  $n_1, \dots, n_m \in \mathbb{Z}$  we have  $\sum_i n_i x_i = 0$ , then  $n_k = 0$  for all  $k$ .*

Given an arbitrary set  $S$ , define  $A(S) = \sum_{s \in S} \mathbb{Z}$ . Let  $\{s_i\} \in A(S)$ . We have the following inclusion

$$\begin{aligned} i : S &\longrightarrow A(S) \\ s &\longmapsto \{s_{\alpha}\} \end{aligned}$$

where  $\{s_{\alpha}\}$  is the sequence formed by elements  $s_{\alpha}$  such that  $s_{\alpha} = 0$  for  $\alpha \neq s$  and 1 for  $\alpha = s$ . The set  $A(S)$  has a natural abelian structure and is called the *free abelian group on  $S$* . The universal property is the following:

**Theorem 1.2.** *Let  $S$  be a set and  $A(S)$  its free abelian group. If  $G$  is an abelian group and  $\varphi : S \rightarrow G$  is a map set, then there exists an unique homomorphism  $\phi : A(S) \rightarrow G$  such that  $\phi \circ i = \varphi$ , where  $i$  denotes the inclusion.*

As before,  $A(S)$  is unique up to an isomorphism. Applying the construction for  $S = G$  an abelian group, we have an isomorphism  $A(S)/\text{Ker}(\phi) \simeq G$  for some surjective homomorphism  $\phi : A(G) \rightarrow G$ . We also have a similar definition for an abelian presentation of  $G$ , which exists by the previous theorem. The difference is that every subgroup of  $A(G)$  is already normal.

## 1.2 Cellular and CW Approximations

The CW structure for a topological space allows us to simplify some calculations, specially for homotopy groups. On the other hand, every space can be approximated in some sense by a CW complex. We based on [1, Section 4.1].

**Definition 1.4.** *Let  $X$  and  $Y$  be CW complexes with a given cell complex structure. A map  $f : X \rightarrow Y$  is a cellular map if  $f(X^n) \subseteq Y^n$  for all  $n$ , where  $X^n$  and  $Y^n$  denote the  $n$ -skeleton of  $X$  and  $Y$ , respectively.*

**Theorem 1.3** ([1], Theorem 4.8). *Every map  $f : X \rightarrow Y$  of CW complexes is homotopic to a cellular map. If  $f$  is already cellular on a subcomplex  $A \subset X$  the homotopy may be taken to be stationary on  $A$ .*

We recall the definition of a  $n$ -connected CW pair.

**Definition 1.5.** *A pair  $(X, A)$  is said to be  $n$ -connected if  $\pi_i(X, A, a) = 0$  for all  $0 \leq i \leq n$ ,  $a \in X$  and the induced map  $i_* : \pi_0(A) \rightarrow \pi_0(X)$  is surjective.*

Theorem 1.3 may be applied for CW pairs as follows. Given a map  $f : (X, A) \rightarrow (Y, A)$ , consider the restriction  $f : A \rightarrow B$ . This map is homotopic to a cellular map  $g$ , by the theorem. Using the Homotopy Extension Property [1, Proposition 0.16], we may extend  $g$  to  $X$ , and preserving  $A$ . The extension of  $g'$  is homotopic to a cellular map  $h$ , and this homotopy may be taken to be stationary on  $A$ . Then, the map  $h$  is cellular.

**Corollary 1.1** ([1], Corollary 4.12). *A CW pair  $(X, A)$  is  $n$ -connected if all the cells in  $X - A$  have dimension greater than  $n$ . Hence, the inclusion  $i : A \rightarrow X$  induces isomorphisms*

$$\pi_i(A) \simeq \pi_i(X)$$

for  $i < n$  and a surjection for  $i = n$ .

*Proof.* Let  $[\varphi] \in \pi_i(X, A)$ ,  $i \leq n$ , represented by  $\varphi : (\mathbb{D}^i, \partial\mathbb{D}^i) \rightarrow (X, A)$  cellular. By hypothesis and the previous paragraph,  $\varphi(\mathbb{D}^i, \partial\mathbb{D}^i) \subset A$ , therefore its class is null. Consider the exact sequence for this pair with  $i < n$ :

$$\dots 0 = \pi_{i+1}(X, A) \xrightarrow{\partial} \pi_i(A) \xrightarrow{i_*} \pi_i(X) \xrightarrow{j_*} \pi_i(X, A) = 0 \dots$$

Then  $\text{Ker}(i_*) = \text{Im}(\partial) = 0$  and  $\text{Im}(i_*) = \text{Ker}(j_*) = \pi_i(X)$ , this latter holds if  $i \leq n$ . The result follows.  $\square$

**Remark 1.1.** The inclusion map  $i : \pi_i(A) \longrightarrow \pi_i(X)$  in the above sequence is also a surjection by cellular approximation of spheroids.

Different versions of this result are the following.

**Corollary 1.2.** *The pair  $(X, X^n)$  is  $n$ -connected, hence the inclusion  $i : X^n \longrightarrow X$*

$$\pi_i(X^n) \simeq \pi_i(X)$$

for  $i < n$  and a surjection for  $i = n$ .

**Corollary 1.3.** *If a space  $Y$  is obtained from a space  $X$  by attaching cells of dimension  $n + 1$ , then the inclusion  $i : X \longrightarrow Y$  induces isomorphisms*

$$\pi_i(X) \simeq \pi_i(Y)$$

for  $i < n$  and a surjection for  $i = n$ . In this latter case, the kernel is generated by the classes of the attaching maps.

*Proof.* We shall verify the last statement. The boundary operator  $\partial : \pi_{i+1}(Y, X) \longrightarrow \pi_i(X)$  operates restricting relative spheroids  $[\varphi_\alpha] \in \pi_{i+1}(Y, X)$  to  $\pi_i(X)$ . On the other hand, the inclusions  $i_\alpha : e_\alpha^{n+1} \longrightarrow Y$  generate the relative group  $\pi_{n+1}(Y, X)$  by cellular approximation. But their images under  $\partial$  are exactly the attaching maps  $[\varphi_\alpha]$ . See Figure 1. We conclude that  $\text{Im}(\partial)$  is generated by the classes of attaching maps. By the exact sequence for the pair,  $\text{Im}(\partial) = \text{Ker}(i_*)$ , and the result follows.

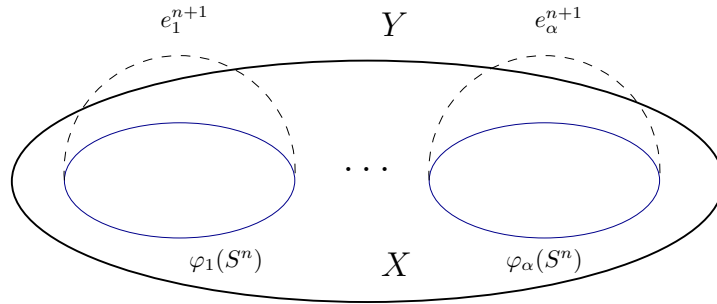


Figure 1:  $Y$  is formed by  $n + 1$ -cells attached to  $X$

□

**Definition 1.6.** *A map  $f : X \longrightarrow Y$  is called a weak homotopy equivalence if it induces isomorphisms  $\pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$  for all  $n$  and all choices of basepoints.*

**Definition 1.7.** *A CW approximation for a topological space  $X$  is a weak homotopy equivalence  $f : Z \longrightarrow X$  where  $Z$  is a CW complex.*

The construction in the next theorem is of particular interest.

**Theorem 1.4.** *Every space  $X$  has a CW approximation.*

*Proof.* We begin with a map  $f : A \rightarrow X$ , where  $A$  consists of 0-cells points for each path-component of  $X$ . We denote these points by  $a_j$ . For each  $k \geq 1$  consider, for each component, the map

$$f_* : \pi_{k-1}(A, a_j) \rightarrow \pi_{k-1}(X, f(a_j)).$$

Choose maps  $\varphi_\alpha : (\mathbb{S}^{k-1}, s_0) \rightarrow (A, a_j)$  which represent the nontrivial elements of  $\text{Ker}(f_*)$ . By Theorem 1.3, we may suppose  $\varphi_\alpha$  a cellular map with  $\mathbb{S}^{k-1}$  equipped with the standard CW structure (see Section 3). For each  $\alpha$ , we attach a cell  $e_\alpha^k$  via the maps  $\varphi_\alpha$ , see Figure 1 below.

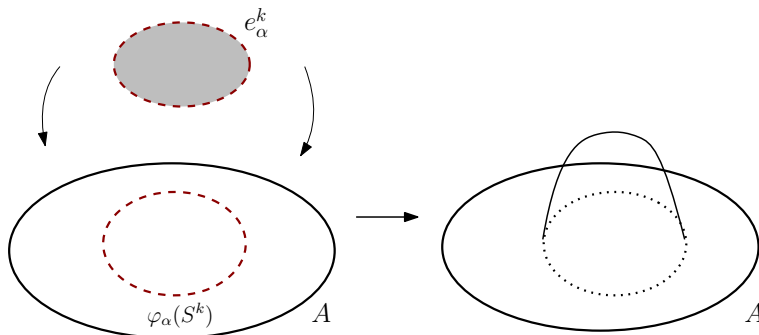


Figure 2: Attaching cells via  $\varphi_\alpha$

For each  $\alpha$ , the procedure cancels the element in the kernel of  $f_*$ . Moreover, if  $B$  is the new space,  $B - A$  has  $k$ -cells, and by Corollary 1.1 the lower homotopy groups are isomorphic, that is,

$$\pi_i(B) \simeq \pi_i(A)$$

for  $i < k$ . We should extend  $f$  on each  $e_\alpha^k$ , but any extension on the interior is nullhomotopic. Therefore, the maps should agree in the boundary  $\varphi_\alpha(\mathbb{S}^{k-1})$ , and this is the case, since  $[f \circ \varphi_\alpha]$  is the trivial class in  $\pi_{k-1}(X, f(a_j))$ .

Let  $B$  the resulting CW complex. If  $h : \mathbb{S}^{k-1} \rightarrow B$  is a map such that  $[h] \in \text{Ker}(f_*) \subset \pi_{k-1}(B, a_j)$ , we may suppose it cellular. Therefore,  $h(\mathbb{S}^{k-1}) \subset A$ , since we obtain  $B$  from  $A$  attaching  $k$ -cells. Thus,  $[h] = [\varphi_\alpha]$  for some  $\alpha$ , but these classes are trivial in  $B$ . We conclude that  $f_* : \pi_{k-1}(B, a_j) \rightarrow \pi_{k-1}(X, f(a_j))$  is injective.

Let  $f_* : \pi_k(A, a_j) \rightarrow \pi_k(X, f(a_j))$  and  $\varphi_\beta$  a nontrivial element in  $\pi_k(X, f(a_j))$ . We attach cells  $e_\beta^{k+1}$  to  $A$  via constant maps

$$i_\beta : \partial(e_\beta^{k+1}) \rightarrow a_j.$$

Now we extend  $f$  over the sphere  $\mathbb{S}^k = e_\beta^k \cup_{i_\beta} a_j$  making  $f$  on  $\mathbb{S}^k$  as  $\varphi_\beta$ . The extension is surjective, because

$$f_*([\text{Id} : \mathbb{S}^k \rightarrow \mathbb{S}^k]) = [f|_{\mathbb{S}^k}] = [\varphi_\beta].$$

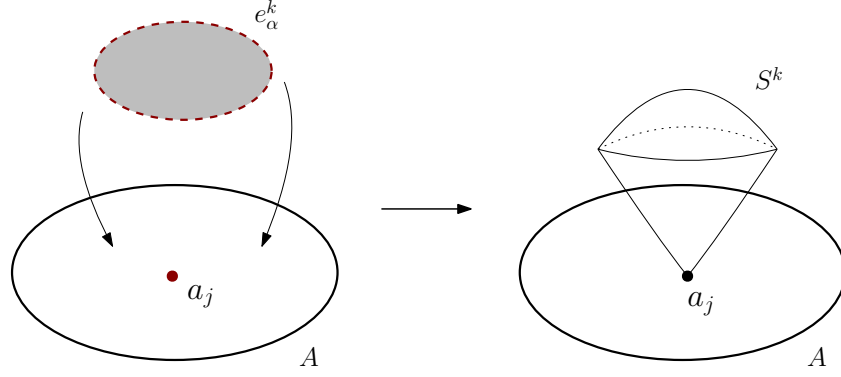


Figure 3: Attaching cells via constant maps

After this procedure,

$$f_* : \pi_i(B, a_j) \longrightarrow \pi_i(X, f(a_j))$$

is injective for  $i = k - 1$  and surjective for  $i = k$ . We repeat this process inductively to obtain the desired CW complex.  $\square$

A similar application of this process is canceling higher homotopy groups  $\pi_i$  of a space from some index  $k < i$ .

**Proposition 1.2** ([2], Teorema 2.3.3). *Let  $X$  be a connected CW complex. For each integer  $n \geq 1$  there exists a CW complex  $X_n$  containing  $X$  as a subcomplex such that:*

- $\pi_i(X_n) = 0$  for  $i > n$ ;
- $i_* : \pi_i(X) \longrightarrow \pi_i(X_n)$  is an isomorphism for  $i \leq n$ , where  $i$  denotes the inclusion.

*Proof.* Let  $[\varphi_\alpha]$  be a generator of  $\pi_{n+1}(X)$ . As before, we attach a cell  $e^{n+2}$  according to this map to cancel this element. Then the new space  $Y$  has the same lower homotopy groups as  $X$ , by Corollary 1.1. Now, let  $[h] \in \pi_{n+1}(Y)$ . By cellular approximation, since  $Y - X$  has cells of dimension  $n + 2$ ,  $[h] \in \pi_{n+1}(X)$ . That is,  $[h] = [\varphi_\alpha]$ , for some generator  $\varphi_\alpha$ . But these maps are nullhomotopic in  $Y$ . Therefore,  $[h] = 0$  and  $\pi_{n+1}(X_n) = 0$ . We repeat this process for  $k \geq n + 3$  on  $Y$  to obtain the space  $X_n$  with the required properties.  $\square$

## 2 Existence and Uniqueness

In this section, we show the existence and uniqueness, up to homotopy, of CW complexes of type  $K(G, n)$  for every  $n$  and  $G$ . The main references are [2, Capitulo 3], [3, Section 11.7], and [1, Section 4.2].

## 2.1 The case $n = 1$

For the case  $n = 1$ , the group  $G$  can be any group, so we start with it. The next lemma can be verified by Seifert-van Kampen Theorem.

**Lemma 2.1.** *Let  $X = \bigvee_{\alpha \in S} \mathbb{S}^1$ , where  $S$  is an arbitrary set of indexes. Then  $\pi_1(X) = F(S)$ , the free group on  $S$ .*

**Theorem 2.1** ([1], Corollary 1.28). *Let  $G$  be any group. Then there exists a space of type  $K(G, 1)$ .*

*Proof.* We will construct a CW complex with fundamental group being  $G$  and then apply Proposition 1.2. Choose a presentation  $(S, R)$  for  $G$ . We define

$$X = \bigvee_{\alpha \in S} \mathbb{S}^1.$$

By Lemma 2.1,  $\pi_1(X) = F(S)$ , the free group on  $S$ . Since  $R \subset F(S)$ , each  $\beta \in R$  corresponds to a class  $[\varphi_\beta]$  represented by a spheroid  $\varphi_\beta : \mathbb{S}^1 \rightarrow X$ . We attach a cell  $e_\beta^2$  via these maps. By Corollary 1.3, the resulting space  $Y$  satisfies

$$\pi_1(Y) \simeq F(S)/T \simeq G,$$

where  $T$  is the smallest normal subgroup containing the subgroup generated by the relations  $\beta$ . □

## 2.2 The General Case

The next lemma is the analogous in higher dimensions of Lemma 2.1.

**Lemma 2.2.** *Let  $n \geq 2$  and  $X = \bigvee_{\alpha \in S} \mathbb{S}^n$ , where  $S$  is an arbitrary set of indexes. Then*

1.  $X$  is  $(n - 1)$ -connected.
2.  $\pi_i(X) \simeq \sum_{\alpha \in S} \mathbb{Z}_\alpha$ , the free abelian group on  $S$ .

*Proof.* The proof of the second item can be seen in [1, Example 4.26]. For the first item, let  $j \leq n - 1$  and  $[\varphi] \in \pi_j(X)$  a class represented by  $\varphi : \mathbb{S}^j \rightarrow X$ , which we may suppose cellular with respect to the standard CW structure on  $\mathbb{S}^j$ . Note that the  $j$ -th-skeleton of  $X$  is only the base point of the wedge sum. Therefore  $[\varphi] = 0$  and the result follows. □

**Theorem 2.2** ([2], Proposition 3.2.2). *Let  $n \geq 2$  and  $G$  be an abelian group. There exists a CW complex of type  $K(G, n)$ .*

*Proof.* We first show the existence of a  $n - 1$ -connected CW complex with the  $n$ -th homotopy group being exactly  $G$ . We then apply Proposition 1.2 to conclude. Let  $G = (S, R)$  an abelian

presentation of  $G$ . For each generator  $\alpha \in S$  we choose a sphere  $\mathbb{S}_\alpha^n$  to construct the space  $X^n = \bigvee_{\alpha \in S} \mathbb{S}_\alpha^n$ . By Lemma 2.2, we have that

$$\pi_n \left( \bigvee_{\alpha \in S} \mathbb{S}_\alpha^n \right) \simeq \sum_{\alpha \in S} \mathbb{Z}_\alpha = A(S).$$

Let  $\beta \in R$  a relation. Since  $R \subset A(S) = A(G)$ , there exists a class  $[\varphi_\beta]$  represented by a map  $\varphi_\beta : \mathbb{S}^n \rightarrow X^n$  in correspondence with  $\beta$ . For each  $\beta$ , we attach a cell  $e_\beta^{n+1}$  via the map  $\varphi_\beta$  to obtain

$$X = X^n \bigcup_{\varphi_\beta} e_\beta^{n+1}.$$

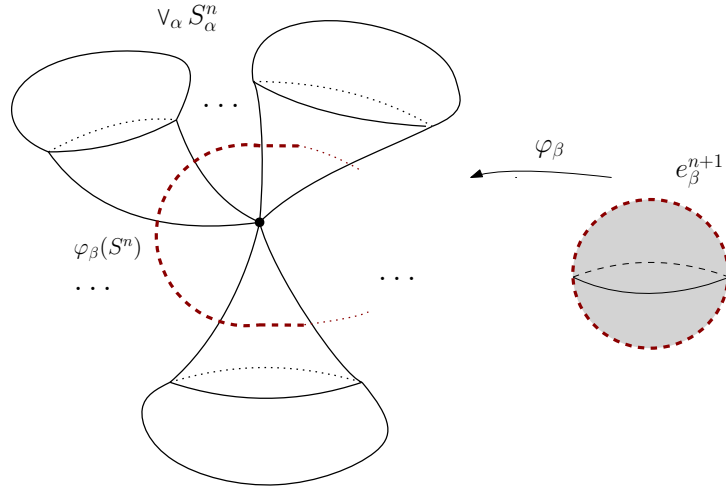


Figure 4: Attaching cells  $e_\beta^{n+1}$  via maps  $\varphi_\beta$

By Corollary 1.3, we have that

$$\pi_n(X) = \pi_n(X^n) / \text{Im}(\partial) = A(S) / A(R) = A(S) / \langle R \rangle \simeq G.$$

Finally, note that the natural CW structure for  $X$  has skeletons of dimension 0,  $n$ , and  $n+1$ . By cellular approximation, the cancellation of homotopy classes in  $\pi(X^n)$ , and the fact that  $X^n$  is  $(n-1)$ -connected, we have that  $X$  is  $(n-1)$ -connected.  $\square$

The existence of spaces of type  $K(G, n)$  allows us to show the existence of CW complexes with all homotopy groups prescribed.

**Corollary 2.1** ([4], Corollary 8.6). *Let  $\{G_n\}_{n \geq 1}$  be a sequence of groups which are abelian for  $n \geq 2$ . There exists a path connected CW complex  $X$  with  $\pi_n(X) \simeq G_n$  for all  $n \geq 1$ .*

*Proof.* For each pair  $(G_n, n)$  we apply the previous Theorem to obtain a space  $X_n$  of type  $K(G_n, n)$ . The space

$$X = \prod_n X_n$$

has the desired properties.  $\square$

**Theorem 2.3** ([3], Theorem 11.8). *Any two spaces of type  $K(G, n)$  are weakly homotopy equivalent. Hence, any two CW complexes of type  $K(G, n)$  are homotopy equivalent.*

*Proof.* Without loss of generality, we may suppose  $X$  the CW complex constructed in Theorem 2.2 and  $Y$  a space of type  $K(G, n)$ . The space  $X$  consists of a bouquet of spheres  $\mathbb{S}^n$  with cells  $e^k$  of dimension  $k \geq n + 1$  attached. By cellular approximation, the inclusion maps

$$i_\alpha : \mathbb{S}_\alpha^n \longrightarrow X$$

generate the group  $\pi_n(X)$ . Since  $\pi_n(Y) \simeq \pi_n(X)$ , each class  $[i_\alpha]$  corresponds to a generator  $[j_\alpha] \in \pi_n(Y)$ . We define the map on each sphere by

$$\begin{aligned} f : \mathbb{S}_\alpha^n &\longrightarrow Y \\ x &\longmapsto j_\alpha(x) \end{aligned}$$

Thus

$$f_*[i_\alpha] = [f(\mathbb{S}_\alpha^n)] = [j_\alpha],$$

that is,  $f$  induces an isomorphism  $\pi_n(Y) \simeq \pi_n(X)$ . For each  $k \geq n + 1$  we shall extend the map  $f$  over the  $k$ -cells.

We should only verify that on each boundary, the map is nullhomotopic. In this case, the homopy defines a natural extension of the map. But each spheroid  $\varphi : \mathbb{S}^n \longrightarrow X$  which is an attaching map for each cell is nullhomotopic in  $\pi_n(X)$ , hence in  $\pi_n(Y)$ . That is,  $[f \circ \varphi] = 0$ . Then we extend  $f$  over this cell, and after on each  $k$ -th skeleton. See Figure 5. Of course,  $f$  induces trivially isomorphisms on all homotopy groups  $\pi_i$  with  $i \neq n$ . This shows the first statement. If  $Y$  is a CW complex, the result follows from Whitehead Theorem [3, Theorem 11.3].

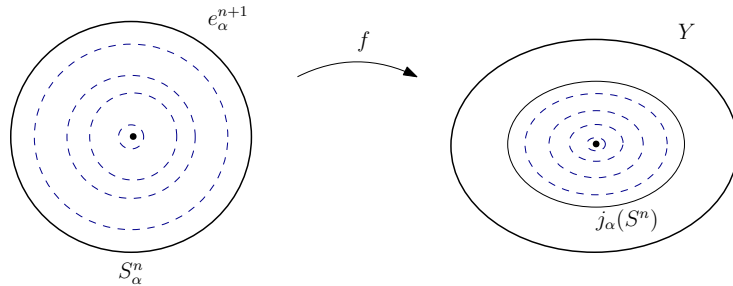


Figure 5: Extension for  $f$  on each attached cell

□

**Remark 2.1.** One can ask for an example of two topological spaces of type  $K(G, n)$  which are not CW complexes and are not homotopy equivalent.

### 3 Examples

We introduce some examples of spaces of type  $K(G, n)$  for  $n \leq 2$ . Indeed, the general cases do not have an explicit geometric construction, as stated in [3, p.139]. As expected, we have CW complexes of infinite dimension.

#### 3.1 Closed Surfaces

We will show that the classical closed surfaces are spaces of type  $K(G, 1)$ . These spaces can be constructed from bouquet of circles, as we shall see. We based on [3, Exercises 3 and 4, Section 8.3] and [1, Section 1A].

**Lemma 3.1.** *Let  $X = \bigvee_{\alpha \in S} \mathbb{S}^1_\alpha$ , where  $S$  is an arbitrary set. Then  $X$  is a space of type  $K(F(S), 1)$ .*

*Proof.* We may regard the bouquet as a graph. Recall that a tree in a graph is a contractible subgraph, and every graph contains a maximal tree, which contains all the others. The fundamental group of a graph is generated by the edges which are out of trees [1, Proposition 1A.2]. On the other hand, a covering space of a graph is also a graph, therefore, a universal covering space must consists of a tree [1, Lemma 1A.3]. We conclude by considering the isomorphism between higher homotopy groups.  $\square$

We may represent the bouquet  $\mathbb{S}^1 \vee \mathbb{S}^1$  as the boundary of a rectangle with the vertices identified. In general, the bouquet  $\bigvee_{\alpha=1}^n (\mathbb{S}^1_\alpha \vee \mathbb{S}^1_\alpha)$  is represented by the boundary of a polygon of  $4n$  edges with the same identification of vertices. See Figures 6 and 7.

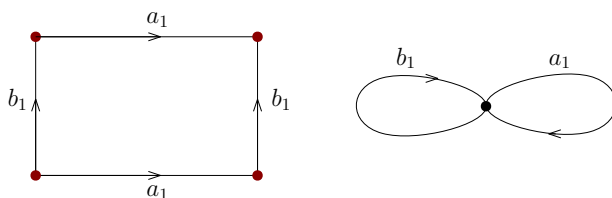


Figure 6

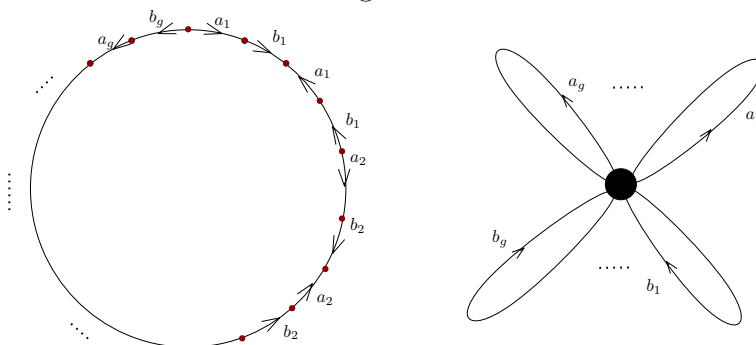


Figure 7

Alternatively,  $\mathbb{S}^1 \vee \mathbb{S}^1$  is given by the boundary of a rectangle with only one pair identified, as shows the next figure. We generalize in the same way. See Figures 8 and 9.

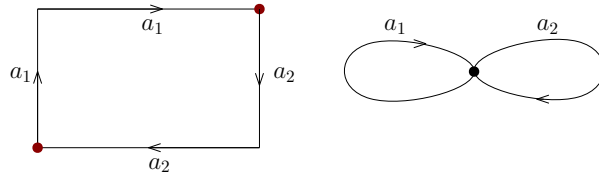


Figure 8

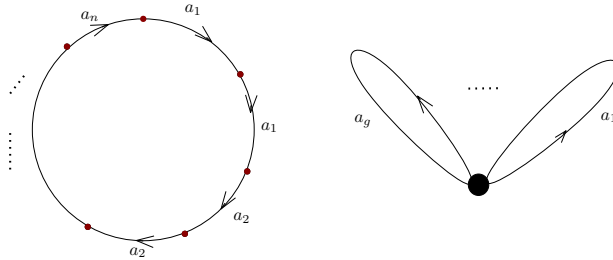


Figure 9

Recall that closed surfaces (compact surfaces without boundary) are classified, up to homeomorphisms, as those that are connected sums of torus or projective planes, according to orientation. They are represented by polygons with edges identified. See Figure 10 below.

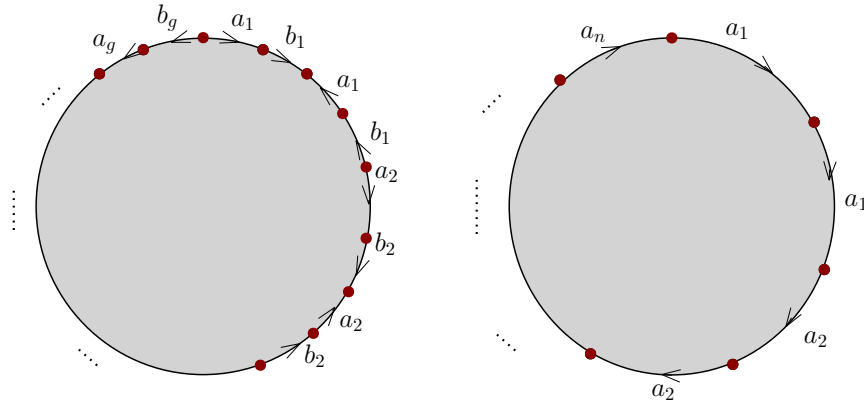


Figure 10: Closed surfaces represented by polygons with a proper identification

The representations of closed surfaces via polygons are exactly the resulting of attaching a 2-cell  $e^2$  to the representations of the bouquet according to a suitable relation. More specifically, in the orientable case of genus  $g$ , we attach the cell via the map  $\varphi_\beta \in \pi_1(X)$ , where  $X = \bigvee_{\alpha=1}^g (\mathbb{S}_\alpha^1 \vee \mathbb{S}_\alpha^1)$ , which corresponds to the relation

$$\prod_{i=1}^g [a_i, b_i]$$

in the free group  $\pi_1(X)$ . In the non-orientable case of genus  $g$ , we consider the bouquet  $\bigvee_{\alpha=1}^g \mathbb{S}_\alpha^1$  and the relation

$$\prod_{i=1}^g a_i a_i.$$

Finally, note that we do not have an analogous description for the cases  $g = 0$ , that is, the sphere  $\mathbb{S}^2$  and the projective plane  $\mathbb{R}\mathbb{P}^2$ . Indeed, they have non trivial and isomorphic higher homotopy groups. We review some definitions to give a proper relation between the bouquet of circles and the closed surfaces. The reference is [12, Chapter 2].

**Definition 3.1.** *Let  $X$  a topological space and  $A \subset X$  a subspace.*

1. *A map  $r : X \rightarrow A$  is said to be retraction if  $r$  is continuous and  $r(a) = a$  for all  $a \in A$ .*
2. *The subspace  $A$  is a deformation retraction of  $X$  if there exists a homotopy  $H : X \times I \rightarrow X$  between a retraction  $r : X \rightarrow A$  and the identity map  $\text{Id}_x$  such that  $H(a, t) = a$  for all  $a \in A$  and  $t \in I$ .*

**Proposition 3.1.** *A deformation retraction is a homotopy equivalence.*

*Proof.* Let  $i : A \rightarrow X$  denote the inclusion. It holds that  $(r \circ i)(x) = \text{Id}_A(x)$  for all  $x \in A$ , in particular,  $(r \circ i) \simeq \text{Id}_A$ . On the other hand,  $(i \circ r)(x) = r(x)$  for all  $x \in X$ . But  $r(x) \simeq \text{Id}_X$ , by the previous definition. Hence,  $A$  is homotopy equivalent to  $X$ .  $\square$

**Proposition 3.2.** *The closed surfaces of positive genus are spaces of type  $K(G, 1)$ .*

*Proof.* Consider Figure 9 and denote the closed disc, centered at the origin, by  $\mathbb{D}$ , and the boundary with the proper identifications, which is a bouquet of circles, by  $\partial\mathbb{D}$ . We shall see that  $\partial\mathbb{D}$  is a deformation retraction of  $\mathbb{D} - \{0\}$ . A retraction is given by

$$\begin{aligned} r : \mathbb{D} - \{0\} &\longrightarrow \partial\mathbb{D} \\ v &\longmapsto \frac{v}{\|v\|} \end{aligned}$$

The following map determines the desired homotopy:

$$\begin{aligned} H : \mathbb{D} - \{0\} \times I &\longrightarrow \mathbb{D} - \{0\} \\ (v, t) &\longmapsto (1 - t)v + t \frac{v}{\|v\|} \end{aligned}$$

Note that the surface minus a point is formed by a open disk without this point and the boundary with identifications. We may identify the open disk minus a point by  $(0, 1) \times \mathbb{S}^1$ , via polar coordinates. Since  $\mathbb{R}$  is the universal covering of  $\mathbb{S}^1$ , we conclude that  $\mathbb{R}^2$ , which is homeomorphic to the open disk, is a universal covering. Then we may extend this covering to the identity map on the boundary to conclude that the own surface is a covering space to the surface minus one point. It implies that their higher homotopy groups are isomorphic. We conclude by Lemma 3.1.  $\square$

### 3.2 Spheres and Orbit Spaces

In this subsection we review the constructions of spheres, projective spaces (real, complex and quaternionic), and lens spaces. The infinite version of these orbit spaces are of type  $K(G, 1)$ , as we shall see, and provide interesting examples.

The standard CW structure on  $\mathbb{S}^n$  is given by a 0-cell  $e^0$  and a  $n$ -cell  $e^n$  attached by the map  $\varphi : \mathbb{S}^{n-1} \rightarrow e_0$ . Equivalently,  $\mathbb{S}^n$  is the disc  $\mathbb{D}^n$  with the boundary collapsed into a point:  $\mathbb{S}^n \simeq \mathbb{D}^n / \partial\mathbb{D}^n$ . See Figure 11.

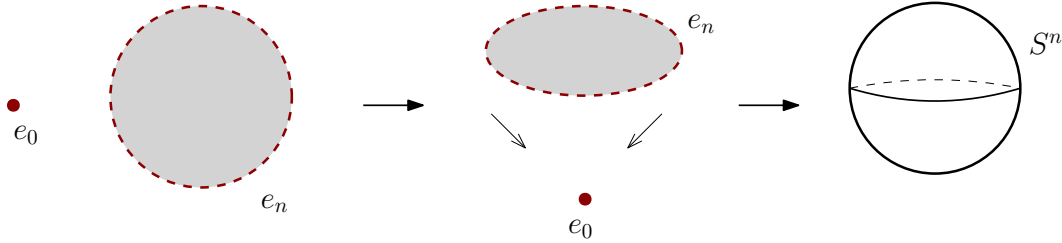


Figure 11: Standard CW structure for  $\mathbb{S}^n$

**Corollary 3.1** ([1], Corollary 4.9). *For all  $i < n$ ,  $\pi_i(\mathbb{S}^n) = 0$ .*

*Proof.* Suppose that  $\mathbb{S}^i$  and  $\mathbb{S}^n$  are endowed with the standard CW structure. Any map  $\varphi : \mathbb{S}^i \rightarrow \mathbb{S}^n$  is homotopic to a cellular map  $\varphi'$ . Since  $i < n$ ,  $\varphi'(\mathbb{S}^i) \subset e^0$ , therefore  $[\varphi] = 0$ .  $\square$

In the above construction we do not have an explicit inclusion of lower dimensional spheres as subcomplexes. An alternative structure is the following. We may obtain  $\mathbb{S}^n$  from  $\mathbb{S}^{n-1}$  attaching two  $n$ -cells  $e^n$  via the identity maps  $\text{Id} : \partial e^n \rightarrow \mathbb{S}^{n-1}$ . In this case,  $\mathbb{S}^n$  has  $\mathbb{S}^{n-1}$  as the equator. See Figure 12.

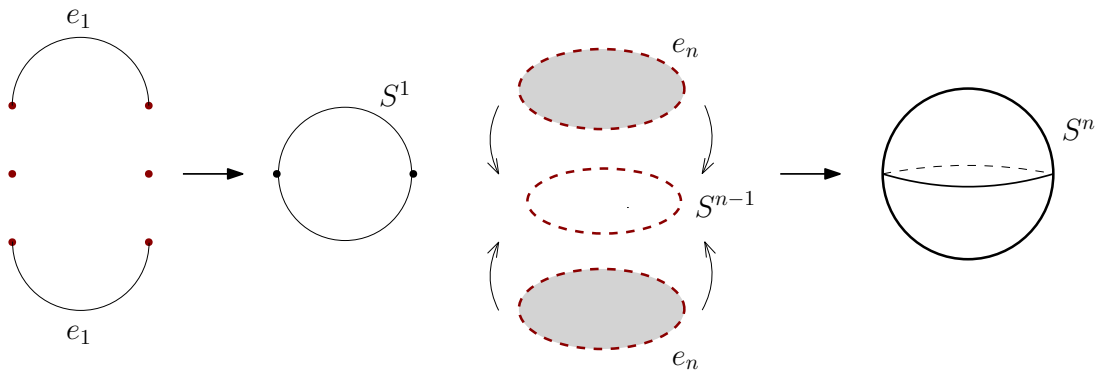


Figure 12: Alternative CW structure for  $\mathbb{S}^n$

Inductively,

$$\mathbb{S}^n = e^0 \cup e^1 \cup \dots \cup e^n \cup e^n.$$

This construction allows us to define the infinite sphere as follows:

$$\mathbb{S}^\infty := \bigcup_{i=1}^{\infty} e^i \cup e^i.$$

**Proposition 3.3.** *The infinite sphere  $\mathbb{S}^\infty$  is a contractible space.*

*Proof.* Let  $j \geq 1$ . Consider the spheres  $\mathbb{S}^j$  with the alternative CW structure. By cellular approximation, the inclusions  $i_j : \mathbb{S}^j \rightarrow \mathbb{S}^\infty$  generate the group  $\pi_j(\mathbb{S}^\infty)$ . But these maps are nullhomotopic in  $\mathbb{S}^\infty$ , because the (contractible) cells  $e^{j+1}$  are attached using them (see Figure 11). Then  $\pi_j(\mathbb{S}^\infty) = 0$  for all  $j \geq 1$ . If  $\{*\}$  is a point space, the constant map  $f : \mathbb{S}^\infty \rightarrow \{*\}$  induces isomorphisms on all the homotopy groups and we conclude by Whitehead Theorem.  $\square$

## Real Projective Spaces

The real projective space  $\mathbb{R}\mathbb{P}^n$  is the sphere  $\mathbb{S}^n$  with antipodal points identified. Equivalently, it is the disk  $\mathbb{D}^n$  with the antipodal boundary points identified. More specifically,  $\mathbb{R}\mathbb{P}^n$  is formed by the interior of  $\mathbb{D}^n$  and  $\partial\mathbb{D}^n$  with antipodal points identified, that is,  $\mathbb{R}\mathbb{P}^{n-1}$ .

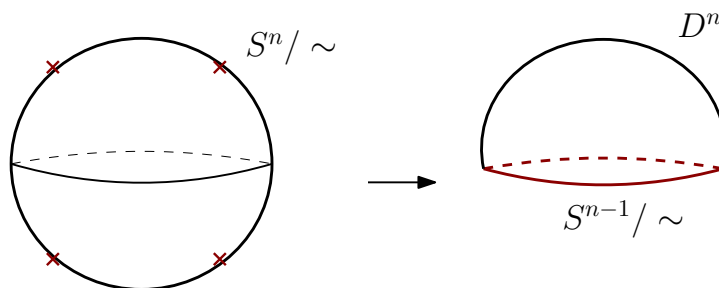


Figure 13: CW structure for  $\mathbb{R}\mathbb{P}^n$

Formally, we conclude that  $\mathbb{R}\mathbb{P}^n$  is the resulting of attach a  $n$ -cell to  $\mathbb{R}\mathbb{P}^{n-1}$  via the projection map  $p : \mathbb{S}^{n-1} \rightarrow \mathbb{R}\mathbb{P}^{n-1}$ . Inductively,

$$\mathbb{R}\mathbb{P}^n = e^0 \cup \dots \cup e^n.$$

Similarly, we may define the infinite projective space:

$$\mathbb{R}\mathbb{P}^\infty := \bigcup_{i=1}^{\infty} e^i.$$

In this case, we also have a projection map between  $\mathbb{S}^\infty$  and  $\mathbb{R}\mathbb{P}^\infty$ , where for each dimension  $k$  the map  $p$  projects the two cells  $e^k \cup e^k$  into  $e^k$ .

**Proposition 3.4.** *The infinite real projective space  $\mathbb{R}\mathbb{P}^\infty$  is a  $K(\mathbb{Z}_2, 1)$  space.*

*Proof.* The  $\mathbb{Z}_2$  action on  $\mathbb{S}^\infty$  is clearly discontinuous. This fact defines the projection as a regular covering map, see [7, Proposição 5.37]. Recall that a contractile space is also simply connected. Then, by Proposition 3.3, this defines  $\mathbb{S}^\infty$  as the universal covering of  $\mathbb{R}\mathbb{P}^\infty$ . The fundamental group has the same cardinality of the fiber [7, Corolário 5.30]. The result follows.  $\square$

## Lens Spaces

The lens spaces generalize the construction of  $\mathbb{R}\mathbb{P}^n$  as quotient space of  $\mathbb{S}^n$ . For instance, they are important in Geometric Topology, in particular, in the classification of some decompositions of 3-manifolds. We briefly introduce them following [10].

**Definition 3.2.** *Let  $l, m$  be positive integers with  $(l, m) = 1$ . The lens space  $L(l, m)$  is the quotient of  $\mathbb{S}^3 \subset \mathbb{C}^2$  modulo the action of  $\mathbb{Z}_m$  given by*

$$(z_1, z_2) \mapsto (z_1 e^{2\pi i/l}, z_2 e^{2\pi i m/l})$$

In particular,  $L(1, 2) \simeq \mathbb{R}\mathbb{P}^3$ . Moreover, since the action of  $\mathbb{Z}_m$  is discontinuous, by an argument similar to that given in the proof of Proposition 3.4, we have that

$$\pi_1(L(1, 2)) \simeq \mathbb{Z}_m.$$

A more geometric and equivalent construction is the following.

**Definition 3.3** (Construction from a ball). *Let  $l, m \in \mathbb{Z}$ ,  $1 \leq l < m$  and  $(l, m) = 1$ . In the 3-ball, divide the equator in  $m$  segments  $[v_j, v_{j+1}]$ , where*

$$v_j = (\cos(2\pi j/m), \sin(2\pi j/m), 0).$$

*We apply a rotation by  $2\pi l/m$  in the north hemisphere. After, identify it with the south hemisphere. The triangle  $\Delta_{j+l} = [v_{j+l}, v_{j+l+1}, N]$  is identified with the triangle  $\Delta_j = [v_j, v_{j+1}, S]$ . The result space is the lens space  $L(l, m)$ .*

We may extend the definition for higher dimensions.

**Definition 3.4.**

1. *Let  $m, l_1, \dots, l_{n-1}$  be positive integers such that  $(l_i, m) = 1$  for all  $i$ . The lens space  $L(l_1, \dots, l_{n-1}, m)$  is defined as the quotient of  $\mathbb{S}^{2n-1} \subset \mathbb{C}^{2n}$  modulo the action of  $\mathbb{Z}_m$  given by*

$$(z_1, z_2, \dots, z_{2n}) \mapsto (z_1 e^{2\pi i/l_1}, \dots, z_{2n} e^{2\pi i m/l_{n-1}})$$

2. *Let  $(l_n)_n$  and  $m$  be a sequence of positive integers such that  $(l_i, m) = 1$  for all  $i$  and some positive integer  $m$ . The infinite lens space  $L((l_n), m)$  is the quotient of  $\mathbb{S}^\infty \subset \mathbb{C}^\infty$  by the action of  $\mathbb{Z}_m$  given by*

$$(z_1, \dots, z_i, \dots) \mapsto (z_1 e^{2\pi i/l_1}, \dots, z_i e^{2\pi i m/l_i}, \dots)$$

Similarly to Proposition 3.4, we have

**Proposition 3.5.** *The infinite lens space  $L((l_n), m)$  is a space of type  $K(\mathbb{Z}_m, 1)$ .*

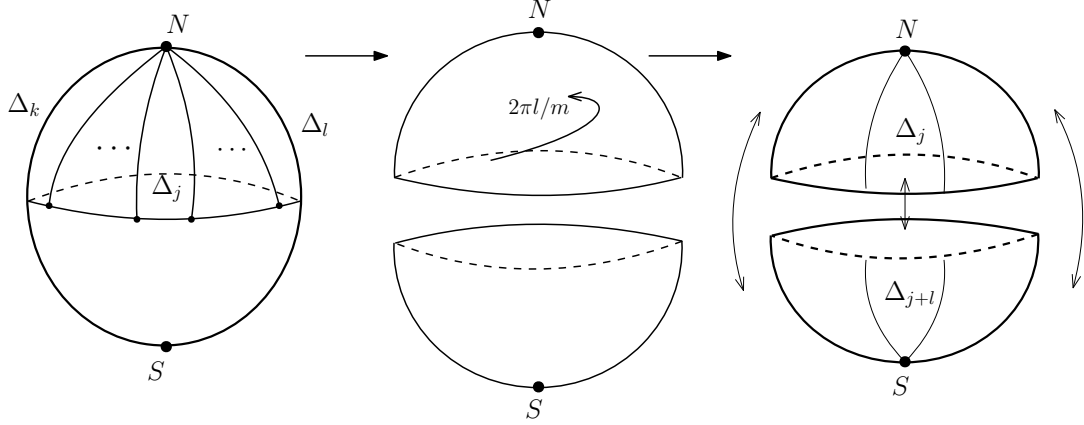


Figure 14: Constructing the lens space from a ball

### Complex Projective Spaces

The complex projective space  $\mathbb{C}\mathbb{P}^n$  is defined as the quotient of the sphere  $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$  by the action of the group  $\mathbb{S}^1$  of norm 1 complex numbers. We shall see a CW structure for this space. Consider the set

$$\Gamma = \{(w, \sqrt{1 - |w|^2}) \in \mathbb{C}^n \times \mathbb{C} : |w| \leq 1\}.$$

This set is the graphic of  $w \mapsto \sqrt{1 - |w|^2}$  and it is homeomorphic to the disk  $\mathbb{D}^{2n}$ . On the other hand, if  $(w, v) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^n \times \mathbb{C}$  satisfies  $v \neq 0$  we may choose a representative of the class  $[(w, v)]$  of the form  $(w, \sqrt{1 - |w|^2})$ . This correspondence is one-to-one. If  $v = 0$ , then the class of  $(w, 0)$  is a point of  $\mathbb{S}^{2n-1} = \partial\Gamma$  up to an action of  $\mathbb{S}^1$ , that is, a point in  $\mathbb{C}\mathbb{P}^{n-1}$ . We conclude that  $\mathbb{C}\mathbb{P}^n$  is obtained from  $\mathbb{C}\mathbb{P}^{n-1}$  attaching a  $2n$ -cell via the projection map  $\mathbb{S}^{2n-1} \mapsto \mathbb{C}\mathbb{P}^{n-1}$ .

Inductively,

$$\mathbb{C}\mathbb{P}^n = e^0 \cup e^2 \cup e^4 \dots \cup e^{2n}.$$

Then we define the infinite complex projective space as

$$\mathbb{C}\mathbb{P}^\infty := \bigcup_{i=0}^{\infty} e^{2i}.$$

**Proposition 3.6.** *The infinite projective space  $\mathbb{C}\mathbb{P}^\infty$  is a space of type  $K(\mathbb{Z}, 2)$ .*

*Proof.* The construction above shows that the projection map  $p : \mathbb{S}^{2n+1} \rightarrow \mathbb{C}\mathbb{P}^n$  defines a fiber bundle with fiber  $\mathbb{S}^1$ . This fact is also true for  $n = \infty$ . By [1, Proposition 4.48] a fiber bundle is a fibration with respect to all CW pairs. Then we apply [1, Theorem 4.41] to obtain the following exact sequence:

$$\dots \pi_i(\mathbb{S}^1) \rightarrow \pi_i(\mathbb{S}^\infty) = 0 \xrightarrow{p_*} \pi_i(\mathbb{C}\mathbb{P}^\infty) \xrightarrow{m_0} \pi_{i-1}(\mathbb{S}^1) \xrightarrow{m_1} \pi_i(\mathbb{S}^\infty) = 0 \dots$$

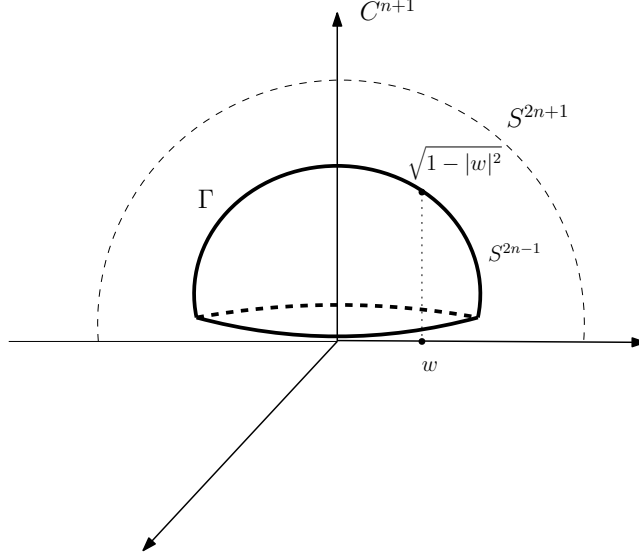


Figure 15: CW structure for  $\mathbb{C}\mathbb{P}^n$

Therefore

$$\begin{aligned} 0 &= \text{Im}(p_*) = \text{Ker}(m_0); \\ \text{Im}(m_0) &= \text{Ker}(m_0) = \pi_{i-1}(\mathbb{S}^1), \end{aligned}$$

which implies  $\pi(\mathbb{C}\mathbb{P}^\infty) \simeq \pi_{i-1}(\mathbb{S}^1)$ . We know from Example 1.1 that  $\mathbb{S}^1$  is a space of type  $K(\mathbb{Z}, 1)$ , so the result result follows.  $\square$

### Quaternionic Projective Spaces

The quaternionic projective space  $\mathbb{K}\mathbb{P}^n$  is defined as the quotient of the sphere  $\mathbb{S}^{4n+3} \subset \mathbb{K}^{n+1} \simeq \mathbb{R}^{4n+4}$  by the action of the group  $\mathbb{S}^3$  of norm 1 quaternions. The same construction for  $\mathbb{C}\mathbb{P}^n$  may be applied in proper dimensions. Therefore, we have that  $\mathbb{K}\mathbb{P}^n$  is formed from  $\mathbb{K}\mathbb{P}^{n-1}$  attaching a  $4n$ -cell via the projection map  $p : \mathbb{S}^{4n-1} \mapsto \mathbb{K}\mathbb{P}^{n-1}$ . Inductively,

$$\mathbb{K}\mathbb{P}^n = e^0 \cup e^4 \cup e^{16} \dots \cup e^{4n}.$$

Similarly,

$$\mathbb{K}\mathbb{P}^\infty := \bigcup_{i=0}^{\infty} e^{4i}.$$

In this case, the projection map also defines a fiber bundle, but applying the exact sequence for this bundle, we obtain that

$$\pi_{i-1}(\mathbb{K}\mathbb{P}^\infty) = \pi_i(\mathbb{S}^3),$$

which are non-trivial groups in general. Our conclusion is that  $\mathbb{K}\mathbb{P}^\infty$  is 3-connected.

**Example 3.1** (Octonionic Projective Plane). The fourth normed division algebra obtained from the quaternions is the algebra of octonions  $\mathbb{O}$ , which is non-associative. Therefore, the sphere  $\mathbb{S}^8$ , which consists of octonions with norm 1, does not have a natural group structure (namely, none!). For the construction of this space, which exists and only in projective dimension 2, we refer the reader for [9]. The geometric realization is quite easy. Recall that the previous projective spaces in projective dimension 1 are spheres. So, the real, complex, and quaternionic projective planes are obtained from spheres attaching cells via an identity map. Since  $\mathbb{O}\mathbb{P}^1 \simeq \mathbb{S}^8$ , our desired space  $\mathbb{O}\mathbb{P}^2$  may be recovered as

$$\mathbb{O}\mathbb{P}^2 = \mathbb{S}^8 \cup e^{16}.$$

### 3.3 Further Examples

Spaces of type  $K(G, 1)$  are very common and came from several contexts. We provide more examples, see also [3, Section 11.7].

**Example 3.2** (Relation with Loop Spaces). Let  $X$  and  $Y$  be pointed spaces. The reduced suspension of  $X$  and the loop space of  $Y$  are related by the following equation:

$$[\Sigma X, Y] \simeq [X, \Omega Y],$$

where we have omitted the basepoints. Now, let  $X = \mathbb{S}^i$  and  $Y$  a space of type  $K(G, n)$ . Then

$$\pi_{i-1}(Y) = \pi_{i-1}(\Omega Y),$$

and the loop space  $\Omega Y$  is a space of type  $K(G, n - 1)$ . A result due to Milnor states that the loop spaces have a CW structure. Therefore, considering the uniqueness in Theorem 2.3, every space of type  $K(G, n)$  with  $G$  abelian and  $n \geq 2$  is an  $H$ -group, see [7, Proposição 6.18]. For more informations, see [3, Exercise 13, p.140].

**Example 3.3** (Manifolds with Non-positive Curvature). Let  $M$  be a path-connected complete Riemannian manifold with nonpositive sectional curvature. By Hadamard Theorem [11, Teorema 3.1], the exponential map at any point will be a covering map, therefore the universal covering of this space is  $\mathbb{R}^n$ . Recall that  $\mathbb{R}^n$  is contractible, and by the induced isomorphism on higher homotopy groups [7, Proposição 8.10], we conclude that  $M$  is a space of type  $K(G, 1)$  for some  $G$ . From the theory of covering spaces, see [7, Corolário 5.30], we know that

$$\pi_1(M) \simeq \text{Aut}(\mathbb{R}^n),$$

where the automorphism group  $\text{Aut}$  consists of covering morphisms, with respect to the space  $M$ , which are also isomorphisms.

**Example 3.4** (Finitely Generated Abelian Groups). An abelian group is finitely generated if it has an abelian presentation  $(S, R)$  with  $S$  finite. In particular, each element of  $G$  is written as a finite sum  $\sum_k n_k s_k$  for  $n_k \in \mathbb{Z}$  and fixed elements  $s_k \in S$ . A well-known theorem

[5, Theorem 2.1] states that a finitely generated abelian group is isomorphic to a finite direct sum of the form

$$\mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_n}.$$

Therefore, considering Proposition 1.1 as well as infinite lens spaces  $L((l_n), m)$ , we may construct an explicit  $K(G, 1)$  space for  $G$  abelian and finitely generated.

## 4 Spaces of type $K(G, n)$ and Cohomology

In this section, we introduce the relation between Eilenberg-MacLane spaces and Cohomology. Namely, the cohomology group with coefficients in an abelian group  $G$  of an arbitrary space  $X$  determines the set of homotopy classes from  $X$  to a space of type  $K(G, n)$ . In order to see this, we state a result in categorical language without proof. We review some basic concepts and theorems from Cohomology to establish some free homotopy classes related to the spaces of Section 3. Our basic reference is [4, Section 8.1].

Given two spaces  $X$  and  $Y$ , recall that the set of free homotopy classes  $[X, Y]$  is the set of continuous functions where two maps are equivalent if they are homotopic with the same basepoints. It is possible constructing an action of  $\pi_1(Y)$  on  $\pi_n(Y)$  in such way that

$$[X, x_0; Y, y_0] \simeq [X; Y]/\pi_1(Y). \quad (1)$$

As consequence we have the next proposition.

**Proposition 4.1.** *Let  $(X, x_0)$  and  $(Y, y_0)$  be CW spaces with  $X$  path-connected. If  $\pi_1(Y, y_0)$  is trivial or abelian, then there exists a bijection*

$$[(X, x_0), (Y, y_0)] \simeq [X, Y].$$

*Proof.* See [1, Proposition 4.A.2 and Exercise 4.A.2] or [4, p.427]. □

Thus, in most of our examples the basepoints are irrelevant.

### 4.1 Preliminaries

We describe briefly basic definitions and notations from Singular Homology and Cohomology. The set of  $n$ -chains  $C_n(X, G)$  is formed by formal and finite sums of maps  $\alpha_i : \Delta^n \rightarrow X$  with coefficients in an abelian group  $G$  and arbitrary lengths. That is:

$$C_n(X, G) = \left\{ \sum_{i=1}^k r_i \alpha_i : k \in \mathbb{N}, r_i \in G, \alpha_i : \Delta^n \rightarrow X \right\}.$$

If  $G = \mathbb{Z}$ , we write  $C_n(X, G) = C_n(X)$  or  $C_n$  only. The boundary map  $\partial$  restricted to  $C_n(X, G)$  is

$$\partial_n : C_n(X, G) \rightarrow C_{n-1}(X, G).$$

The homology groups are defined by

$$H_n(X, G) = \text{Ker}(\partial_n) / \text{Im}(\partial_{n+1}).$$

In order to define the cohomology groups, we consider the dual spaces. Fix an abelian group  $G$  and the  $n$ -complex chain  $C_n$  with coefficients in  $\mathbb{Z}$ . We define

$$C_n^* = \text{Hom}(C_n, G).$$

The dual boundary operator is given by

$$\begin{aligned} \partial_n^* : C_{n-1}^* &\longrightarrow C_n^* \\ F &\longmapsto F \circ \partial_n \end{aligned}$$

Then

$$H^n(X, G) = \text{Ker}(\partial_n^*) / \text{Im}(\partial_{n-1}^*).$$

Given a map  $f : X \longrightarrow Y$  between topological spaces, there are the following induced maps:

$$\begin{aligned} f_* : H_n(X, G) &\longrightarrow H_n(Y, G) \\ \left[ \sum_i r_i \alpha_i \right] &\longmapsto \left[ \sum_i r_i (f \circ \alpha_i) \right] \\ f^* : H^n(Y, G) &\longrightarrow H^n(X, G) \\ [F] &\longmapsto [F(f_*)] \end{aligned}$$

**Theorem 4.1** (Hurewicz - [7], Teorema 9.14). *Let  $X$  be a  $(n - 1)$ -connected space. Then  $H_i(X) = 0$  for all  $0 \leq i \leq n - 1$  and the Hurewicz map*

$$\varphi : \pi_n(X, x_0) \longrightarrow H_n(X)$$

*defines an isomorphism.*

For an abelian group  $G$ , let  $[F] \in H^n(X, G)$  represented by  $F : C_n \longrightarrow G$ . Then

$$\partial_{n+1}^*(F) = F(\partial_{n+1}) = 0.$$

In particular,  $F$  is null in  $\text{Im}(\partial_{n+1})$ . Recall that

$$\begin{aligned} \text{Ker}(\partial_n) &\subset C_n \\ \text{Im}(\partial_{n+1}) &\subset C_n. \end{aligned}$$

Consider the restriction  $F_0$  of  $F$  to  $\text{Ker}(\partial_n)$ . Therefore, we have a well-defined homomorphism

$$\begin{aligned} \overline{F}_0 : H_n(X, \mathbb{Z}) &\longrightarrow G \\ [\alpha] &\longmapsto F_0(\alpha) \end{aligned}$$

In general, we have the following map:

$$h : H^n(X, G) \longrightarrow \text{Hom}(H_n(X), G).$$

The next theorem relates homology and cohomology groups and is called *The Universal Coefficient Theorem*.

**Theorem 4.2** ([1], Theorem 3.2). *The following sequence is exact:*

$$0 \longrightarrow \text{Ext}(H_{n-1}(X), G) \xrightarrow{i} H^n(X, G) \xrightarrow{h} \text{Hom}(H_n(X), G) \longrightarrow 0,$$

where the map  $i$  is induced by inclusion.

For details about the group  $\text{Ext}$ , we refer the reader to [1, Section 3.1]. The basic result of this section is the following.

**Lemma 4.1** ([4], Lemma 7.8.1). *Let  $F : H \longrightarrow H'$  be a natural transformation between homotopy functors which induces an isomorphism  $H(\mathbb{S}^k) \longrightarrow H'(\mathbb{S}^k)$  for all  $k < n$  and a surjection if  $k = n$ , where  $1 \leq n \leq \infty$ . For any path-connected pointed CW complex  $W$  the map*

$$F(W) : H(W) \longrightarrow H'(W)$$

is a bijection if  $\dim(W) \leq n - 1$  and a surjection if  $\dim(W) \leq n$ .

**Definition 4.1.** *Let  $Y$  be a path-connected space and  $G$  abelian. An element  $\nu \in H^n(Y, y_0, G)$  is said to be  $n$ -characteristic if*

$$\pi_n(Y, y_0) \xrightarrow{\varphi} H_n(Y, y_0) \xrightarrow{h(\nu)} G$$

is an isomorphism.

**Lemma 4.2.** *Let  $Y$  be an  $(n - 1)$ -connected space, where  $n \geq 2$ . There exists an  $n$ -characteristic element if and only if  $G \simeq \pi_n(Y)$ .*

*Proof.* If an such element exists, of course  $G \simeq \pi_n(Y)$ . On the other hand, by Hurewicz Theorem 4.1,

$$\varphi : H_n(Y) \longrightarrow \pi_n(Y)$$

is an isomorphism. In particular,  $H_n(Y) \simeq G$ , and there exists an isomorphism in  $\text{Hom}(H_n(Y), G)$ . By Theorem 4.2,  $h$  is surjective and the result follows.  $\square$

**Remark 4.1.** The surjection of  $h$  also follows directly from our construction, just defining extensions.

## 4.2 Cohomology Groups

We are in conditions to state the main theorem.

**Theorem 4.3** ([4], Theorem 8.1.8). *Let  $G$  be an abelian group,  $Y$  a space of type  $K(G, n)$ , and  $\nu$  an  $n$ -characteristic element for  $Y$ . Consider the map*

$$\begin{aligned} \Psi : [.; Y] &\longrightarrow H^n(., G) \\ [f] &\longmapsto f^*\nu \end{aligned}$$

Then, for each path-connected CW complex  $X$ ,  $\Psi(X)$  is a bijection. Formally,  $\Psi$  is a natural transformation which defines a natural equivalence on the category of path-connected CW complexes.

**Remark 4.2.** If two maps  $f, g$  are homotopic, the induced maps  $f_* = g_*$  are the same and so  $f^*, g^*$  as well, see [1, Theorem 2.10]. Thus  $\Psi$  is well-defined.

*Proof.* We apply Lemma 4.1 for  $n = \infty$ . Since the only non-trivial homotopy group is  $\pi_n(Y)$ , we must verify that

$$\Psi : \pi_n(Y) \longrightarrow H^n(\mathbb{S}^n, G)$$

is an isomorphism. Consider the map

$$\begin{aligned} \phi : H^n(\mathbb{S}^n, G) &\longrightarrow G \\ v &\longmapsto (h(v))\varphi([1_{\mathbb{S}^n}]) \end{aligned}$$

where  $[1_{\mathbb{S}^n}] \in \pi_n(\mathbb{S}^n)$  is the identity map. Recall that the sphere  $\mathbb{S}^n$  is  $(n-1)$ -connected from Corollary 3.1, then the Hurewicz map is an isomorphism. Since  $H_n(\mathbb{S}^n) \simeq \mathbb{Z}$ , we conclude that  $\varphi([1_{\mathbb{S}^n}])$  is a generator of  $H_n(\mathbb{S}^n)$ . Therefore, any map  $F \in \text{Hom}(H_n(\mathbb{S}^n), G)$  is uniquely determined by its image on this generator. On the other hand,  $h$  is surjective, so  $\phi$  is an isomorphism. Let  $[f] \in \pi_n(\mathbb{S}^n)$ , then:

$$\begin{aligned} \phi \circ \Psi(\mathbb{S}^n)([f]) &= \phi(f^*\nu) \\ &= (h(f^*\nu))(\varphi([1_{\mathbb{S}^n}])) \\ &= h(\nu f_*)(\varphi([1_{\mathbb{S}^n}])) \\ &= (h(\nu))(f_*)(\varphi([1_{\mathbb{S}^n}])) \\ &= (h(\nu) \circ \varphi)[f], \end{aligned}$$

where we have used the induced maps on homology and cohomology groups and the following remark. The element  $\nu \in H^n(Y)$  vanishes on  $\text{Im}(\partial_{n+1})$  and the image of  $f_*$  lies on the quotient  $\text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$ . So the composite is the restriction of  $\nu$  applied on  $f_*$ . Considering our construction above, it means that

$$\nu(f_*) = \bar{\nu}|_{\text{Ker}(\partial_n)}(f_*) = h(\nu)(f_*).$$

Since  $\nu$  is  $n$ -characteristic,  $\phi \circ \Psi(\mathbb{S}^n)$  is an isomorphism. Because  $\phi$  is an isomorphism as well, we conclude the same for  $\Psi(\mathbb{S}^n)$ .  $\square$

For spaces of type  $K(G, 1)$  we can relate the free homotopy classes and homomorphisms between fundamental groups as follows.

**Theorem 4.4** ([4], Theorem 8.1.9). *Let  $Y$  be a space of type  $K(G, 1)$ , where  $G$  is an arbitrary group  $G$ . Let  $H$  be the functor which assigns to a pointed space  $X$  the set of  $\text{Hom}(\pi_1(X), \pi_1(Y))$ . Consider the map*

$$\begin{aligned} \bar{\Psi} : [.; Y] &\longrightarrow H(\mathbb{S}^1) \\ [f] &\longmapsto f_{\#} \end{aligned}$$

*Then, for each path-connected CW complex  $X$ ,  $\bar{\Psi}(X)$  is a bijection. Formally,  $\bar{\Psi}$  is a natural transformation which defines a natural equivalence on the category of path-connected CW complexes.*

*Proof.* We apply Lemma 4.1 with  $n = \infty$ . We should verify the isomorphism for  $k = 1$ . Define the following map:

$$\begin{aligned}\bar{\phi} : \text{Hom}(\pi_1(\mathbb{S}^1), \pi_1(Y)) &\longrightarrow \pi_1(Y) \\ F &\longmapsto F[1_{\mathbb{S}^1}]\end{aligned}$$

As before, a homomorphism  $F : \pi_1(\mathbb{S}^1) \simeq \mathbb{Z} \longrightarrow \pi_1(Y)$  is uniquely determined by its image on the generator  $[1_{\mathbb{S}^1}]$ . Thus  $\bar{\phi}$  is an isomorphism. On the other hand, for any  $[\gamma]$  and  $F$ :

$$\begin{aligned}(\bar{\varphi} \circ \bar{\Psi})[\gamma] &= \bar{\phi}(\gamma_{\#}) = \gamma_*[1_{\mathbb{S}^1}] = \gamma; \\ (\bar{\Psi} \circ \bar{\phi})(F) &= \bar{\Psi}(F[1_{\mathbb{S}^1}]) = (F([1_{\mathbb{S}^1}]))_{\#} = F.\end{aligned}$$

Therefore  $\bar{\Psi}$  is the inverse of  $\bar{\phi}$  and the result follows.  $\square$

### 4.3 Applications

Theorem 4.3 allows us to determine some free homotopy classes. To show more specific examples, we do some computations of cohomology groups for some spaces.

**Proposition 4.2.** *Let  $X$  be an arbitrary CW complex. There exist the following bijections:*

1.  $[X; \mathbb{S}^1] \simeq H^1(X)$ ;
2.  $[X; \mathbb{T}^n] \simeq H^1(X, \mathbb{Z}^n)$ ;
3.  $[X; \mathbb{RP}^{\infty}] \simeq H^1(X, \mathbb{Z}_2)$ ;
4.  $[X; L((l_n), m)] \simeq H^1(X, \mathbb{Z}_m)$ ;
5.  $[X; \mathbb{CP}^{\infty}] \simeq H^2(X, \mathbb{Z})$ ;

**Proposition 4.3.** *Let  $Y = \bigvee_{\alpha \in S} \mathbb{S}_{\alpha}^1$  and  $X$  be an arbitrary CW complex. Then*

$$[X, Y] \simeq \text{Hom}(\pi_1(X), F(S)).$$

The Universal Coefficient Theorem allows us to relate homology and cohomology groups. Considering the Hurewicz homomorphism for  $n = 1$ , we may relate the first homology and cohomology groups with the fundamental group.

**Proposition 4.4.** *Let  $G$  be abelian, there exist the following isomorphisms:*

$$\text{Hom}(\pi_1(X), G) \simeq \text{Hom}(H_1(X), G) \simeq H^1(X, G).$$

*Proof.* The Hurewicz Theorem for  $n = 1$  establishes an isomorphism between the abelianization of the fundamental group and the first homology group, see [7, Teorema 3.10]. In

general, if  $\theta : H \rightarrow G$  is a homomorphism with  $G$  abelian and  $H' = [H, H]$  is the commutator subgroup, we may define an induced homomorphism by

$$\begin{aligned}\bar{\theta} : H/H' &\rightarrow G \\ hH' &\mapsto \theta(h)\end{aligned}$$

On the other hand, given  $\tau : H/H' \rightarrow G$ , define

$$\begin{aligned}\bar{\tau} : H &\rightarrow G \\ h &\mapsto \tau(hH')\end{aligned}$$

It shows the first identification. For the second one, recall that for any space  $X$ ,  $H_0(X) \simeq \mathbb{Z}$ . A result states that  $\text{Ext}(F, G) = 0$  if  $F$  is a free abelian group. The result follows from Theorem 4.2.  $\square$

**Remark 4.3.** In the case of  $G$  abelian, Theorems 4.3 and 4.4 agree by Proposition 4.4.

For compact manifolds without boundary the most remarkable Theorem relating cohomology and homology groups is Duality Theorem, also called Poincaré Duality Theorem.

**Theorem 4.5** (Duality, Teorema 15.3 - [8]). *Let  $X$  a connected closed orientable  $n$ -dimensional manifold and  $G$  abelian. Then there exists an isomorphism:*

$$H^k(X, G) \simeq H_{n-k}(X, G)$$

for all  $k \leq n$ .

We are in conditions to exhibit more examples.

### Closed Orientable Surfaces

Let  $X = \mathbb{M}_g$  be the closed orientable surface of genus  $g \geq 1$ . By Hurewicz Theorem, the first homology group is the abelianization of the fundamental group. Thus, one can show that:

$$H_1(\mathbb{M}_g) \simeq \mathbb{Z}^{2g}.$$

On the other hand, duality shows that

$$\begin{aligned}H^1(\mathbb{M}_g) &\simeq H_1(\mathbb{M}_g) \\ H^2(\mathbb{M}_g) &\simeq H_0(\mathbb{M}_g)\end{aligned}$$

If  $G$  is abelian, by Corollary 4.4, we have that:

$$H^1(\mathbb{M}_g, G) \simeq \text{Hom}(\mathbb{Z}^{2g}, G) \simeq G,$$

because  $\mathbb{Z}^{2g}$  is a ring and the image on the unity determines the homomorphism. We summarize some results:

**Proposition 4.5.**

1.  $[\mathbb{M}_g; \mathbb{S}^1] \simeq \mathbb{Z}^{2g}$ ;
2.  $[\mathbb{M}_g; \mathbb{T}^n] \simeq \mathbb{Z}^n$ ;
3.  $[\mathbb{M}_g; L((l_n), m)] \simeq \mathbb{Z}_m$ ;
4.  $[\mathbb{M}_g; \mathbb{C}\mathbb{P}^\infty] \simeq \mathbb{Z}$ ;

**Projective Spaces and the Infinite Sphere**

We start with a result about homology groups of the complex projective space.

**Theorem 4.6** ([12], Section 9.4). *Let  $G$  be an abelian group. Then*

$$H_k(\mathbb{C}\mathbb{P}^n, G) = \begin{cases} G, & \text{for } k \leq 2n, k \equiv 0 \pmod{2} \\ 0, & \text{otherwise} \end{cases}$$

**Proposition 4.6.** *Let  $Y$  be a space of type  $K(G, 1)$  with  $G$  abelian and  $1 \leq n \leq \infty$ . Then*

$$[\mathbb{C}\mathbb{P}^n; Y] = 0.$$

*Proof.* The complex projective plane is a compact differentiable manifold of real dimension  $2n$ . The van Kampen Theorem may be used to show that it is simply connected, see [9, Theorem 2.1]. Thus, we may apply the Poincaré duality to conclude that

$$H^1(\mathbb{C}\mathbb{P}^n, G) \simeq H_{2n-1}(\mathbb{C}\mathbb{P}^n, G) = 0.$$

Now, the result follows from Proposition 4.2. □

The remainder case is studying the homotopy classes with the infinite complex projective space.

**Proposition 4.7.** *Let  $1 \leq n \leq \infty$ . Then*

$$[\mathbb{C}\mathbb{P}^n; \mathbb{C}\mathbb{P}^\infty] \simeq \mathbb{Z}.$$

*Proof.* Consider first  $n < \infty$ . Then

$$H^2(\mathbb{C}\mathbb{P}^n) \simeq H_{2n-2}(\mathbb{C}\mathbb{P}^n) \simeq \mathbb{Z}.$$

If  $n = \infty$ , recall that  $\mathbb{C}\mathbb{P}^\infty$  is simply connected. Hence, Hurewicz Theorem shows that  $H_2(\mathbb{C}\mathbb{P}^\infty) \simeq \pi_2(\mathbb{C}\mathbb{P}^\infty) \simeq \mathbb{Z}$ . Therefore

$$H^2(\mathbb{C}\mathbb{P}^\infty) \simeq \text{Hom}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}.$$

□

For the quaternionic projective space  $\mathbb{K}\mathbb{P}^n$  the results are similar.

**Theorem 4.7** ([12], Section 9.4). *Let  $G$  be an abelian group. Then*

$$H_k(\mathbb{K}\mathbb{P}^n, G) = \begin{cases} G, & \text{for } k \leq 4n, k \equiv 0 \pmod{4} \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the quaternionic projective plane is a simply connected compact differentiable manifold of real dimension  $4n$ . The same arguments as before show the next proposition.

**Proposition 4.8.** *Let  $Y$  be a space of type  $K(G, 1)$  with  $G$  abelian and  $1 \leq n \leq \infty$ . Then*

$$[\mathbb{K}\mathbb{P}^n; Y] = 0.$$

The remainder case is

**Proposition 4.9.** *Let  $1 \leq n \leq \infty$ . Then*

$$[\mathbb{K}\mathbb{P}^n; \mathbb{C}\mathbb{P}^\infty] = \begin{cases} G, & \text{for } 4n \equiv 2 \pmod{4} \\ 0, & \text{otherwise, including } n = \infty \end{cases}$$

*Proof.* Let  $n < \infty$ . Then  $H^2(\mathbb{K}\mathbb{P}^n) \simeq H_{4n-2}(\mathbb{K}\mathbb{P}^n)$ . For  $n = \infty$ , recall that  $\mathbb{K}\mathbb{P}^\infty$  is a 3-connected space. By Hurewicz Theorem, the first two homology groups are trivial, then

$$H^2(\mathbb{K}\mathbb{P}^\infty) = \text{Hom}(H_2(\mathbb{K}\mathbb{P}^\infty), \mathbb{Z}) \simeq 0,$$

and the result follows. □

**Theorem 4.8.** *Let  $n \geq 1$ . It holds that  $H_n(\mathbb{S}^\infty) = 0$  for all  $n$ . Hence,  $H^n(\mathbb{S}^\infty, G) = 0$  for all  $G$  abelian.*

*Proof.* The result follows applying Hurewicz theorem for all  $n$ , because  $\mathbb{S}^\infty$  is contractible. The second statement follows from the Universal Coefficient Theorem:

$$H^n(\mathbb{S}^\infty, G) \simeq \text{Hom}(H_n(\mathbb{S}^\infty), G) = 0.$$

□

**Proposition 4.10.** *Let  $Y$  be any space of type  $K(G, n)$ . Then*

$$[\mathbb{S}^\infty; Y] = 0.$$

*Proof.* If  $G$  is abelian, the result follows from Theorem 4.8. If  $G$  is not abelian (in this case,  $n = 1$ ), we should recall the relation (1). The result follows from Theorem 4.4, since  $\pi_1(\mathbb{S}^\infty) = 0$ . □

## 5 Further Remarks

We shall comment some more advanced directions on the theory of Eilenberg-MacLane spaces as well as give references.

**Remark 5.1** (Postnikov Towers and Fibrations). By Proposition 1.2 we have constructed a sequence of spaces  $\{X_n\}$  such that

- $\pi_i(X)\pi_i(X_n)$  for all  $i \leq n$  and
- $\pi_i(X_n) = 0$  for all  $i > n$ .

It defines a sequence of fibrations  $p_n : X_n \longrightarrow X_{n-1}$  with fiber being a space of type  $K(\pi_n(X), n)$ . This sequence is understood as an approximation for  $X$ . More details can be seen in [1, Example 4.16]. The general idea is that there exists a sequence of fibrations

$$\dots X_n \xleftarrow{p_{n+1}} X_{n+1} \xleftarrow{p_{n+2}} \dots,$$

which is convergent in some sense such that for each  $n$  the fiber is a space of type  $K(G, n)$ . See also [4, Section 8.3] about Moore-Postnikov systems.

**Remark 5.2** (Extension of a Theorem). Theorem 4.3 may be extended to CW pairs as well as non path-connected spaces, see [4, Theorem 8.1.10]. An even better result is considering instead of a space of type  $K(G, n)$  an  $(n - 1)$ -connected space under a proper condition, see [4, Theorems 8.1.15 and 16].

**Remark 5.3** (Cohomology Operations). Perhaps the most powerful result on Eilenberg-MacLane spaces is related to Cohomology operations, where they play a role of classifying spaces.

**Definition 5.1.** A cohomology operation of type  $(n, q, G, H)$  is a natural transformation  $\theta : H^n(., G) \longrightarrow H^q(., H)$ . The set of all operations of type  $(n, q, G, H)$  is denoted by  $\Theta(n, q, G, H)$ .

**Theorem 5.1** ([4], Theorem 8.1.13). There is an isomorphism:

$$\Theta(n, q, G, H) \simeq H^q(K(G, n), H).$$

We also refer the reader to [3, Chapter 4].

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