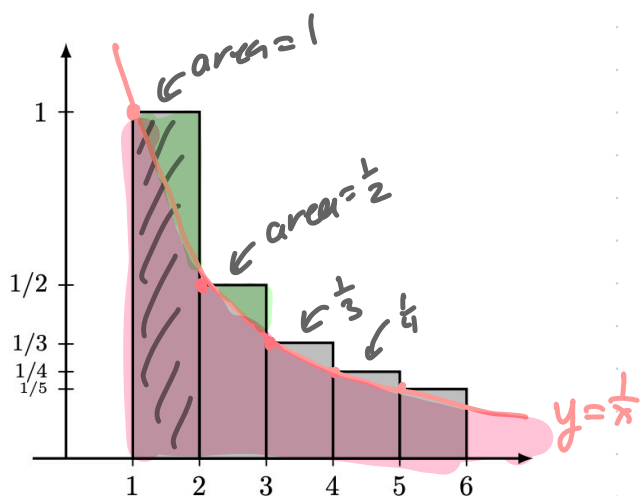


Integral Test:

ex: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

* called the harmonic series



Recall: $\int_1^{\infty} \frac{1}{x} dx$ diverges

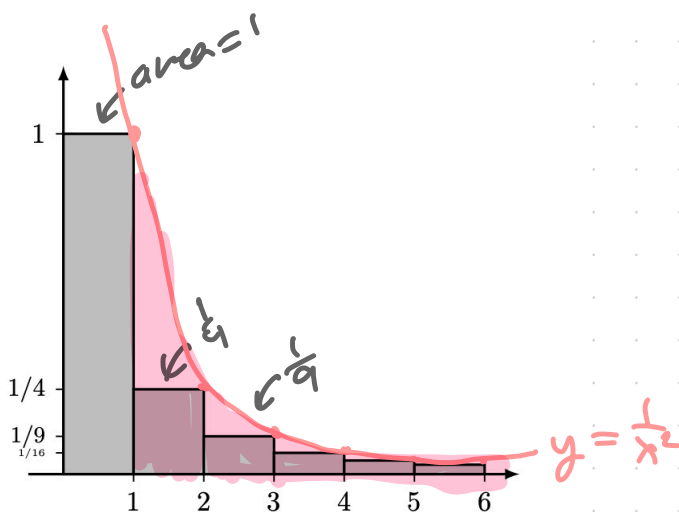
Compare:

$$\infty = \int_1^{\infty} \frac{1}{x} dx < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$$

also goes to ∞

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

ex: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$



Recall: $\int_1^{\infty} \frac{1}{x^2} dx$ converges ($p=2>1$)

Compare:

$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2} dx$$

$$\frac{+ 1}{\quad} \quad \frac{+ 1}{\quad}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \underbrace{\int_1^{\infty} \frac{1}{x^2} dx}_{\text{a number}}$$

$\Rightarrow$ series converges.

The Integral Test:

Given a series $\sum_{n=k}^{\infty} a_n$, if $f(x)$ is a function such that $f(n) = a_n$ for each natural number n and

- f is continuous
 - f is positive
 - f is decreasing
- } for $x \geq k$

then $\sum_{n=k}^{\infty} a_n$ converges if and only if the improper integral $\int_k^{\infty} f(x) dx$ converges.

HW hint

"if and only if" is saying 2 things:

① if $\int_k^{\infty} f(x) dx$ converges then $\sum_{n=k}^{\infty} a_n$ converges

② if $\int_k^{\infty} f(x) dx$ diverges, then $\sum_{n=k}^{\infty} a_n$ diverges

ex: $\sum_{n=1}^{\infty} \underline{\underline{n e^{-n^2}}}$

converges or diverges?

Int Test: Let $f(x) = x e^{-x^2}$.

Check the 3 conditions: cts, positive and decreasing on $[1, \infty)$

f is continuous ✓

f is positive ✓ on $[1, \infty)$ because $x > 0$ on $[1, \infty)$ and so is e^{-x^2}

f is decreasing?

$$\begin{aligned} f'(x) &= e^{-x^2} + x(-2x e^{-x^2}) \\ &= \underbrace{e^{-x^2}}_{>0} (\underline{\underline{1 - 2x^2}}) \end{aligned}$$

$$f'(x) = 0 \quad \text{if} \quad 1 - 2x^2 = 0$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \sqrt{\frac{1}{2}}$$



pt.	f'
∞	-

f is decreasing ✓ on $[\frac{1}{\sqrt{2}}, \infty)$

so also on $[1, \infty)$

So Int. Test applies!

Now compute

$$\int_1^{\infty} x e^{-x^2} dx = \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx$$

$$u = -x^2$$
$$du = -2x dx$$

$$= \lim_{t \rightarrow \infty} \int_{-1}^{-t^2} e^u \left(-\frac{1}{2}\right) du$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} \left(e^u \Big|_{-1}^{-t^2} \right)$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} \left(\underbrace{e^{-t^2}}_{\downarrow 0} - e^{-1} \right)$$

$$= -\frac{1}{2} \left(-\frac{1}{e} \right) = \frac{1}{2e}$$

converges ↗

So the series converges by the Int. Test.

(Note: the value of the series is NOT $\frac{1}{2e}$!)

Showing f is decreasing:

- use derivative

- or its obvious: $\frac{1}{x}$

- or use $\frac{1}{\text{increasing}}$ is decreasing

e.g. $\frac{1}{x \ln x}$

ex: $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$$f(x) = \frac{\ln x}{x}$$

↑
looks
integrable

Int. Test: Let $f(x) = \frac{\ln x}{x}$.

f is continuous on $[1, \infty)$

f is positive if $\ln x > 0$
 $x > e^0 = 1$

f is decreasing?

$$f'(x) = \frac{\left(\frac{1}{x}\right) \cdot x - \ln x \cdot 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

←
always ≥ 0

$$f'(x) < 0 \text{ if } 1 - \ln x < 0$$
$$1 < \ln x$$
$$e^1 < e^{\ln x} = x$$

f is decreasing $[e, \infty)$

↑ good enough ✓
not $[1, \infty)$
but that's ok.

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx$$

$$u = \ln x$$

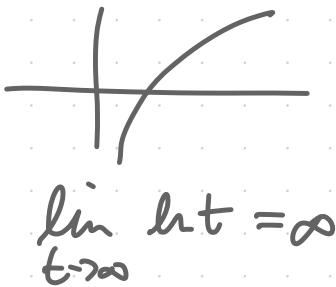
$$du = \frac{1}{x} dx$$

$$= \lim_{t \rightarrow \infty} \int_0^{\ln t} u du$$

$$= \lim_{t \rightarrow \infty} \left(\frac{1}{2} u^2 \right) \Big|_0^{\ln t}$$

$$= \lim_{t \rightarrow \infty} \frac{1}{2} (\ln t)^2$$

$$= \infty$$



the integral diverges so the series
diverges by Int. Test.

The decreasing issue:

decreasing after some point is good enough

$$\underbrace{\sum_{n=1}^{\infty} \frac{\ln n}{n}}_{\text{diverges too}} = \underbrace{a_1}_{\uparrow} + \underbrace{a_2}_{\uparrow} + \underbrace{\sum_{n=3}^{\infty} \frac{\ln n}{n}}_{\text{diverges}} *$$