

Error Bounds for Midpoint & Trapezoidal Rules

approximating $\int_a^b f(x) dx$

error using Midpt Rule: $E_M = \underbrace{\int_a^b f(x) dx}_{\text{true value}} - \underbrace{M_n}_{\text{approx.}}$

error using Trap. Rule: $E_T = \int_a^b f(x) dx - T_n$

If there exists a number K such that $|f''(x)| \leq K$ for all x in $[a, b]$ then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad \text{and} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2}$$

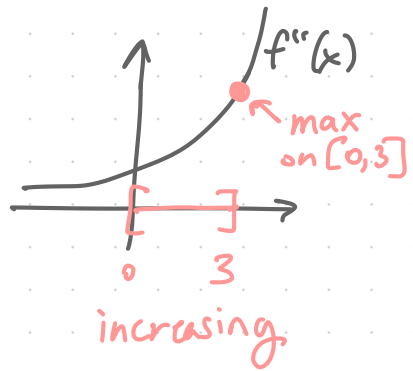
ex: Find a bound for the error in approx. $\int_0^3 2^x dx$ using $n=10$ trapezoids. (T_{10})

Given: $f(x) = 2^x$, $a=0$, $b=3$, $n=10$

Need to determine K .

$$f'(x) = 2^x \cdot \ln 2$$

$$f''(x) = 2^x \cdot \underbrace{(\ln 2)^2}_{\text{constant}}$$



Take K to be the max of $f''(x)$ on the interval $[a, b]$. If $f''(x)$ is incr. then max is the right end point.

$$K = f''(3) = 2^3 (\ln 2)^2 = 8 (\ln 2)^2$$

$$\begin{aligned} \text{Now } |E_T| = |\text{error}| &\leq \frac{K (b-a)^3}{12 n^2} \\ &= \frac{(8 (\ln 2)^2)(3-0)^3}{12 \cdot 10^2} \end{aligned}$$

$$\approx 0.0865$$

example: How large does n need to be so that M_n is within 0.001 of $\int_0^1 e^{x^2} dx$?

Given: $f(x) = e^{x^2}$, $a=0$, $b=1$

Want: n

$$\underbrace{|E_n|}_{\text{error}} \leq \underbrace{\frac{K(b-a)^3}{24n^2}}_{\text{bound that we can guarantee}} \leq \underbrace{0.001}_{\text{desired error}}$$

solve for this

Find K : $f(x) = e^{x^2}$

$$f'(x) = e^{x^2} \cdot 2x$$
$$f''(x) = (2 + 4x^2)e^{x^2}$$

Is $f''(x)$ increasing / decreasing on $[0,1]$?

Look at f''' to see.

$$\begin{aligned}
 f'''(x) &= (12x + 8x^3)e^{x^2} \\
 &= x(12 + 8x^2)e^{x^2}
 \end{aligned}$$

$\uparrow$
 ≥ 0
 on $[0, 1]$

≥ 0 ≥ 0 ≥ 0

Means $f'''(x) \geq 0$ on $[0, 1]$

$\Rightarrow f''(x)$ is increasing on $[0, 1]$

$\Rightarrow$ Max occurs at right endpoint

Choose $K = f''(1) = 6e$

$$|E_m| \leq \frac{(6e)(1-0)^3}{24 n^2} \leq 0.001$$

$$\frac{6e}{24 n^2} \leq \frac{1}{1000}$$

$$1000 \cdot \cancel{4n^2} \cdot \frac{e}{\cancel{4n^2}} \leq \frac{1}{\cancel{1000}} \cdot \cancel{1000} \cdot \cancel{4n^2}$$

$$1000e \leq 4n^2$$

$$\sqrt{\frac{1000e}{4}} \leq \sqrt{n^2}$$

$$26.07 \approx \sqrt{\frac{1000e}{4}} \leq n$$

$$n = 27$$

next largest
integer