

Ratio Test: For the series $\sum a_n$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \leftarrow \text{important}$$

① If $L < 1$, the series is absolutely convergent (hence convergent)

② If $L > 1$, the series is divergent
 $\uparrow$ includes if $L = \infty$

③ If $L = 1$, the test is inconclusive
(means you know NOTHING)

Two ways that Ratio Test can fail:

$L = 1$
inconclusive

OR

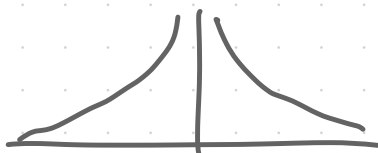
$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \text{ DNE}$
(no L to make a conclusion with)

$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ DNE}$$



different

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$



If a limit equals ∞ , it exists (limits can equal ∞)

ex: $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

$$a_n = \frac{n^5}{5^n}$$

$$a_{n+1} = \frac{(n+1)^5}{5^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^5}{5^{n+1}}}{\frac{n^5}{5^n}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^5}{5^{n+1}} \cdot \frac{5^n}{n^5} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{n^5} \cdot \frac{\cancel{5^n}}{\cancel{5^n} \cdot 5}$$

every thing $\rightarrow 0$
so drop

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{5n^5}$$

$$= \frac{1^5}{5} = \frac{1}{5} = L$$

$$L < 1 \text{ so}$$

the series converges
(~~absolutely~~)

↑
all positive
terms
anyways

Handouts: limits of rational
sequences

by the Ratio Test

ex: $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

Ratio Test works really
well on series involving
factorials!

$$\begin{aligned} \text{Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{10^{n+1} \cdot \cancel{n!}}{10^n \cdot (n+1) \cdot \cancel{n!}} \right| \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\cancel{10^n} \cdot 10}{\cancel{10^n} \cdot (n+1)} \right|$$

★ important to include!
↓

$$= \lim_{n \rightarrow \infty} \frac{10}{n+1} = 0 < 1$$

series converges by Ratio Test

ex: $\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!}$

$$a_n = \frac{(-2)^n}{(2n+1)!}$$

$$a_{n+1} = \frac{(-2)^{n+1}}{(2(n+1)+1)!}$$

$$= \frac{(-2)^{n+1}}{(2n+3)!}$$

Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-2)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\cancel{(-2)^n} (-2) \cdot \cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!} \cancel{(-2)^n}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{-2}{(2n+3)(2n+2)} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2}{(2n+3)(2n+2)} = 0 < 1$$

converges by Ratio Test.

non-ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ ← Comp. Test with $\sum \frac{1}{n^2}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^2+1} \cdot \frac{n^2+1}{1} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} = 1 \end{aligned}$$

Ratio Test is inconclusive!

Ratio Test is not a good choice for series like $\sum \frac{p(n)}{q(n)}$ where p, q are polynomials in n .

$$\sum \frac{10^n}{n!}$$

factorial

⇒ use Ratio Test

$$\sum \frac{n^5}{5^n}$$

5^n piece will make Ratio Test

Why does the Ratio Test work?

In Ratio Test, we compute

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

For a geometric series, $r = \frac{a_{n+1}}{a_n}$ ← common ratio

converges if $|r| < 1$
diverges if $|r| \geq 1$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

like $|r|$

← tests if your series behaves like a geometric series "in the long run" (as n goes to ∞)

ex: $\sum_{n=1}^{\infty} \frac{n^{100} \cdot 100^n}{n!}$

← expand to $100^n \cdot 100$

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^{100} \cdot 100^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{100} \cdot 100^n} \right|$$

↑ expand to $(n+1) \cdot n!$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{100} (\cancel{100^n} \cdot 100) \cancel{n!}}{((n+1) \cdot \cancel{n!}) n^{100} \cdot \cancel{100^n}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{100} \cdot 100}{(n+1) \cdot n^{100}} \right|$$

$$= 100 \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{\cancel{100}^{99}}}{(n+1) \cdot n^{100}} \right|$$

$$= 100 \lim_{n \rightarrow \infty} \frac{(n+1)^{99}}{n^{100}}$$

$$= 0 < 1$$

converges by the Ratio Test

The Divergence Test: If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum a_n$ diverges.

know NOTHING

Note: You can only conclude *divergence* using the Divergence Test. If $\lim_{n \rightarrow \infty} a_n = 0$, the test is *inconclusive*.

The Geometric Series Test: This test only applies to geometric series, i.e. series of the form $\sum_{n=1}^{\infty} ar^{n-1}$. This series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Note: If the series converges, then we can compute the value using the formula $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. OR first term $\frac{a}{1-r}$

The p-Series Test: This test only applies to p-series, i.e. series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. A p-series converges if $p > 1$ and diverges if $p \leq 1$. and if $p = 1$, $\sum \frac{1}{n}$ is harmonic $\Rightarrow$ diverges

The Integral Test: Given $\sum a_n$, if $f(n)$ is a function such that $f(n) = a_n$ for each n and f is continuous, positive and decreasing, then

usually take the derivative $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x) dx$ converges.

function not the terms of the series

The (Direct) Comparison Test: Suppose $\sum a_n$, $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for each n , then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $b_n \leq a_n$ for each n , then $\sum a_n$ is also divergent.

e.g. $\sum \frac{1}{n^2}$

$\frac{1}{x^2}$ is cts, pos. decreasing
NOT $\frac{1}{n^2}$

The Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \quad \text{where } c \text{ is positive and finite}$$

then either both series converge or both diverge.

Note: for both types of Comparison Test, good choices of series to compare to are p-series or geometric series.

The Alternating Series Test: This test applies only to alternating series, i.e. series that can be written $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$ where $b_n > 0$. If

- $\lim_{n \rightarrow \infty} b_n = 0$, and
- $\{b_n\}$ is decreasing,

$\leftarrow$ show $b_{n+1} < b_n$ for all n (not just first few terms!)

then the series is convergent.

OR take the derivative

Note: If one of the two conditions above is not true, Alternating Series Test is *inconclusive*. However, if the Alternating Series Test fails, the Divergence Test is a good idea to try next.

The Ratio Test: Suppose we have a series $\sum a_n$ and

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

- If $L < 1$, the series is absolutely convergent (hence convergent).
- If $L > 1$, the series is divergent.
- If $L = 1$, the test is inconclusive.

$\leftarrow$ means you know nothing about the series, try something else!

stay organized, two series involved so keep them separate, look at notes and follow format

1. Select the best test for determining whether the following series converge or diverge. Hint: try to eliminate the tests that do not apply. (You do not need to determine whether the series actually do converge or diverge.)

(a) $\sum_{n=1}^{\infty} \frac{3^{2n-1}}{(2n+1)!}$

~~A.~~ p -series Test

B. Divergence Test

~~C.~~ Geometric Series Test

☒ D. Ratio Test

$\uparrow$ inconclusive b/c

$\lim_{n \rightarrow \infty} \frac{3^{2n-1}}{(2n+1)!} = 0 \leftarrow$ by comparing growth rates of numerator and denominator

(b) $\sum_{n=2}^{\infty} \frac{5n^3}{\sqrt{n^7-4}}$

~~A.~~ Alternating Series Test

~~B.~~ Divergence Test

not alt.

~~C.~~ Ratio Test

☒ D. Limit Comparison Test

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{5n^3}{\sqrt{n^7-4}} = \lim_{n \rightarrow \infty} \frac{5n^3}{n^{7/2}} = \lim_{n \rightarrow \infty} \frac{5}{n^{1/2}} = 0$$

inconclusive

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n^2-2}$

☒ A. Alternating Series Test

B. Divergence Test

~~C.~~ Integral Test

~~D.~~ Ratio Test

$\leftarrow$ need positive terms

(d) $\sum_{n=2}^{\infty} \frac{10}{n(\ln(n))^2}$

$$f(x) = \frac{10}{x(\ln x)^2}$$

$$\int \frac{10}{x(\ln x)^2} dx \quad u = \ln x$$

~~A.~~ Alternating Series Test

~~B.~~ Divergence Test

☒ C. Integral Test

D. Ratio Test

more conceptual questions

2. Find an example of each of the following or explain why such an example does not exist.

(a) A sequence $\{a_n\}$ such that $\{a_n\}$ converges to 0, but $\sum_{n=1}^{\infty} a_n$ diverges.

(b) A sequence $\{a_n\}$ such that $\{a_n\}$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges.

(c) A sequence such that $\left| \frac{a_{n+1}}{a_n} \right| < \frac{1}{2}$ for all n and the series $\sum_{n=1}^{\infty} a_n$ diverges.

3. Suppose $a_n > 0$ for all n and the series $\sum_{n=1}^{\infty} a_n$ converges.

(a) Must the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converge? Explain why or why not.

(b) Why is it true that $a_n < 1$ for all n after a certain point?

(c) Show that $\sum_{n=1}^{\infty} a_n^2$ must converge. (Hint: use the Comparison Test)