

The Invertible Matrix Theorem. Let A be an $n \times n$ matrix.* The following are equivalent:

- (a) A is invertible
- (b) $A \sim I$ (i.e. A is row equivalent to the $n \times n$ identity matrix)
- (c) A has n pivot positions
- (d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- (e) The columns of A are linearly independent
- (f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
- (g) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
- (h) $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^n$
- (i) The columns of A span $\mathbb{R}^n$
- (j) There exists an $n \times n$ matrix C such that $CA = I$ (i.e. A has a left inverse)
- (k) There exists an $n \times n$ matrix D such that $AD = I$ (i.e. A has a right inverse)
- (l) A^T is invertible
- (m) The columns of A form a basis of $\mathbb{R}^n$
- (n) $\text{Col}(A) = \mathbb{R}^n$
- (o) $\text{rank}(A) = n$
- (p) $\text{Nul}(A) = \{\mathbf{0}\}$
- (q) $\text{nullity}(A) = 0$
- (r) $\det(A) \neq 0$
- (s) 0 is not an eigenvalue of A

*It's important that A is a square matrix!

Theorem 8 (*Linear Transformations that are One-to-One*). Suppose A is an $m \times n$ matrix and $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the associated linear transformation, $T(\mathbf{x}) = A\mathbf{x}$. The following are equivalent:

- (a) T is one-to-one
- (b) The range of T has dimension n
- (c) For each $\mathbf{b} \in \mathbb{R}^m$, $T(\mathbf{x}) = \mathbf{b}$ has at most one solution
- (d) For each $\mathbf{b} \in \mathbb{R}^m$, either $A\mathbf{x} = \mathbf{b}$ has a unique solution or is inconsistent
- (e) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- (f) The columns of A are linearly independent
- (g) A has a pivot position in every column

Theorem 9 (*Linear Transformations that are Onto*). Suppose A is an $m \times n$ matrix and $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the associated linear transformation, $T(\mathbf{x}) = A\mathbf{x}$. The following are equivalent:

- (a) T is onto
- (b) The range of T is equal to the entire codomain, i.e. $\text{range}(T) = \mathbb{R}^m$
- (c) $T(\mathbf{x}) = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbb{R}^m$
- (d) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$
- (e) The columns of A span $\mathbb{R}^m$
- (f) A has a pivot position in every row